Applied Linear Algebra
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Week 02
Algebraic operations on linear maps

Hello and welcome to this final lecture for week two. This is on an interesting topic which deals with algebraic operations on linear maps. Right. So first of all you think of linear maps themselves as operations on vectors, right? So linear maps take one vector, spit out another vector. Input, output. It turns out you can take these linear maps themselves and keep doing operations on them. It should not be very surprising to you because linear maps are represented by matrices and you know you can do operations on matrices, right? You can add matrices, you can multiply matrices, you can do a lot of things. Maybe these were taught to you as just operations. Okay, here is the matrix. This is how you add a matrix, this is how you multiply a matrix... So in this lecture you will sort of see a motivation for why matrix operations are defined in that fashion. It turns out they are all related to how linear maps have to be operated upon, how algebraic operations work on linear maps and how that corresponds to matrices in some sense, okay? So let us get started.

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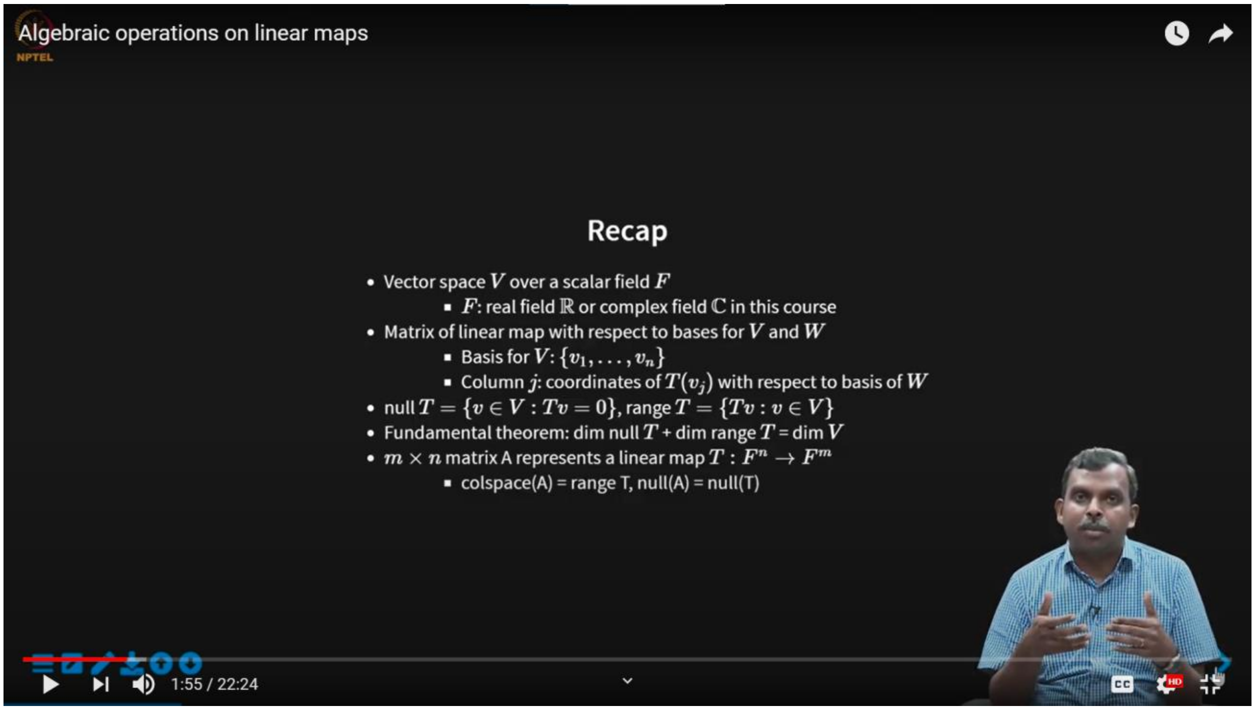

Okay, a quick recap. We have seen what linear maps are. We have seen this fundamental theorem of linear maps which is very powerful. Dimension of V equals dimension of null space plus dimension of range space, right? So splitting up the linear map itself as, how it works, you know, to understand the set that it brings down to zero and to understand the set of all elements of the range that it covers... So all of these things give you some illuminating ideas on what the linear map is. And we saw that to do calculations with it, it's good to think of a linear map in terms of a matrix, right? So once you associate a matrix with it, there are standard things like column space, nullity, rank, all of that have corresponding elements in linear map and you have to think of a matrix as really representing a linear map. All of it, all of that is, I mean all of the things that you do with matrices are sort of related to what happens when you do, when you operate on linear maps also, okay? So we'll now proceed and talk about algebraic operations on linear maps and how that connects to matrix operations, okay?

So the first two things we'll study are addition and scalar multiplication of linear maps, okay? So it's sort of easy to see, but I want to emphasize some things and be careful about some things here, okay? So you have two linear maps between the same pair of vector spaces V to W, you have two linear maps S and T. So you can do addition of the two linear maps which I will call $S+T$. Remember now, this operation plus is on the two linear maps. S is a linear map, T is a linear map. I'm adding the two linear maps, okay? So how do I define linear maps? All I have to say is - for a particular input what output it will produce. So once I say that, I have defined my linear map. So how do I define this $S+T$ linear map? $S+T$ operating on any vector v simply gives me $Sv + Tv$, okay? So it is a direct and simple definition. This is how we define addition of linear maps.

The same way we can define scalar multiplication in linear maps, okay? So if you have a linear map T and a scalar lambda, lambda belonging to the field F, you can multiply λ and T and define a new linear map called λT, okay? And this λT, how do I define it? I simply have to specify how λT operates on an input to produce an output. How do I do that? I say λT working on an input vector v simply produces λ times Tv, okay? So T operates on the vector v. I know, I have defined T already. So you simply do this λT in this fashion, okay? So some very simple results. First of all $S+T$ and λT are valid linear maps from V to W. You can take it as an exercise and prove it. It's very easy to prove, okay? And then there is this interesting set you can consider. $L(V,W)$ which is the set of all linear maps from V to W, okay? So there are many linear maps. The set of all possible linear maps you can put them into this set $L(V,W)$ and you can show that this $L(V,W)$ is in fact a vector space over F under the addition and scalar multiplication that have been defined here, okay? And in fact the additive identity will be the zero map, okay? So all of these things you can do.

So notice what we have done here. We first defined vector spaces. And then we defined linear maps from one vector space to another. And now we are looking at the space of all possible linear maps. And guess what? That again is a vector space, okay? So you can see how this thing keeps recurring again and again. So vector spaces will occur in so many places and that helps you tremendously in simplifying your thought process and thinking about how to classify, you know, operators and all that, okay? So it is very important to know this idea. So this idea is crucial. I am not doing a proof for why this is a vector space, it is quite easy to write down. So you just have to mechanically write down and check all those properties that are true, okay? So with respect to this addition and this scalar multiplication defined in this slide, the $L(V, W)$, the set of all linear maps from V to W is a vector space, okay? Okay, so in addition, so usually in this world of linear maps it turns out you can also define the product of two linear maps, okay? So this takes a little bit of setup carefully, okay? So it's not very simple and direct, the way the product is defined. So let me just walk you through the definition, okay? First thing is you need three vector spaces, okay? So see, remember when you think of a linear map, it goes from V to W. Now I want to multiply two linear maps. So it turns out I need three vector spaces and all of them over the same field F, and I need two maps. I mean of course I have to multiply two maps, I need two maps. So T is a map that goes from U to V, okay? And S is a map that goes from V to W, okay? U, V, W. T is a map that goes from U to V and S is a map which goes from V to W.

Maybe a picture here is not out of order. So you have U, you have V, you have W, and this T map goes from U to V and this S map goes from V to W, okay? So keep that in mind, once I have a situation like this, then I can define a product ST, okay? So notice how the ordering goes, right? So you do T from U to V and S from V to W. And then I define the product ST, okay? ST how do I define? First of all, ST it turns out eventually will be a linear map from U to W, okay? And how do I define ST? I say I have to only specify how it operates on a vector from U and produces a vector from W, right? Once I do that I have defined ST. So how do I define ST? ST operating on u is nothing but S operating on Tu, okay? So what is Tu? Tu takes something from U and puts out something from V, and then what will S do? It will take that something from V and put out something from W, okay? So what I am doing here actually, though I call it as a product of a linear map, what I am actually doing is a composition of two maps, okay? I have two maps T and S which are composable, they are properly defined. The T goes from U to V, and then S takes from V to W. So I can of course compose them, okay? But when I compose them, usually it is common to write it in this fashion ST as opposed to TS, okay? Why is that? Because T acts first on the vector, we always put the input on the right, so T acts first on the vector. So you should write it as ST, that's all. Just keep this little twist in mind, it is not too difficult to understand.

But product of two linear maps is nothing but composition of the two maps, okay? And it's defined only when the composition is possible. So for instance if you know this V is not common... See the V is a sort of a common vector space, right? T goes from U to V and S goes from V to W. Only then it makes sense to compose. If this V is not the same, then you can't compose, you know? When you go from here to there. And then, I mean, it's not possible, right? So you cannot compose if it's not the same. So the V needs to be the same, okay? So that condition is there. I mean, not all maps are composable. Only when this V is the same, you have a composible thing. So not all products are defined, okay? So once you define this, you can come up with lots of nice algebraic properties. So for instance, multiplication is associative. You can prove associativity, okay? I am not going to prove it in this class. But you can write down the proof if you like. So if you do

$T_1T_2T_3$, it is unambiguously defined. It does not matter whether you multiply T_1T_2 first and then you multiply T_3, or you multiply T_2T_3 first, and then you multiply T_1. Both ways you get the same answer, so you don't need to put all these brackets, you can just write down multiplication, okay?

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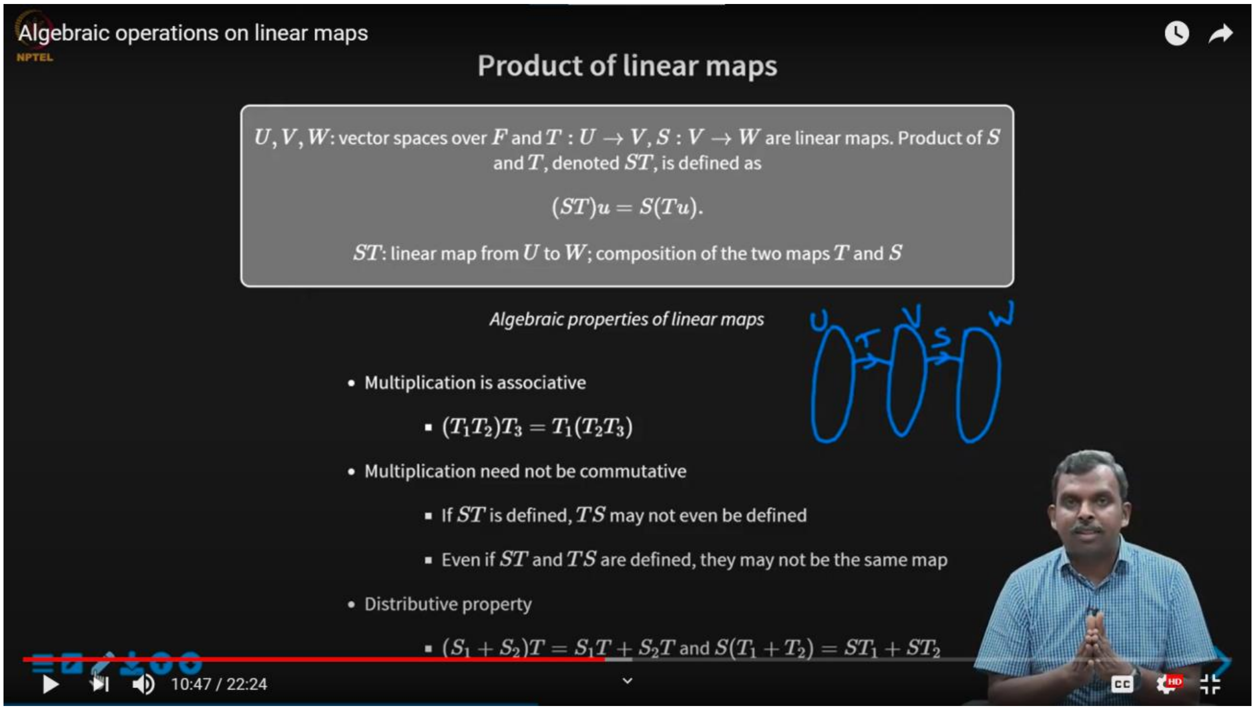

But you have to be very careful. Linear maps need not be commutative under multiplication, okay? So we are all used to multiplying numbers, you always think ab is ba, ab is ba. But in this case, in fact, even if ST is defined, TS need not even be defined. Take a look at this U, V, W. If you want to compose, you know, T first and then S next, you can do. But if you have to do S first and then T next, it's not even properly defined, right? Because S gives you, it takes an input from V and puts out W. T takes inputs from U, then what will it do with W? It can't do anything with W, right? So all these problems are there when you compose, right? So just composing, changing the order of composition may not even be defined. And even if it is defined, they need not be the same map, okay? So we will see maybe some examples later on. Maybe in the, you know, assignments you'll see some examples of this and precisely prove it. So commutativity will not necessarily hold for this product or composition of linear maps, but distributivity will hold, okay? So if you do $(S_1 + S_2)T$, it's the same as $S_1T + S_2T$. You can see why these things have to be true, okay? So look at what we can do with linear maps, right? So linear maps are a vector space. Plus there is a product defined on that vector space. Usually vectors may not have products and that product is very nice. It plays well with the plus of the distributive, it has some associativity property. It's not commutative. Maybe if it was commutative, in fact things would be very different. So it’s not commutative. So it becomes an interesting thing at that level, okay? So I'll leave it at this. As far as the algebra is concerned, think about what it is and maybe later on if you do some advanced courses in linear algebra you'll see more properties of such things, okay?

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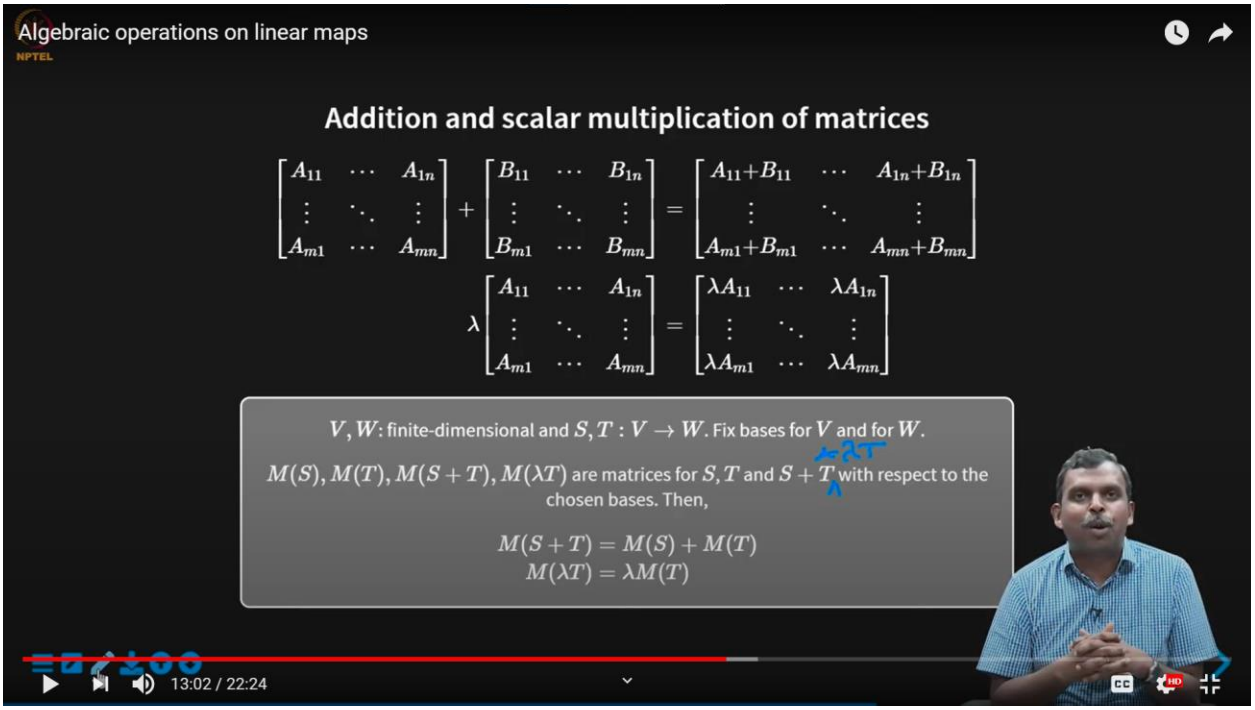

So what we will do next in this lecture is to connect these things to properties of matrices, right? When you have matrices, you can add matrices, you can do scalar multiplication of matrices, you can multiply matrices. And how are these things connected to these notions that we defined with linear maps? Addition of linear maps, scalar multiplication of linear maps, product of linear maps, what is the connection, okay? So let us see that next. So here is addition and scalar multiplication of matrices, okay? So you have two matrices, they have to be of the same size, right? $m \times n$, otherwise you can't add them. When you add them, what do you do is just coordinate-wise addition. So we know how to add them. Same thing with scalar multiplication. Any $m \times n$ matrix, multiply with a scalar, what do you do? You take each scalar inside, every element gets multiplied. So this is the rule for addition and scalar multiplication. Where do these things come from? It turns out, here is the interesting result, these things come because of this property, okay? So what is this property? It turns out if you have two finite dimensional vector spaces V, W and two linear maps S, T from V to W, and if you fix the same basis. When you keep the bases fixed for V, and W and find the matrix of S, matrix of T, matrix of $S+T$, and matrix of λT, right? Corresponding to each of these things... I think I missed out one of these things, so maybe I should add that... and λT with respect to this chosen basis, then it turns out the matrix of $S+T$ is nothing but the matrix of S plus matrix of T, okay? The usual matrix addition that you are used to do. So this is the reason why the matrices are added in that fashion, okay? So you add the corresponding elements because when you add the linear map, this is the corresponding thing to do the same with $M(\lambda T)$, okay? When you look at the matrix of λT, it is exactly λ times the matrix $M(T)$ okay? So one can prove these things. I am not going to prove it. It's not very hard to prove this. You have to write it down carefully. But this is the motivation for addition and scalar multiplication, okay?

What about multiplication of two matrices? All of you have seen the definition. Probably been confused by it, probably given complicated problems based on the multiplication. But where did the multiplication come from? Why do people define multiplication for matrices in that complicated fashion, or simple fashion, whatever fashion you think of? Once again that's because of the connection to linear maps, okay? So let's look at matrix multiplication. It's a bit complicated if you have not seen it. But there are some rules, right, important rules to keep in mind. The first matrix, okay, when you multiply two matrices, the first matrix has to be... Let us say, if it is $m \times n$, the second matrix better have the same number of rows, right? This n has to be the same. If the n is not the same, then you know you don't even multiply, you can't multiply, you say product doesn't exist, right? It's not correct multiplication, okay? Lots of reasons for that, we'll see now, okay? So how is that product exactly defined? The C_{ij}, the ijth element of the product - you take the ith row and the jth column and do sort of a dot product, right? So summation l equals 1 to n, you can do $A_{il} \times B_{lj}$, right? So that's the, that's what you do and then that gives you the, that gives you the value for the ijth element of the product, okay? And the reason why you do it… We won't see a big proof for this. There is a proof in your book. You can go back and look at it if you like. But it turns out if you have three finite dimensional vector spaces U, V, W and if you have a T going from U to V and S which is another linear transformation going from V to W, and if you fix bases for U, V, W, find matrices for $M(S)$, $M(T)$ and $M(ST)$, okay? You know what ST is, right? ST goes from U to W, right? And if you find those bases, then it turns out if you define matrix multiplication in this fashion the matrix corresponding to ST becomes equal to matrix of S times matrix of T. So in fact this is the reason why matrix multiplication is defined in this fashion, okay? So if you define it like that, the product, the composition of the linear transformation is well respected.

So what does that mean now? So it means that this guy represents S, let us say which goes from $\mathbb{F}^m \to \mathbb{F}^n$ and this guy represents T which goes from $\mathbb{F}^k \to \mathbb{F}^n$, okay? I got this wrong I think, its not $\mathbb{F}^m \to \mathbb{F}^n$, it's $\mathbb{F}^n \to \mathbb{F}^m$, right? So this goes from $\mathbb{F}^k \to \mathbb{F}^n$. So this one is nothing but ST which goes from $\mathbb{F}^k \to \mathbb{F}^m$, isn't it? So that is the trick here. So you see that only when n is common this $\mathbb{F}^n$ becomes the common space that comes in the middle. So S and T become composable. So you can do all this, okay? And the definition also there is a little proof that you have to do to calculate and show that, you know, you write the... Pick a basis, write it etc. and then C_{ij} has to be equal to this summation, okay? So you can prove it, actually, that - this way of defining the matrix multiplication is the correct thing to do if you want the composition of the linear maps, the product of the linear maps to correspond to the product of matrix multiplication, okay? So we will see more details of the matrix multiplication. I will give you a couple of examples to give you some thinking about what matrix multiplication is, how does it work, etc. okay? So let us go on, okay?

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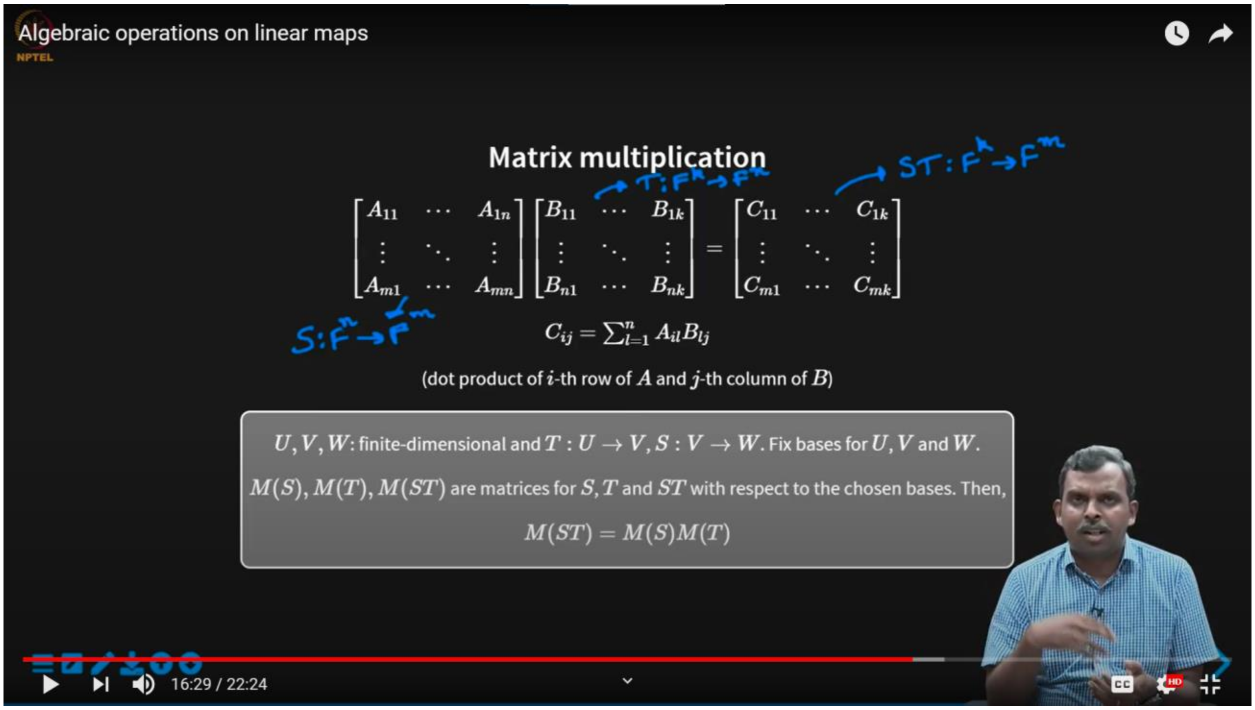

So the first two things I will talk about is a couple of interesting matrix multiplications. One interesting matrix multiplication is - if you multiply a row matrix with a column matrix, both having the same number of coordinates, okay? So this is like the familiar, you know, dot product, scalar product as it's called. The reason is - the output, when you, after you multiply, is simply a scalar, right? So $a_1b_1 + a_2b_2 + \cdots + a_nb_n$. Now if you think in terms of the composition of linear maps, if you have a, you know, row matrix, right? So this is S. This S is right here, okay? This matrix, just the $b_1, b_2, \ldots, b_n$, actually corresponds to transformation from $\mathbb{F}^n \to \mathbb{F}$, right? Now this one is a transformation from $\mathbb{F} \to \mathbb{F}^n$, okay? So overall when you do the composition ST, it becomes a transformation from $\mathbb{F} \to \mathbb{F}$, okay? So that is why, that is what is going on here. This is one way to sort of view it.

The other contrasting sort of situation is here, okay? So you have a column vector multiplying a row vector, okay? So that, you know, will give you a rectangular matrix, $m \times n$ matrix, and that also has a nice correspondence with linear maps. This S here is a $\mathbb{F} \to \mathbb{F}^m$ map. This T here is a $\mathbb{F}^n \to \mathbb{F}$ map. So when you compose ST you go from $\mathbb{F}^n \to \mathbb{F}^m$, okay? So these are all things to nicely think about. You can see how, you know, even though this ST goes from $\mathbb{F}^n \to \mathbb{F}^m$, this has got like, you know, dimension of range of ST is actually 1, okay? So this matrix has only rank 1, okay? You might have studied this in other context but that has only rank 1, you can see why it has rank 1, right? So you see every column is simply a multiple of the first column, right? b_1 multiplies a_1... I mean, not the first column, the original column a_1 through a_m, right? So b_1 multiplies the first column, b_2 multiplies like the a_1 through a_m column. So everything is simply a multiple of just one column. So the rank is just one, okay? So if rank is one, you know that nullity will be... So nullity or the null space dimension will be n-1, right? So that's another result which is sort of interesting in this, in outer product. It's also called outer product by the way, column multiplying a row, okay?

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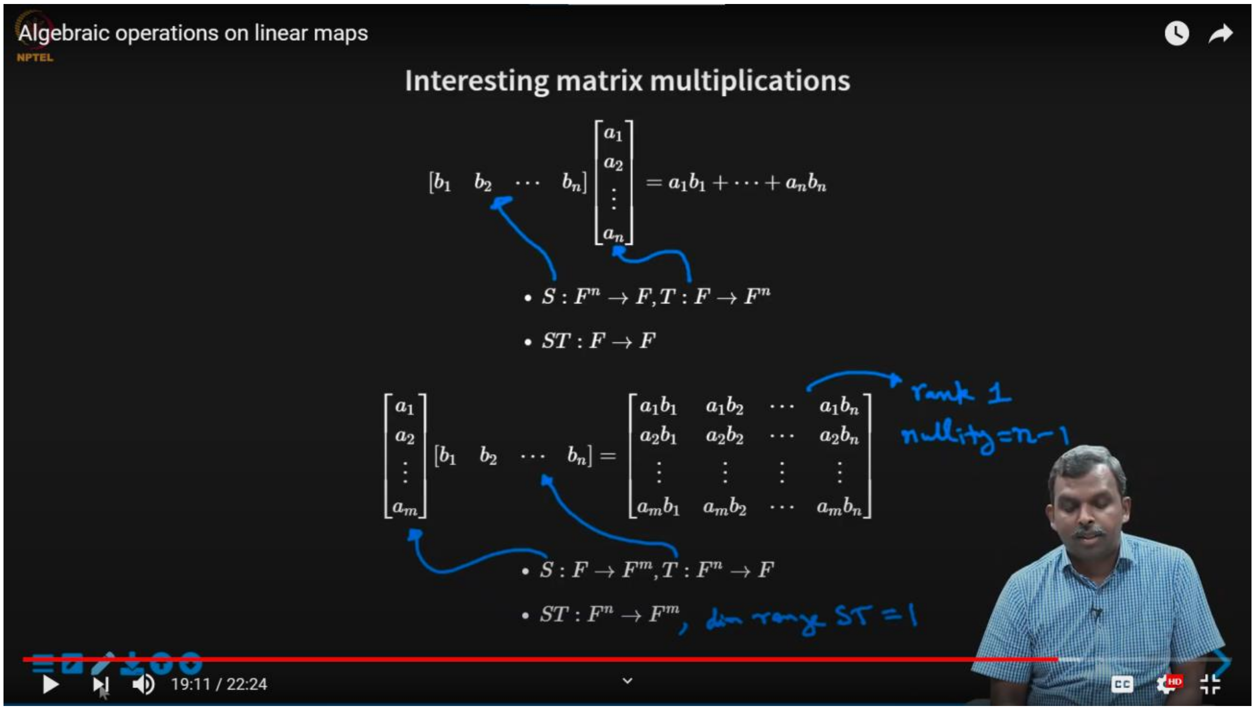

So you have some interesting multiplications of this sort. And also in the next slide, I want to talk about some general properties of matrix multiplication, some things which you might have seen, may not have seen before. I want to emphasize that because these will come back and will be important to us later on, particularly composing two operators or composing two linear maps is very important for us as we go along later in this class, okay? So we saw the definition of matrix multiplication. There are lots of interesting observations that one can make of course. C_{ij} is the ith row of A times the jth column of B, right? So both of them will correspond in the number of elements. You're just doing a dot product. So that will work out, okay? So that's correct. What about the ith row of C? If you look at the entire ith row, it turns out the ith row of C is nothing but the ith row of A multiplied by the matrix B, okay? So if you take the ith row of A and multiply by the matrix B, you will get the ith row of C. So every row of C is actually a linear combination of the rows of B, okay? Something to think about, okay? What are the linear combination coefficients? Those are from the ith row of A, okay?

You can also have like a contrasting column view. If you look at the jth column of C, it is nothing but A, the entire matrix A multiplied by the jth column of B, okay? So think about why that works.

If you go back and work it out, you will see if you take the j^th^ column of B and multiply it with on the left, you know, A comes first and then comes the j^{th} column of B, you will get the j^{th} column of C, okay? So that product does what? It actually does a linear combination of the columns of A, right? So the j^{th} column of C will actually be a linear combination of the columns of A, okay? So in one way to put it, if you, if you think about it is... So the j^{th} column of C belongs to the column space of A and the i^{th} row of C belongs to the row space of A. I didn't really define row space, but you can imagine what row space will be, right? What is row space? You take the rows of B and simply do span of that, right? So that is row space. So this is another result which you may not have known. And the last one maybe you have not seen before. This is a way to write the product as a sum of a bunch of outer products, okay? So you take the l^{th} column of A and the l^{th} row of B and just simply do outer products and add them all up, you will get the product C, okay? So this is also a result which is true for matrix multiplication. So you can view matrix multiplication in so many different ways and derive some very nice results from that as well, okay? So this concludes our discussion of the algebra of linear maps. It is very important to know that you can multiply linear maps also together, okay? Compose them when it is allowed, okay? So when it is allowed, you can compose them, and that has some nice connection with matrices. And later on we will see some nice results of what happens to null space, range space, all these things when you compose maps. And that is very important also, okay? Thank you very much.

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