Applied Linear Algebra
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Week 03
Solving Linear Equations

Hello and welcome. We've been studying vector spaces, linear maps, a lot of abstract things in this course and in case you're feeling very worried about any possible application that's going to come or not, this lecture hopefully will solve some of those concerns. In this lecture, we will start looking at solving linear equations. So now linear equations show up so many, so often in engineering and practice and other applications that it's very very important to know how to solve them. Particularly large set of linear equations, when you have, you know, thousands of variables and thousands of equations, how do you go about solving them systematically etc. is very important. So this lecture, we will put to use all that we have learnt about vector spaces, linear maps and their associated properties to see how to go about solving linear equations and you will see the important ideas that we saw in the previous lectures about injectivity, surjectivity invertibility of linear maps. They will play an important role in being able to quickly solve linear equations, you know, in a nice way, okay? So let us proceed.

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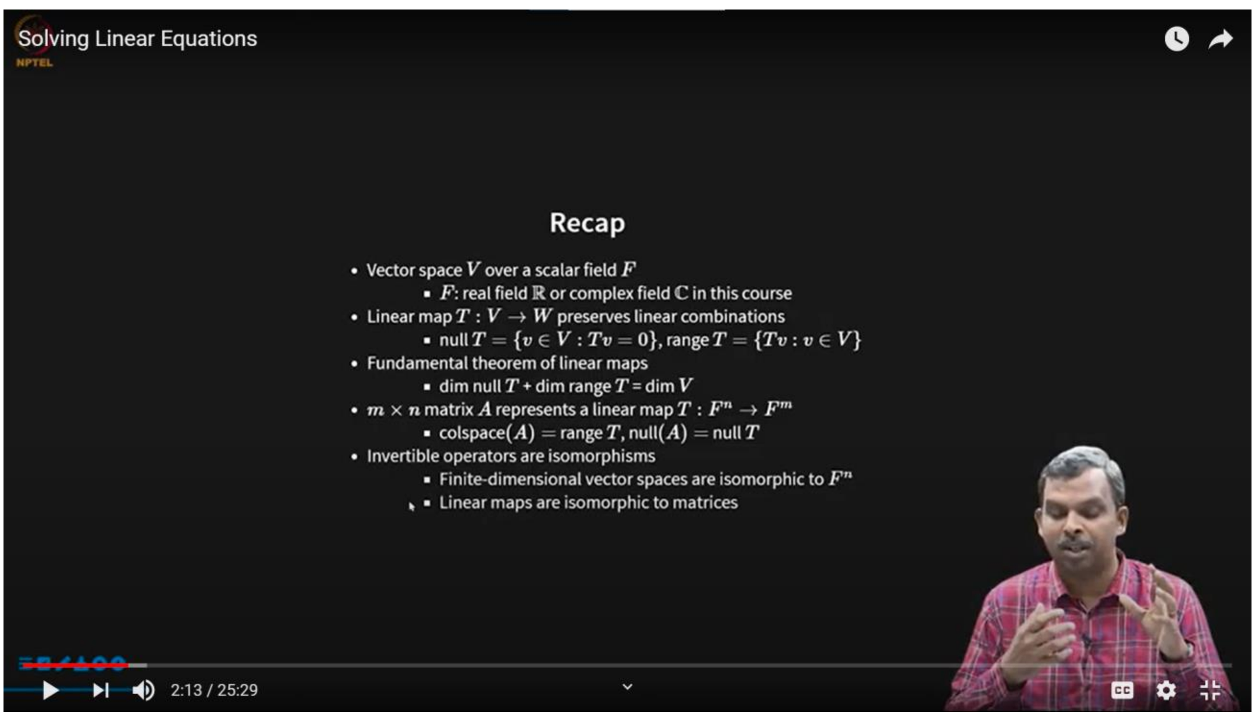

So a quick recap. Once again we always start with the recap. We all know these vector spaces over a particular scalar field $\mathbb{F}$, the $\mathbb{F}$ is usually real or complex in this course at least. We have studied linear maps from one vector space to the other, which preserves linear combinations. We looked at the null space, we looked at the range space etc. And this one, this is this wonderful result called the fundamental theorem of linear maps which relates the dimension of the null and the range to the dimension of the overall initial vector space V. And then we saw this interesting correspondence between mxn matrices over, you know, the field $\mathbb{F}$ and a linear map from $\mathbb{F}^n \rightarrow \mathbb{F}^m$. And we also saw these isomorphisms which, you know, made this connection much stronger. In particular there's this notion of column space which is the same as a range of the linear map, this notion of null space of a matrix which is also the same as the null space for the linear map, okay? And we also saw invertible operators and we saw that they define isomorphisms, things that are the same. And then we saw these powerful isomorphisms that any finite dimensional vector space is like $\mathbb{F}^n$, and linear maps are in fact isomorphic to matrices, all of these results we saw before. Now we will put quite a few of these results to use in trying to solve linear equations.

So what is a linear equation? It is given right here. It's $Ax = b$. If you probably have already seen it before, it's worth emphasizing once again. We'll in general keep A as an $m \times n$ matrix from the scalar field F, okay? Each element a_{ij} is like that, and I will denote the general element as a_{ij}. So that's the ith row and jth column, that's the element a_{ij}. The vector x, we will generally think of it as a column vector, but when I want to write it compactly, I will write it as a row with the comma. So hopefully that is clear enough to you. I have done it quite a few times so far. So x is a vector of length n, right? $x_1 ... x_n$. And b is a vector of length m, okay? So this matrix A takes an input from $\mathbb{F}^n$ and puts out an output of $\mathbb{F}^m$, right? After multiplication by the vector on the right. So $Ax = b$. So what is given usually in a linear equation? A is given, the matrix A is given. x is unknown, you have to find x. And then b is given, okay? So to find x, you have to solve for x such that Ax becomes equal to b, okay? So simple enough to state. It shows up in so many applications. I mentioned quite a few of them, those of you in electrical engineering would have seen that to solve linear circuits you need to use linear equations, and so many other applications today in the world, they use linear equations. This is bread and butter. Without this you can't really, you know, implement many things today.

Okay. So now that we know this connection between linear maps and matrices, what is the interpretation in terms of linear maps, okay? So you have a linear equation here. So how would you think about it in terms of linear maps and, you know, use maybe some of the properties of linear maps to help solve your equation? So is that possible, okay? So we know that this matrix A, once I think of the standard basis... So see, when somebody gives a linear equation, nobody's going to mention basis or anything. So you just assume standard basis, right? So you take the standard basis and let A represent a linear map T in the standard basis. That's easy enough for us to do, right? So how do you do this? We have these vectors in the standard basis. I will use this notation for vectors in the standard basis. You can see that notation here. So this is the jth vector in the standard basis for length n, right? So e_{jn}. So this sort of captures everything. Quite often this n will be clear. So if this n is clear, I will simply say e_j, okay? But if it is not clear, I will put in this case. And in this case you can have both n and m, so it's a bit confusing, so we put the n explicitly in this notation, okay? So hopefully that's clear. So that's the standard basis vector which has 1 at the j^{th} position, 0 everywhere else, right? So that's what the standard basis means. And then how do we do this matrix to linear map correspondence? We simply say that T acting on e_j, okay, when the linear map T acts on e_j, its output is the j^{th} column of A, okay? So this is sort of consistent with the matrix vector product and that is exactly what we mean, okay?

So we have seen this already before. So there is this linear map T which is associated with the matrix A, okay? So in terms of the linear map, we are also asking a specific question here. When you solve a linear equation, right, so you're given a w which belongs to the capital W vector space which is nothing but your coordinate vector $b_1, b_2, \dots, b_m$, right? So you pick the standard basis for W also, and so w one can write in terms of what it is, right? So $b_1 e_1 + \dots + b_m e_m$ with length m, right? So that's w. You have to find a v whose coordinates are x_1 through x_n. okay? Find all v, I guess not just one v, you want to solve it entirely, maybe, right? So that's one ambition you may have. So such that $Tv = w$.

So if you like drawing pictures here, we've been drawing pictures to represent linear maps quite a bit... So if you draw this picture here, let's say you want to denote this as V and you want to denote this as W and somebody gives you a w here. Somebody defines a map T from V to W. You need to see if there is a v that will take you to w, or maybe, you know, a set which entirely takes you to w, okay? So this is your question, okay? So this is your solution. So this entire set, there may be multiple vectors that take you to the same w, right? So it is not, every linear map is not one-to-one, right? There are non one-to-one linear maps. So you may have multiple inputs taking you to the same output w. We already know that. So in that case one needs to find out all those inputs. That is the goal of solving a linear equation. So we see that there is this nice correspondence. So quite often when somebody gives you a matrix, maybe you have to think about the linear map associated with the matrix, what kind of properties does it have, can I guess some properties of that linear map. From that, what can I infer about the solutions to the linear equation, okay? So those are interesting ideas. We will explore some of that in this lecture, okay?

There is one particular case which is very important and interesting forms a sub case of the problem. Supposing your b is 0, okay, or w is 0, okay? So in the previous slide I drew that picture. And supposing w is 0. In that case, the solution to that equation $Ax = 0$ is basically the null set, right? Null space, sorry, okay? So the $null\,T$ is given by set of all x such that $Ax = 0$, okay? So the $null\,T$ is also a solution to this linear equation, okay? So you see this nice little interesting connection here. And the fact that this null plays an important role, you will see even when b is not 0, null will play an important role. You will see how it enters the picture here. So this null of this linear map T is very crucial to understand, okay? So it determines a lot of properties of the linear map.

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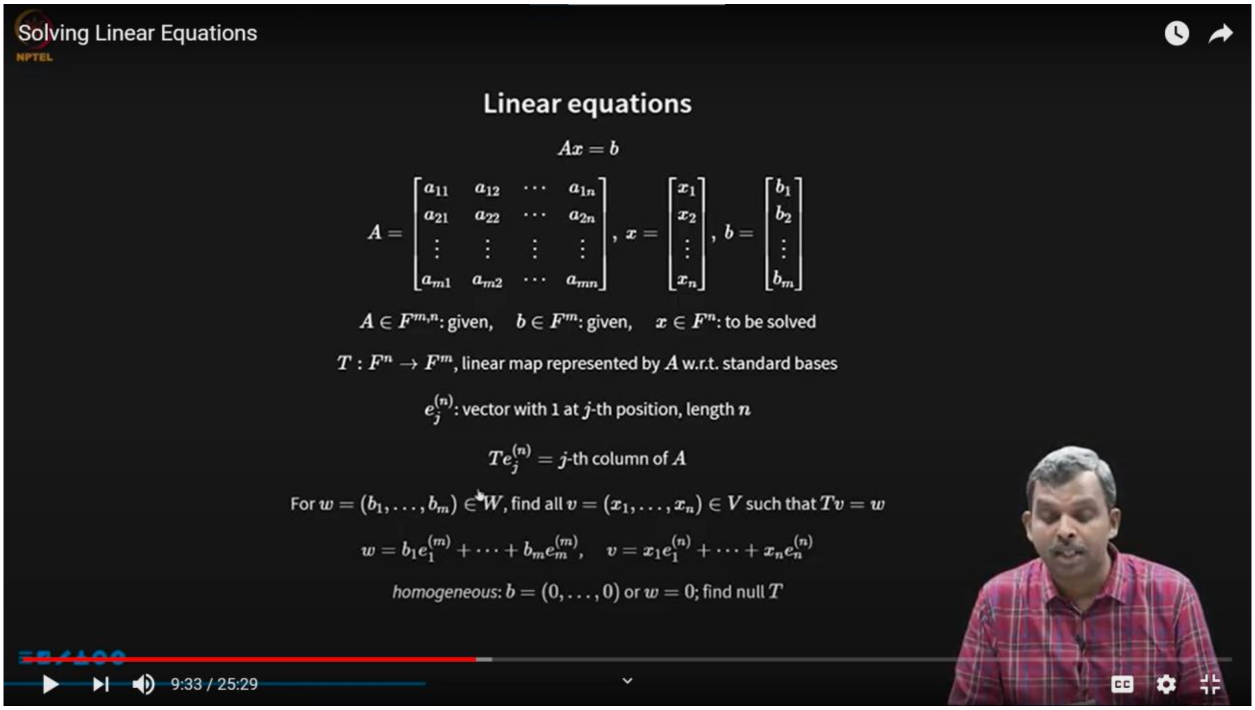

We have seen before, null space is connected to injectivity, right? If null is just the zero, then the map is one-to-one. If it is non-zero, then it is not one-to-one, okay? So likewise range of T is also very important, okay? Range of T maybe it looks like it does not show up explicitly, we will see later on how it will show up. For instance you can see that, you know, the b needs to be in the range of T, right? So if you pick a w which is not in the range of T, then no v is going to take you there, okay? So null of T and range of T you can see already they are going to play an important role and understanding what they are and whether the map is injective or surjective will play a crucial role in the solution. So let's start looking at some such equations and see how to figure out things about the solution. So we'll do mostly by example. I'll pick some simple examples and then work through them, and then I will present the general case and how to go about solving a general case, okay?

So here is a very simple example. You have $Ax = b$. It is a 3×3 example. I have picked the A in a specific simple way, I've picked it in a sort of a triangular way. You can see that there are lots of zeros there in a critical place. So maybe it looks less general to you, but later on we will see how this is good enough, okay? So now when you look at a matrix like this, you will start thinking about the linear map associated with this matrix and see if you can say anything about it, okay? So the first thing you observe is the linear map associated with this matrix works as follows, right? It's going to take the input $(1\ 0\ 0)$ to the output $(1\ 0\ 0)$, it's going to take the input $(0\ 1\ 0)$ to the output $(2\ 4\ 0)$, okay? That's the second column, isn't it? $(2\ 4\ 0)$. It's going to take the input $(0\ 0\ 1)$ to $(3\ 5\ 6)$, that's the third column, right? So that's how this linear map works, isn't it? So that's a very easy description to see. Now what is the range of this linear map? We already saw that the range is important, right? So b needs to be in the range, otherwise there is no hope of a solution, okay? So range of this linear map is simply span of the columns. We know that. So $(1\ 0\ 0)$, $(2\ 4\ 0)$, $(3\ 5\ 6)$. If you work it out, you see that these are linearly independent. It's easy to see that they are linearly independent, right? And $\mathbb{R}^3$, you are in $\mathbb{R}^3$ and you have three linearly independent vectors, that is a spanning set, isn't it? So you span the whole thing, so the span of the range of T becomes $\mathbb{R}^3$. Now once range becomes $\mathbb{R}^3$, you know null is going to be just 0, isn't it? So that is because, you know, you have the fundamental theorem. So 3 has to be equal to 3 plus something, so and that something is going to become 0, okay? So the dimension of null T is going to go to 0. Even otherwise maybe you can conclude null is 0. So null is 0.

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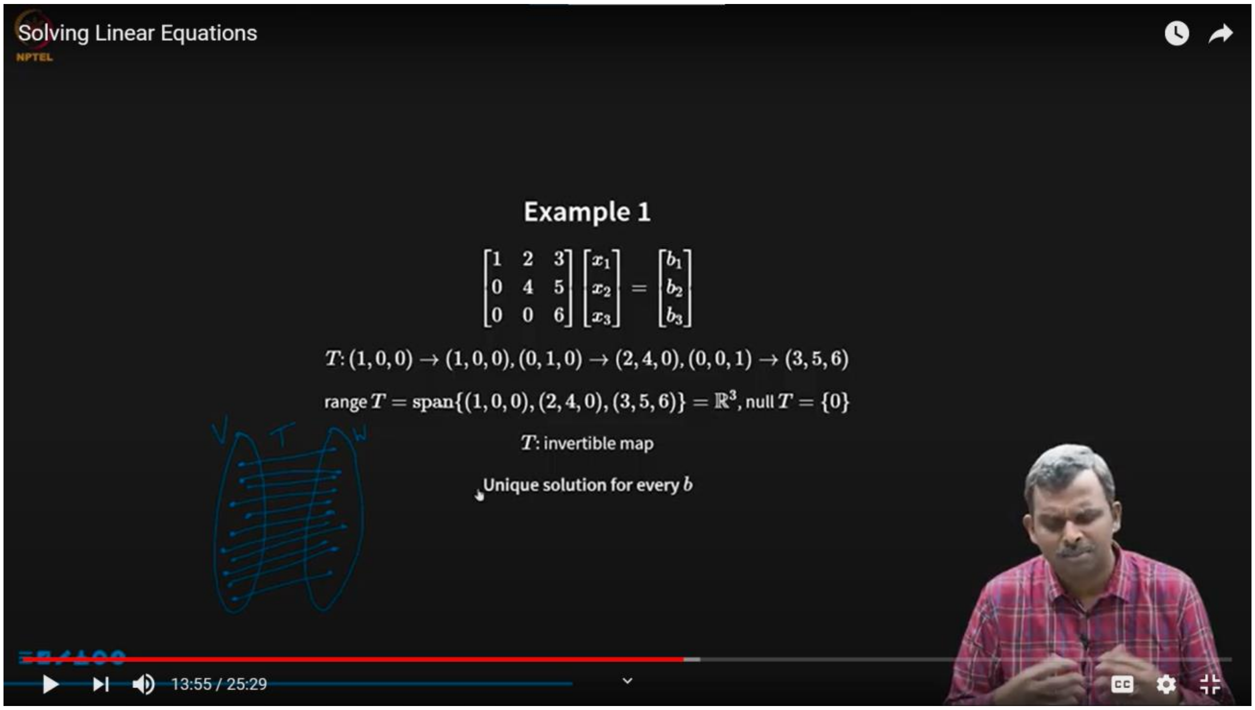

So now you see that the map T is both injective and surjective, okay? So specifically T becomes an invertible map, okay? So just from the matrix A, I'm able to look at the linear map corresponding to it and look at the structure of the matrix A, make some statements about the range, make some statements about the null and then infer that T is invertible. So once I know T is invertible, I can make a statement about the solution. Why is that? I know that there will be a unique solution for every b. Why is that? Let us just think about what an invertible matrix means, okay? So an invertible operator takes V to W, it's one-to-one and onto, right? So it's almost like every point here goes to some unique point there, right? So that is how an invertible map looks. You know, I probably won't be able to draw every single point on the vector space, you can imagine what I am trying to get to, right? So this is what one-to-one invertible T does, okay? So every point in V gets mapped to a unique point in W, okay? So no matter what b I give you, no matter what w I give you, there will be an x which is uniquely being brought from V to W by T, right? So that is the invertible map. So I know for sure that there will be a unique solution for every b, so that's the statement I can make about a linear equation with this matrix A, okay? Why? Because the linear map associated with it is an invertible map, okay? So that's a nice statement we made. So notice what we were able to do. We were able to look at the matrix, infer properties of the linear map, infer properties about, you know, how the linear map works and then, you know, we're able to make statements about the solution without, you know, worrying too much. And there are maybe, you knew these answers before, but maybe this is a different way to view it and this can give you, you know, more insight into what's going on, okay? So that's a simple example. So let's start complicating the examples a little bit more and look at other types of linear maps. But we'll keep them, you know, sort of upper triangular to make our work easy.

Here's another example, here's another matrix. It's a 3×5 matrix, okay? It takes length five vectors to length three vectors, all right? And the matrix is given to you there. You see the numbers. 1, 2, 3, 4, 5, just pick them in some way. You can put other numbers if you like. What's important is the diagonal sort of structure there, the upper triangular structure, okay? So you have zeros in the critical place. So from there you can quickly infer the range of T, you saw the range plays an important role. The range of T is $\mathbb{R}^3$, right? So if you look at it… Similar to the previous argument you have three linearly independent vectors in the column, so that will span the entire $\mathbb{R}^3$. So the range of T is $\mathbb{R}^3$. From the fundamental theorem once again we can quickly infer that the null space is non-trivial. So the dimension of null of T is going to be 2. And we do not know yet how to find the null space. Towards the end of this week you will see clear methods to find the, you know, actual null space itself. But this information is enough to us. Just by looking at this matrix, I know that the range is the entire $\mathbb{R}^3$, the null has dimension 2, so I am able to infer that my map T represented by this matrix A is surjective but it's not injective, okay? So it's not one-to-one, the null space is not trivial but it is surjective, the range occupies the entire W, right? So that's nice to know. So once you have a T which is surjective and not injective, we know a lot of properties about how the map looks, right? So there is a null space but it's surjective, okay? So you can see that there will be infinitely many solutions for every b, okay? So I'll be very precise and clear about it later on, but you can see why this should be true, right? Surjective means every point in w... Maybe I should draw this picture once again. Surjective already means... So I have a V which is maybe much bigger and the W which is slightly smaller in this particular case since the map is surjective. What does it mean? So surjective means - any w I pick here, there is at least one v such that T takes you to w, right? So that is what surjective means, okay? So every w here, there is something on V that will come here. That is true. But why do I say infinitely many solutions? Once I know that there is one solution, I can make many more solutions from it, okay? So we will see precisely why this is so later on, but once you have one solution, you can sort of add the null to it and you will get infinitely many solutions, okay? So at least you can see that there should be one solution for every b, that much is maybe easy for you to see. Maybe you do not quite see where the infinitely many comes from. Later on, I will tell you exactly why it is that. You can also imagine why this should be true, right? So from surjectivity we know that there exists at least one v such that $Tv = w$. Now I know my null space is non-trivial, right? So if you take any x, any u, okay? So you take any u in $null(T)$ and if you look at $v + u$. What will happen if I hit T with $v + u$? So you pick any u in $null(T)$ and then you look at $T(v + u)$, you know $T(v)$ is w. What is $T(u)$? u is in the null space, so $T(u)$ is zero. So this also becomes w, okay? So if you have at least one v which is guaranteed by the surjectivity, and if you have a null space which is big, which is non-trivial, then you will have infinitely many solutions because you can take any one solution and keep adding the null space vectors to it and you will still retain the same property here, okay? So this whole thing is actually v plus... I can do a shortcut and write it as $v + null(T)$, okay? So what is $v + null(T)$? v plus every vector in the null space T. All of those are solutions. So that is why you have this infinitely many solutions, okay? So just the surjectivity and not being injective definitely guarantees you will have infinitely many solutions for every b, okay? So that is a nice property for you to know, okay?

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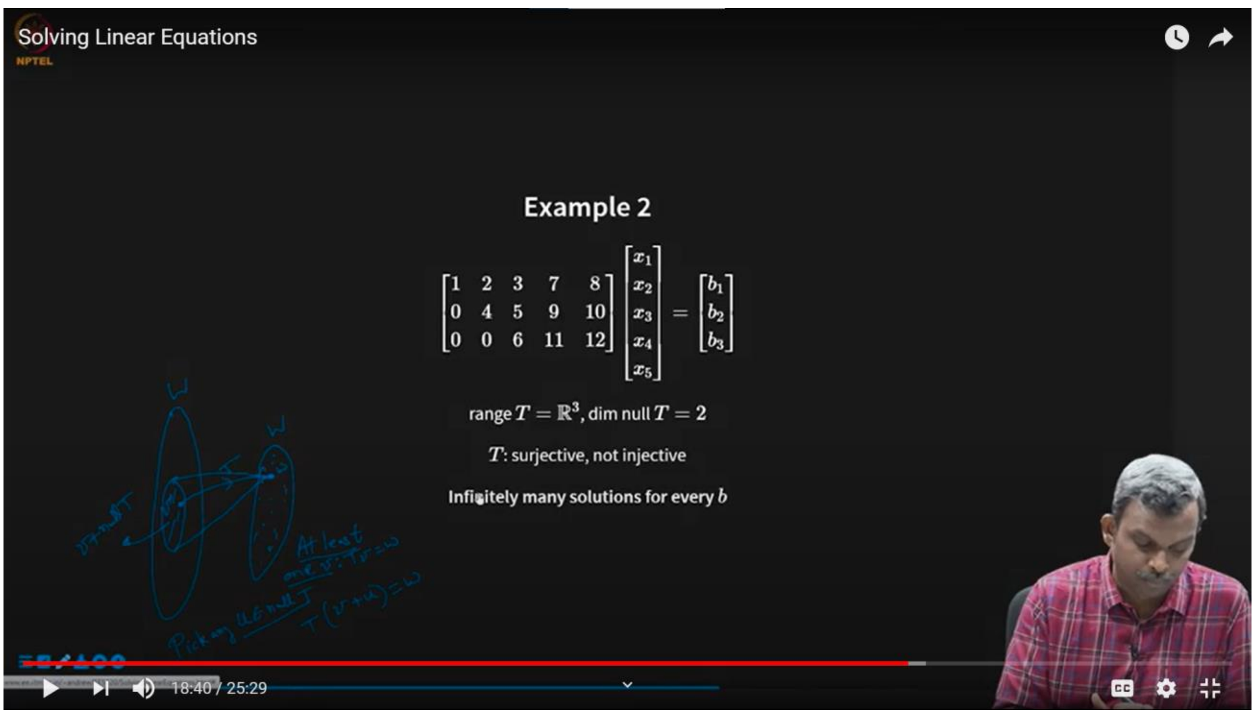

So let us look at other cases. We have seen two cases now. One was a 3×3 case which was invertible. Here is a 3×5 case which was surjective but not injective. You can have so many more combinations, right? Let's see what happens in the other two possible combinations, okay? So here's an example. There is a matrix A once again, but I have made a change here. I made the last row fully zero. It's still a 3×3 case but what happens here is - if you look at the range of T, it's only dimension two, okay? The reason is that the last row is zero, you can never get anything non-zero in the last coordinate, right? So you will never hit the z-axis so to speak, you'll only have the first two values, and the dimension of the range is two. When the dimension of the range is 2, the dimension of null becomes 1, okay? So now here we have a question, a linear map which is not surjective, okay? Why? Because the range is not the entire space. It's not injective also, okay? So the null is non-trivial, okay? So it's not surjective, which means for every b, I'm not guaranteed a solution, right? Because I may be outside the range. If I'm outside the range, I will not have a solution, okay? So... But if you are inside the range, I will definitely have at least one solution and by our previous logic, I can add the null space to all the solutions, any solution I have, so I will have infinitely many solutions, okay? So that is what will happen in this case. And I have written it down below. If you have a $(b_1\ b_2\ b_3)$ in the range of T, then you will have infinitely many solutions. In this particular example. it is easy to write down a condition to check whether or not this b is in the range of T, right? So b_3 has to be 0. If b_3 is 0, then you know that $(b_1\ b_2\ b_3)$ is in the range of T and then you know you will have infinitely many solutions. Once again, infinitely many because it is not injective, okay? And if you are in the range then you have infinitely many solutions. But if you are not in the range, you will have no solution, okay? So that is the sort of picture to keep in mind.

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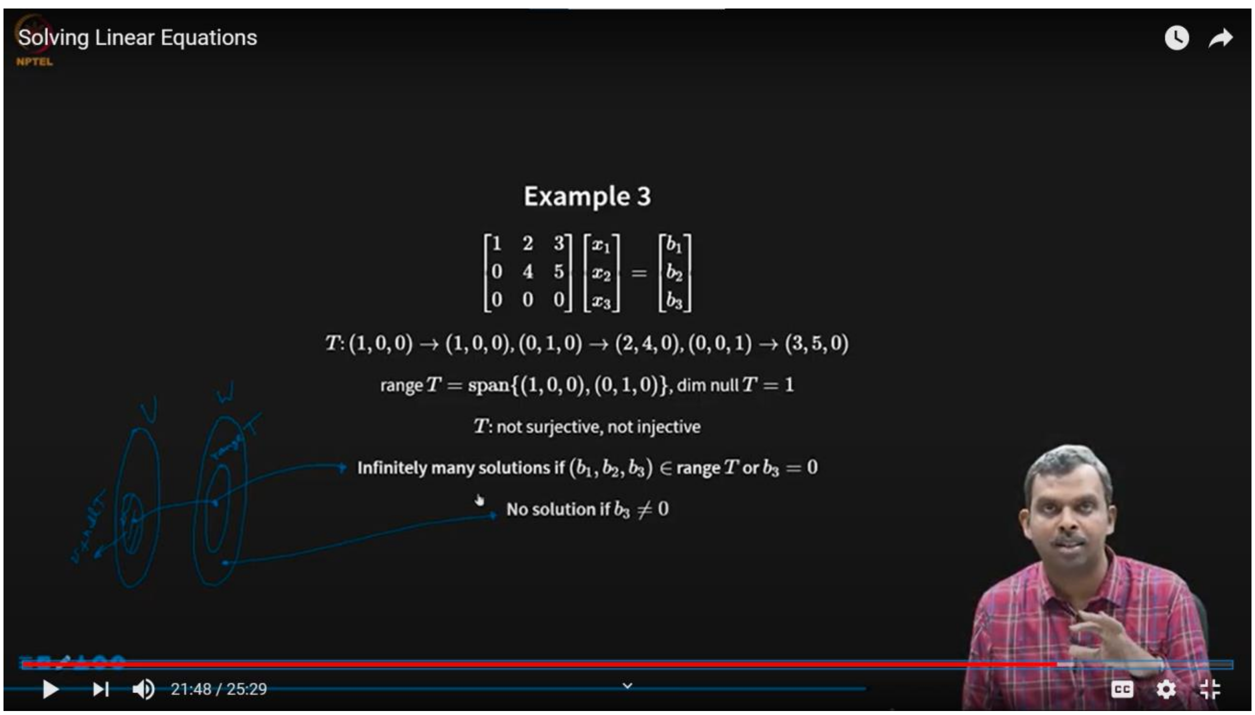

Once again let us draw pictures here. If you have a V and if you have a W and this is your range of T, okay, and if your b is in here, this corresponds to this particular case. If your b is here, this corresponds to this case, okay? Once again, why is that true? There is a v, there is at least one v. Because it is in the range, there is at least one v. And then you do $v + null\ T$ to get your infinitely many solutions, okay? So hopefully that was clear to you. So you see that the properties of the linear map start affecting the nature of the solution. What can happen, what cannot happen. As long as the map was surjective, we know that for any b, there is a solution. Once it becomes not surjective, you have to worry about whether a solution will be there or not. It depends on whether b is in the range or not, okay? And then whether it's injective or not controls how many solutions you have. If it's injective, then you have only one solution, if it's not injective, if there is a non-trivial null space, then you will have infinitely many solutions, okay? So that's the nice sort of, you know, classification of the type of solutions.

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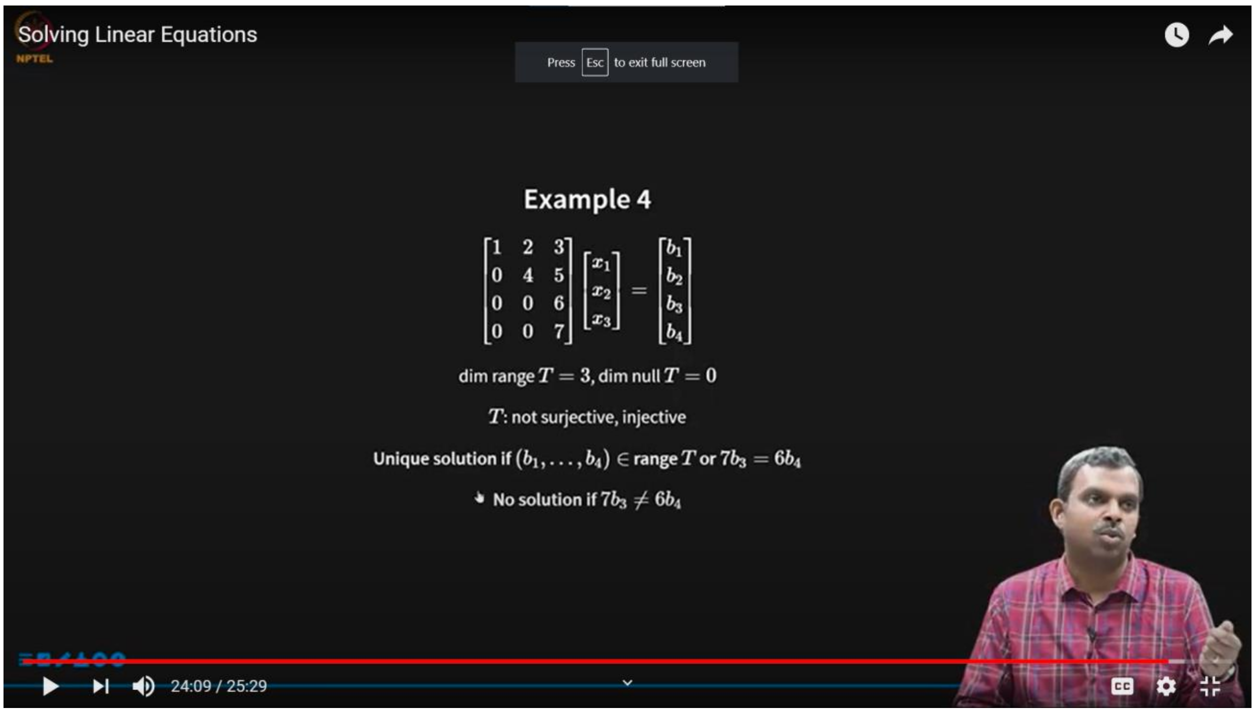

Okay. So there's one more case we missed out and that's given here in this picture. It's sort of a tall matrix. You see once again 0 is all over the place, but this particular matrix has this nice little structure. If you think about it, the dimension of the range is only 3, right? So there are only three vectors. You cannot go more than three. But then the space of W is four dimensional, okay? So the range of that has dimension only 3. So it's not surjective, but the null space is zero, right? So once the dimension of the range becomes three, you use your fundamental theorem, you get null space of dimension zero, so it is injective. It is not surjective but injective, okay? So that sort of a linear map this one is. If you have not surjective but injective, this is what will happen, okay? You will have a unique solution if b is in the range, right? So it's injective but it's not surjective. So if b is inside the range, you will have a unique solution and in this particular case maybe it's not very obvious to you or maybe you can think about it. There is again a condition which you can state in terms of b_3 and b_4 to check whether $(b_1\ b_2\ b_3\ b_4)$ is in the range or not and that is just, that's what I've put there. But generally basically $(b_1\ b_2\ b_3\ b_4)$ has to be in the range for there to be a unique solution. If they are not in the range, there will be no solution to this particular problem, okay? So hopefully these four examples give you a nice feel for how to think of a linear equation, look at the matrix, look at the associated linear map corresponding to that matrix, figure out its properties, see if it's injective, see if it's surjective and then make some statements about what type of solution you will have. Notice I didn't do any solving, right? So I don't even know what the values of b are, right? So based on just the properties of the matrix I am able to infer what kind of solutions I can anticipate, okay? And then given a b, I will be able to come up with some sort of an answer, right? So that is the, hopefully it gave you an example.

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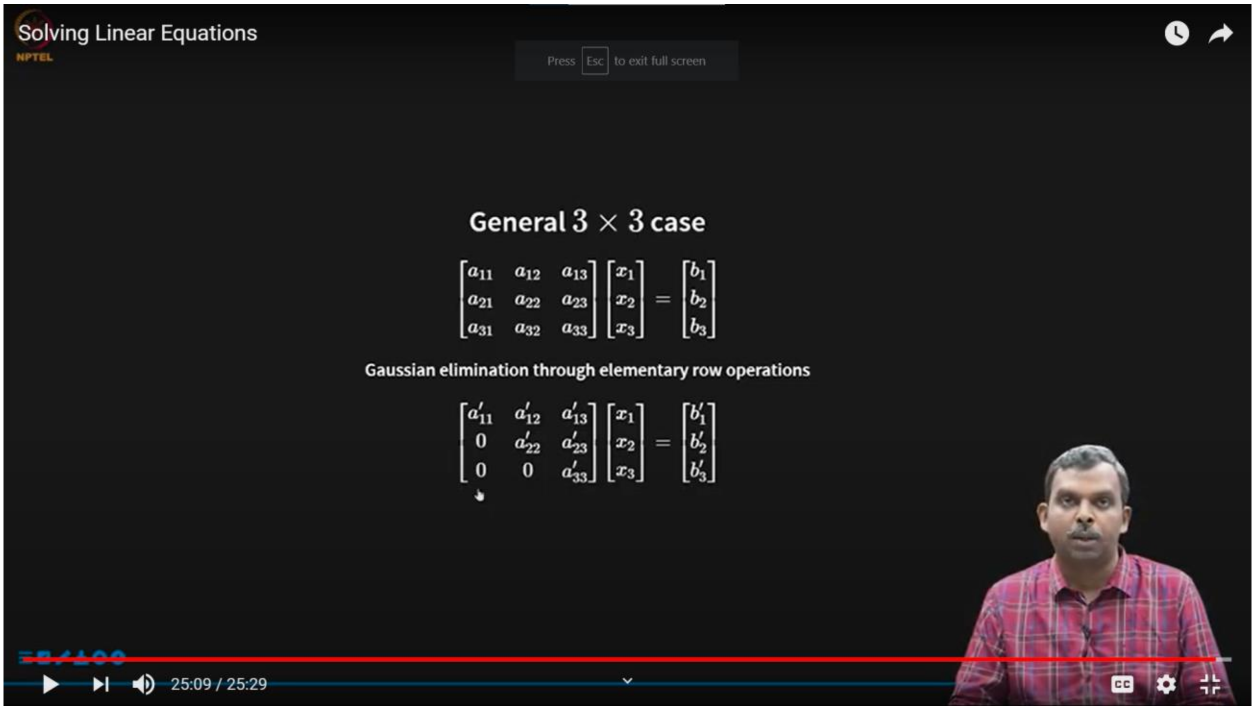

But then you should now ask me this question. I always pick these convenient examples with a lot of zeros where this linear independence was easy to see. I could easily see the dimension of the range, right? You could just eyeball the dimension of the range looking at the zeros and you could use fundamental theorem to find the dimension of the null, so you knew whether it was injective or surjective, and then you could do that. What if your matrix was not like that, what if your matrix had non-zero elements everywhere, all over the place. Random, maybe zero here but nothing useful for you to figure out linear dependence or independence. What do you do in that case? It turns out, if you look for instance at the general 3×3 case, okay? So I can also look at a general bigger case, but, you know, general 3×3 is good enough for you to get a feel for what can happen. It turns out you can do Gaussian Eliminations through these elementary row operations. At least in this course, I will think about elementary row operations in some sense. And get it to a form which looks like that, okay? Just by elementary row operations, you can make sure you have a lower triangular sort of form, okay? So you have this diagonal, and then zeros below that. And once you have that, you're back to your familiar territory, right? So so far, these four examples that I gave cover this lower triangular form and you can make inferences about surjectivity, injectivity and all that, okay? So in the next lecture we will talk about elementary row operations. But before we go there, there is a little quiz I've prepared for you. It will be useful for me if you can go through the quiz and pick some answers. And it'll give me feedback on what you've understood and how things are, okay? Thank you very much.

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