Applied Linear Algebra
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Week 03
Elementary Row Operations

Hello and welcome once again. In the previous lecture, we saw some simple examples of linear equations with, you know, in a form where we could infer the range dimension, the null dimension and all that very easily. So it maybe gave us some nice ways to infer how the solution will look like. Now there was that one issue, that how do we get all those zeros in the matrix, right? So why is it okay to have all those zeros? If you don't have zeros, what do you do? How do you get it to that form and that's what you do with what are called Elementary Row Operations or Gaussian Elimination, the way we describe these things, okay?

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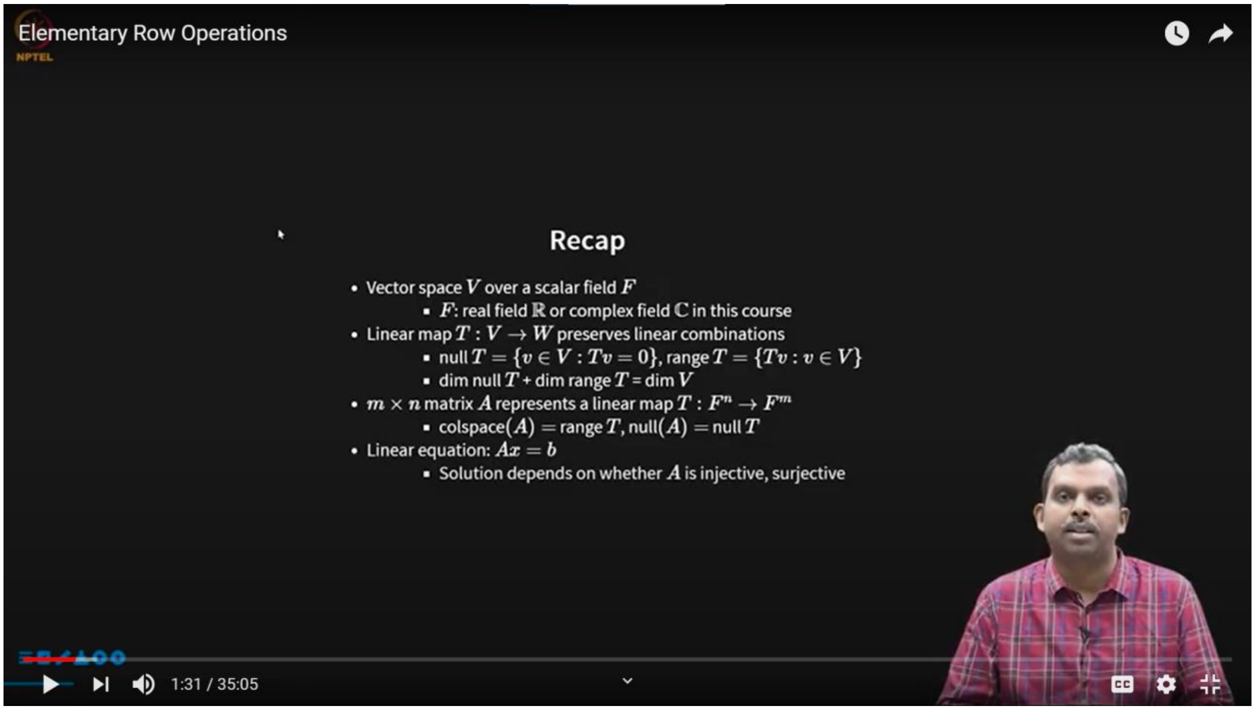

So let's do a quick recap. We've looked at vector spaces, linear maps. We know matrices represent linear maps and all that. In particular in the last lecture, we saw how linear equations, solutions to a linear equation depend very critically on the nature of the linear map represented by the matrix A. Whether it's injective or surjective and all that. And in this particular case, we will now look at elementary row operations to get the matrix to a nice form, and still not affect the solution, okay? So that's the idea and that's where operators will play a very important role. And we'll start using operators in a very direct way, okay? Okay, so here is the, once again a reminder. If you have a general 3×3 equation, it will look like that. It would have non-zero, possibly non-zero entries everywhere. You can do Gaussian Elimination through elementary row operations. And what is that? Basically, the given problem is to find a v such that $Tv = w$. We saw that before T is the linear map and you want to find a vector v. I find all v may be so that $Tv = w$. It turns out this problem is equivalent to another linear equation. What is that another linear equation? You can take an invertible operator $S: W \to W$, okay? So Tv has already taken you to w, now you take an invertible operator $S: W \to W$ and then operate on it, compose that to this equation so to speak, okay? So instead of just looking at $Tv = w$, I look at $STv = Sw$ and these two are in fact equivalent because S is invertible, right? So it's easy to see why they have to be equivalent, right? So when you have an invertible operator, it's like an isomorphism, you know? So it takes you from... Invertible operator takes you from input to some other output, but you can also undo that linear operation, right? So it's an invertible operator. You can also go from output to input. So whether you solve $Tv = w$ or $STv = Sw$ you will get the, you will get the exact same v's for the exact same w. Nothing will change as long as S is invertible and you've done STv equals Sw, then you're the same, okay?

So then ST, you can design your invertible operator S so that ST looks like what you want, okay? So that is the core essential idea behind these elementary row operations and Gaussian Elimination. I know if my linear operator, the matrix of the linear operator, looks in a specific way, lots of zeros, then I can quickly solve it, okay? This notion of using operators on the left, okay? $STv = Sw$ to make ST look like an easier matrix for you to deal with is at the heart of this elementary row operations. But be careful about the condition here, right? S needs to be from W to W and it needs to be an invertible operator, you cannot do whatever you want. If you do some non-invertible operation, then you've lost the solution, okay? Something important to be careful about, okay? All right, so that's the essential idea behind row operations, finding the invertible operator S so that ST is of a form suitable for easy solving.

So how do you go about finding this operator S? You do not find it in one go, you do it in multiple steps slowly, okay? So I will show you how this is done. So once again I will denote, I will show it, illustrate it with an example. One example is good enough. Many of you might be already familiar with this, but let's go through this one example and look at it from this operator point of view, sort of emphasize this in all this knowledge we have about operators and linear maps, and what is it that we are doing as opposed to just the mechanics of how to do it, okay? So here is a 3×3 example. You have $Ax = b$ being your linear equation and this A is the matrix shown right there. $[0\ 2\ 3;\ 3\ -1\ 4;\ 2\ 5\ 1]$. And then you have to solve for the x, okay? So the first operation I will do, which is an invertible operation, is to permute row 1 with row j to get a non-zero pivot at $(1,1)$. So the first step will focus on $(1,1)$, the $(1,1)$ entry of the matrix. If you remember from our examples, you want that to be non-zero, right? So can you make $(1,1)$ non-zero, okay? So here in this example, $(1,1)$ has been chosen as 0 for you to illustrate what to do. But below $(1,1)$ there are non-zero values. So what I will do in the first invertible operation is to interchange row

1 and row 2. Now this is an invertible operation, you can see why this is invertible, right? So you did row 1 and row 2, you can always flip it back, go back to the same old matrix like before when you do, multiply by the inverse of this, which is flipping the rows 1 and 2 back again, okay? So you can do that.

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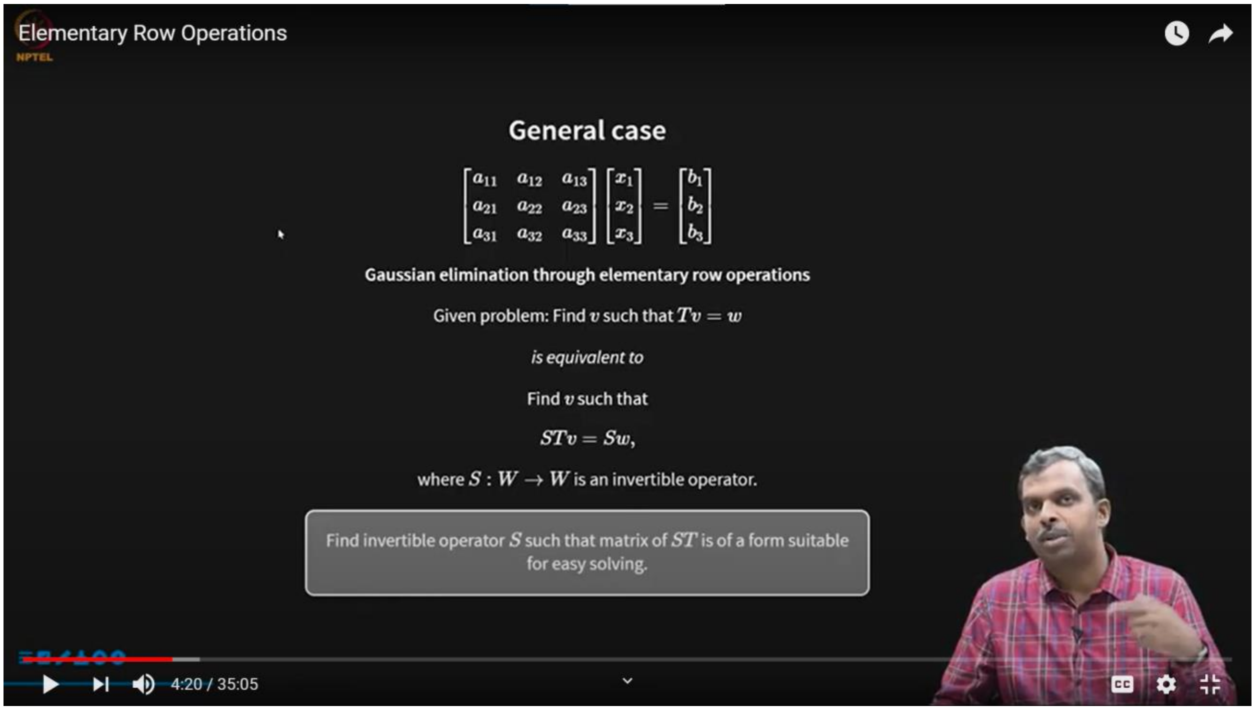

But how do you execute that with the matrix, okay? So for that we can design a nice little invertible operator matrix which represents permuting row 1 and row 2 of anything that's multiplied on the right, okay? So that matrix I will denote as E_1, okay? Just an E_1. You can see what I have done here. $[0\ 1\ 0;\ 1\ 0\ 0;\ 0\ 0\ 1]$. Now if you take this E_1 and multiply with any matrix on the right, what will happen? Row 1 and row 2 are interchanged, okay? And you can see clearly that this E_1 is also an invertible matrix, right? So it's clearly invertible. It's got full range, no null space. There is, it's clearly invertible and also intuitively you can see interchanging row 1 and 2, you do it once again, you go back to the original state, right? So it's invertible, okay? So this E_1 is a nice little invertible operator, elementary row operator which gives me this interchange of rows 1 and 2, okay? And what will I do now? I will take this E_1 and multiply both sides of this equation $Ax = b$, sort of transform this equation into another form. So if I do that, I have instead of $Ax = b$, I have $E_1Ax = E_1b$. But since E_1 is invertible, I know it's exactly equivalent to $Ax = b$. If I solve this new equation, I can do the old equation also, okay? So what happens after I multiply? The first and second row get interchanged, okay? So my matrix becomes $[3\ -1\ 4;\ \ldots]$. Notice the same interchange happens on the right hand side also. See E_1 multiplies b also. So b_2 and b_1 also get interchanged, okay? So this is what happens, this is the first step. There's lots of steps. Like I said, in every step we will use an invertible operator and we'll keep multiplying and we can do it multiple times, right? So it's not enough… See, I did one invertible operator, I can find one more invertible operator and do the multiplication. As long as it's invertible, as long as I keep multiplying on the left I know I can undo it and go back to the original equation. So that's the principle, okay? So you do the first operator, that's good enough, it's looking okay. Now already you see the non-zero value at the $(1, 1)$ place, and then there is a zero below it, right? But you want zeros all the way below, right? If you remember the upper triangular sort of form, I want a non-zero value and then everything should be zero below that. How do we do that?

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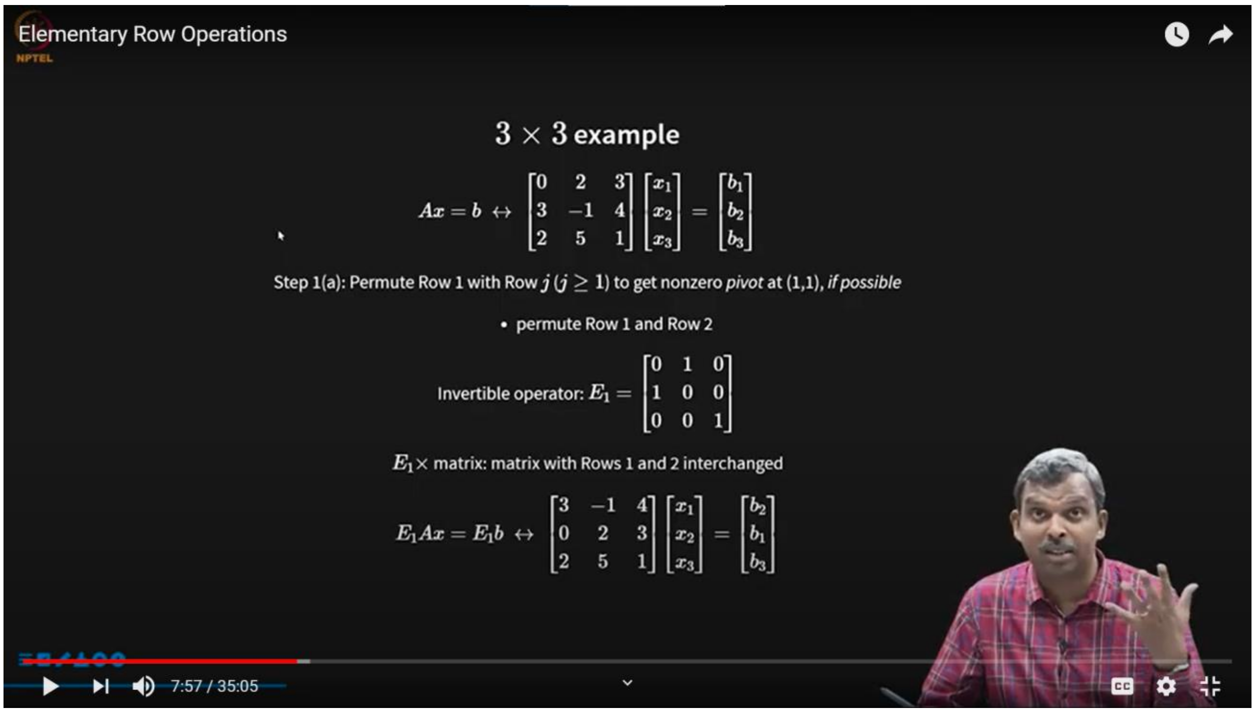

We can once again use an invertible operator for that, okay? So right now, I am in this form. The next step is to make the pivot value equal to 1, okay? So the pivot value at $(1, 1)$ is 3, right? Now maybe I do not like 3, I want to make it 1. So what do we do for that? I can divide row one by 3, okay? Once again that's an invertible operation, right? You divided row one by 3, how do I go back to the original one? You multiply row 1 by 3 again. If you go back to the original... So for that there is an operator and that is a diagonal operator, okay? So this is diagonal. The previous one was a permutation sort of operator, this one is a diagonal operator. You see that it has non-zero values only on the diagonal. 1/3, 1, 1, okay? And you can see that when this E_2 multiplies a matrix, row one alone ends up getting divided by 3, the other two rows are not affected, it's retained exactly as it is, okay? So I will take this E_2 and operate with it on my equation, okay? So I'll do $E_2E_1Ax = E_2E_1b$ and you see the first row got divided by 3, okay? So you got $(1 \ -1/3 \ 4/3)$ and then other rows remain untouched, exactly the same okay? And notice what has happened.

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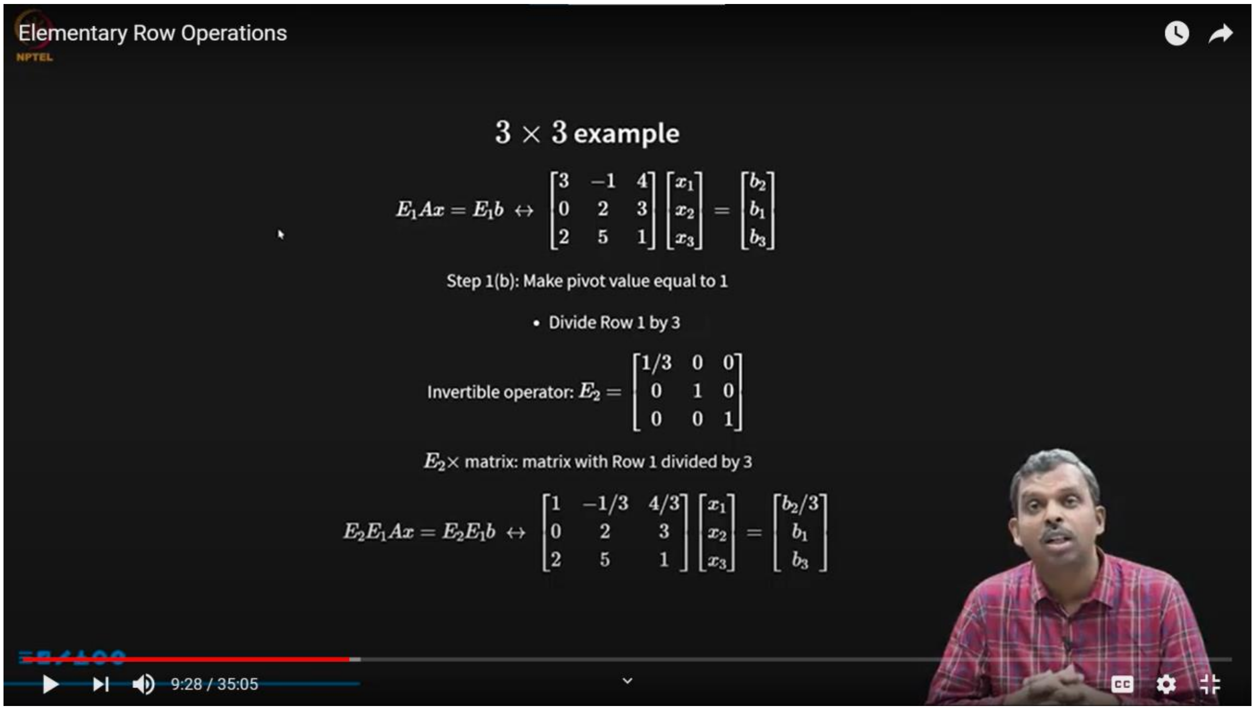

This E_2 works on the right hand side also, so the b_2 became $b_2/3$, okay? So you can see how the equation is getting altered. But, you know, retaining the invertibility thing so that you do not lose out on the actual solution. When you find the final solution it is exactly what you want from the original matrix, okay? So this is step 1b as I have called it. You can do more now, okay?

So I have 1 on the pivot position. I can make all values below the pivot zero. That's the next step, okay? So you can see I have 1 in the $(1,1)$ position. I want everything below it 0. Row 2 already has 0 there, so I do not need to do anything. Row 3 has 2 there, so what will I do? I will write row 3 as row 3 - 2(row 1), okay? Once again, an invertible operation. I can always undo this. I can add 2(row 1) again to go back to row 3 if I want. So it's invertible and it's also represented by a nice little matrix that I'm calling as E_3 here, okay? First row remains as it is. Second row remains as it is. $[1\ 0\ 0;\ 0\ 1\ 0;\ \dots]$. What do we do for the third row? I need to do -2 times the first row and nothing in the second row, and then 1 in the third, okay? So this E_3 times the matrix is exactly the operation that I want to do on the row. This is also an elementary row operation, okay? Once I hit my equation with E_3, I have this picture where I have, I am slowly getting the form that I want, right? So there is 1 on the top and then 0 0 below, okay? So next I want to go, continue on my upper triangular thing. So I will change my pivot. Instead of pivoting at $(1,1)$, I will go to pivot at $(2,2)$. So that is the next step. But notice once again that this E_3 matrix also was operated on the right hand side, okay? So my b_3 became $b_3 - 2b_2/3$, okay? So it will just become a little bit more messy on the right hand side. It will become simpler on the left hand side. But it does not matter, this right hand side is just some value for me, okay? So now we'll proceed. I have, I've gone to the pivot $(2, 2)$ from the pivot $(1, 1)$ and I already see that the pivot is non-zero, right?

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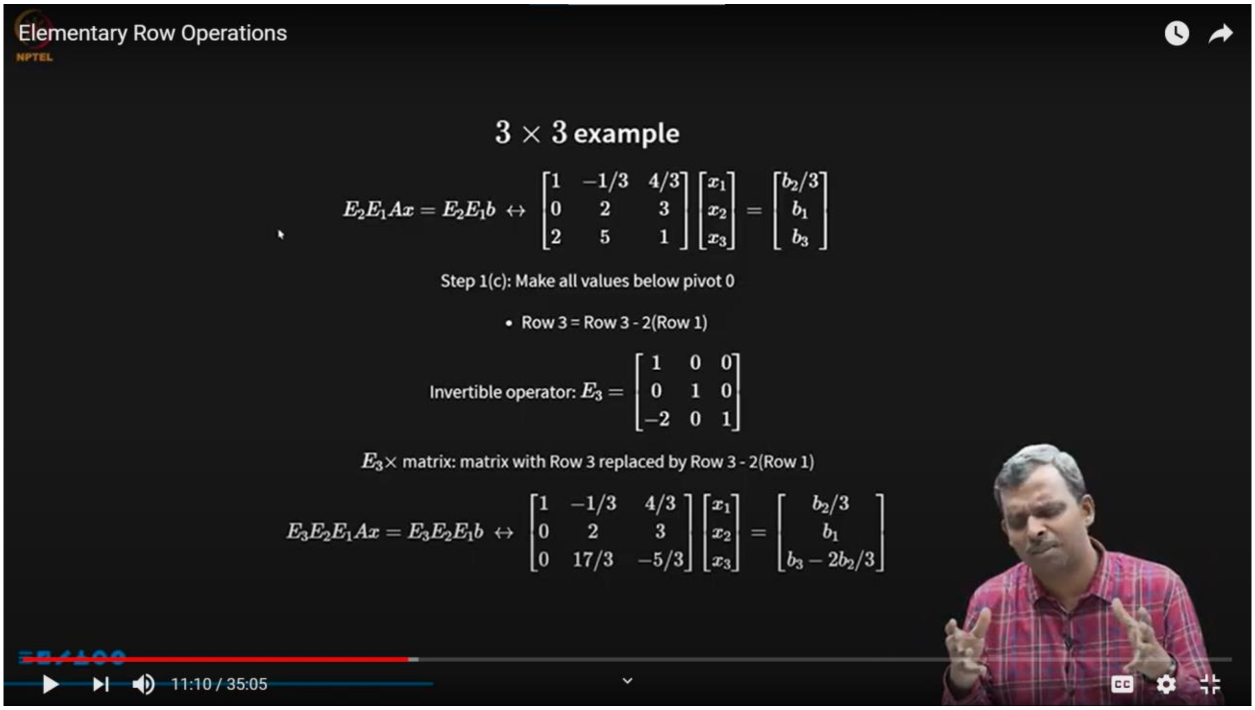

So I don't need to, you know, do any permutation to make that pivot non-zero. But it is 2, I have to make it 1, so I will divide by 2, divide row two by 2. I know already now how to get an operator for that. It is a diagonal operator 1 1/2 1. And when I multiply E_4 by any matrix on the right, the second row gets divided by 2, right? So that is the nice thing to know. Once I do E_4 no, my operators are increasing in numbers. I have just put the dot dot dot to indicate it's all everything from E_4 to E_1. You multiply that, I get this nice looking thing and remember - same operator works on the right, okay? So you get $b_1/2$. So you begin to get this now, you know? You can get the picture. Everything below the pivot has to become zero, okay?

That's my next step and then how do I do that? It's, you know. row 3 equals row 3 - 17/3(row 2). How do I do that? I have an operator for that. I know it's diagonal. Except for this, which row, you, know gets involved in the subtraction, you put a -17/3 and then you multiply with E_5 and you see that you get this sort of a matrix. It's becoming messier, but still, you know, on the left hand side you see the matrix is becoming really clean, right? So $[1\ 0\ 0;\ -1/3\ 1\ 0;\ \ldots]$. And the second column. And what happens in the third column? I have this -61/6. I know how to take care of that. I will make the pivot at $(3, 3)$ 1 by multiplying by -6/61 and I know the operator for that. I know the final form. And look at the final form that we got, right? It is exactly like before. $[(1;\ 0;\ 0)\ (-1/3;\ 1;\ 0)\ (4/3;\ 3/2;\ 1)]$.

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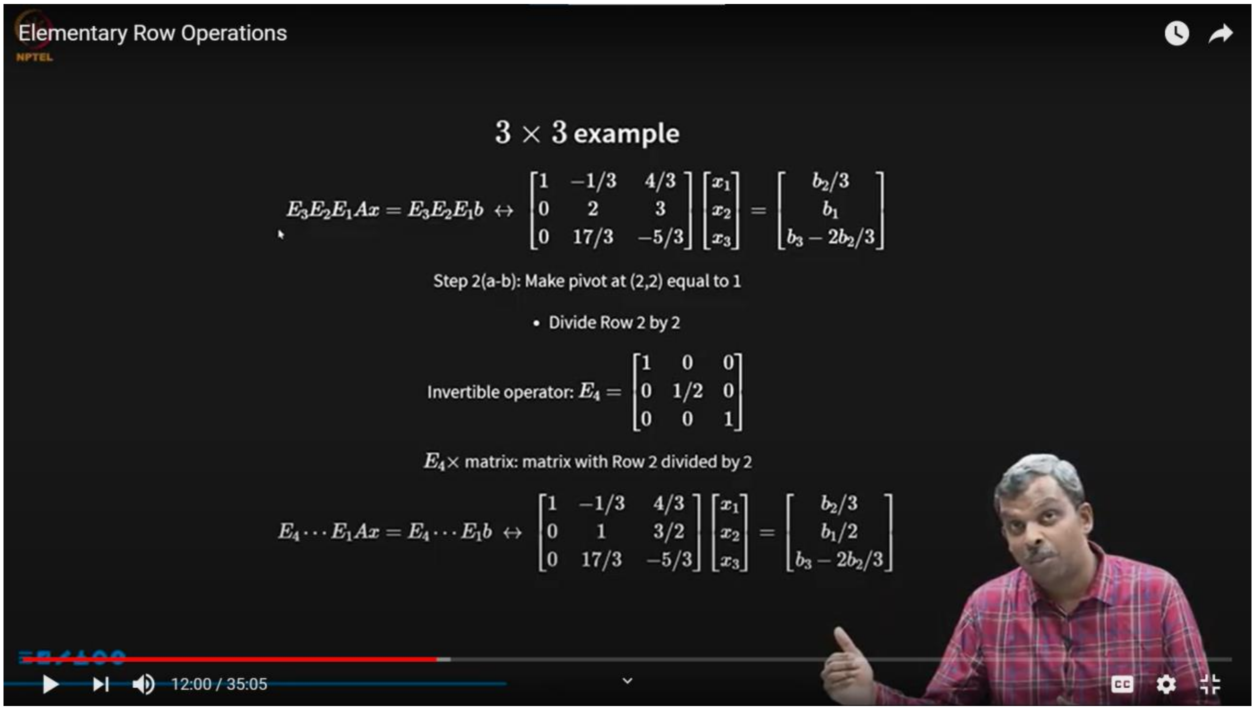

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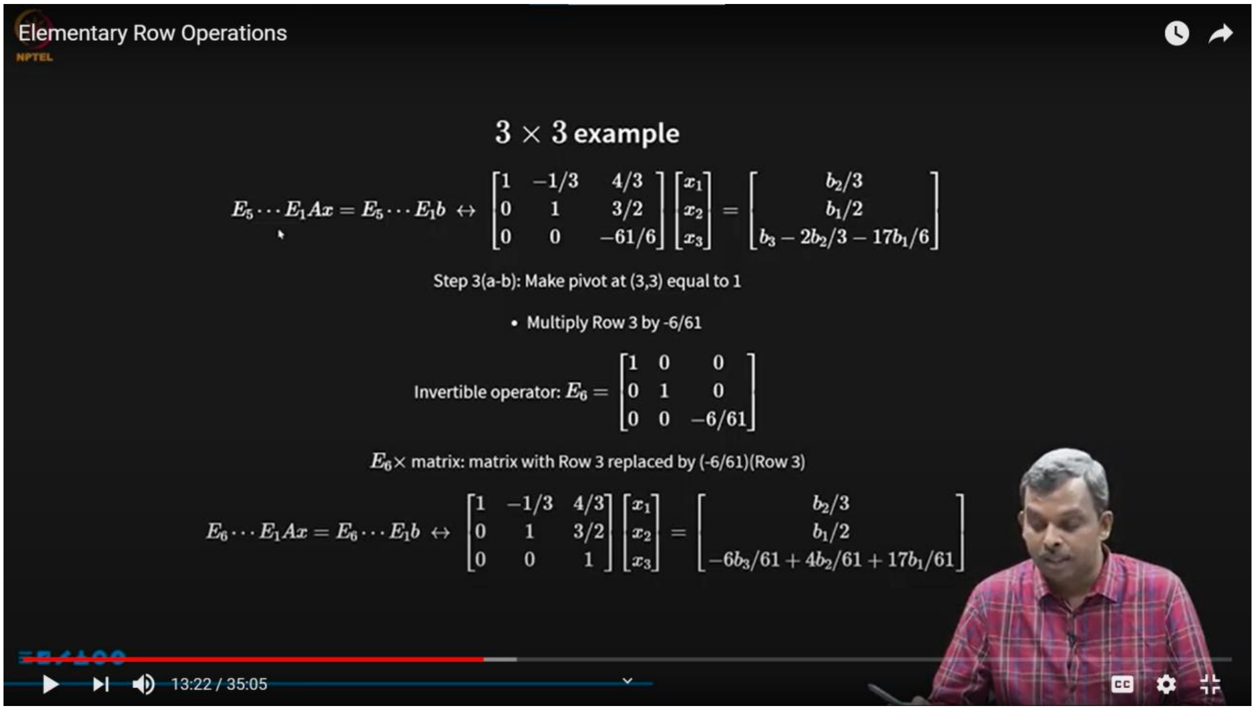

And we accomplish it by doing a series of invertible operations, you know? You know, composing invertible operators one after the other you still... Everything is invertible, right? So you can go back easily from one to the other. So we have not lost anything. And from the original equation, we have come to this really simplified form, okay?

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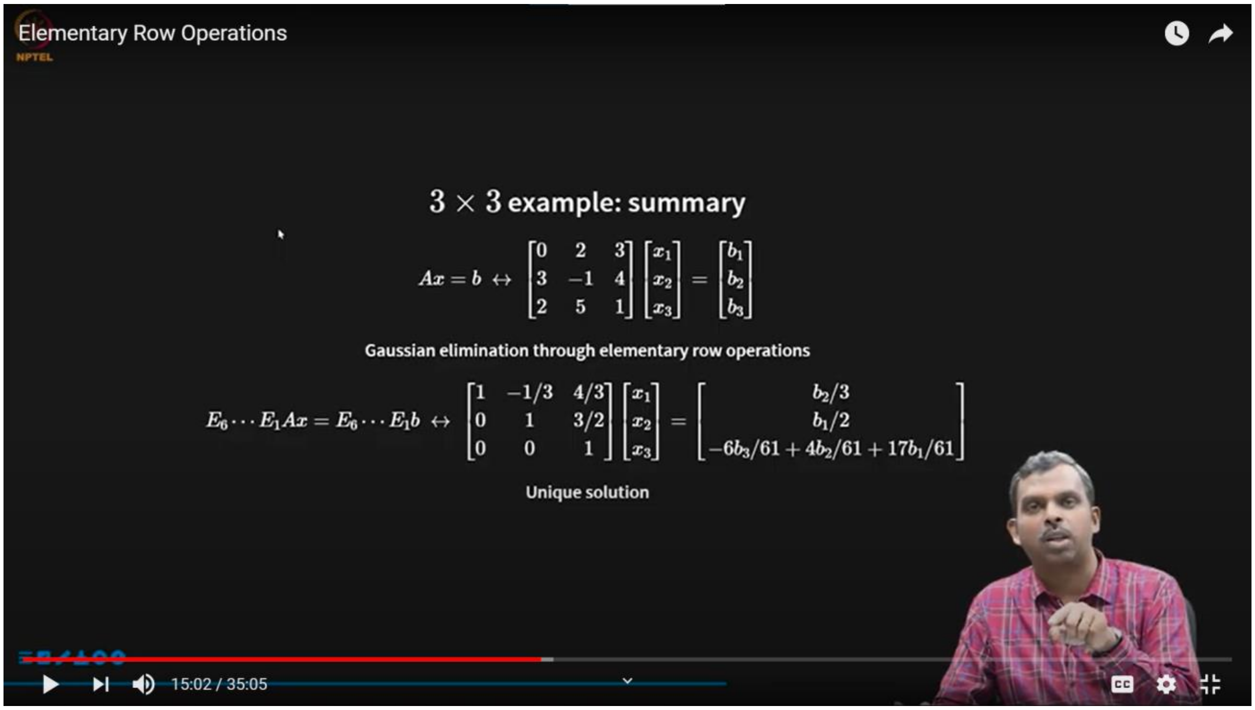

So here is a summary for you to quickly see what we did, right? We did six steps in this and every step corresponded to an invertible operator. And we saw that $Ax = b$ which looked like a form which was difficult for us to handle, but through elementary row operations, we got it down to a form which is very very easy for us to handle. We have done examples exactly like this before and you know what type of a map that is. It's an invertible map. So every possible b I will have a unique solution for. $[x_1;\ x_2;\ x_3]$ I know that much just directly from this, okay? So that's in a nutshell is what elementary row operations are, okay? So you might ask what is it in general, okay? I took a 3×3 case. What if the matrix is 100×500, what do I do, okay? What can happen? Turns out you can do something very similar, right? So it's not very difficult to imagine what it is. You just go, you know, start with $(1, 1)$ as the pivot, then bring a non-zero value into the pivot, make it 1, make everything below 0. And then you go to $(2, 2)$. Like that, like that. So now what happens if at some point you cannot find anything non-zero? Then you have to just skip that column and keep proceeding, okay? So that's the idea. So you look, you go to a particular pivot position, look for a non-zero value there or below, okay? You do not look above, you look there or below, okay? Because I want upper triangular, no? I don't care too much about other things. And if you cannot find a non-zero pivot, you simply skip that column, go to the next position in the same row, okay?

So you don't, you know... $(1, 1)$, $(2, 2)$. Suppose in $(2, 2)$ you cannot find a pivot, you go to $(2, 3)$, okay? You go to the next column in the same thing.

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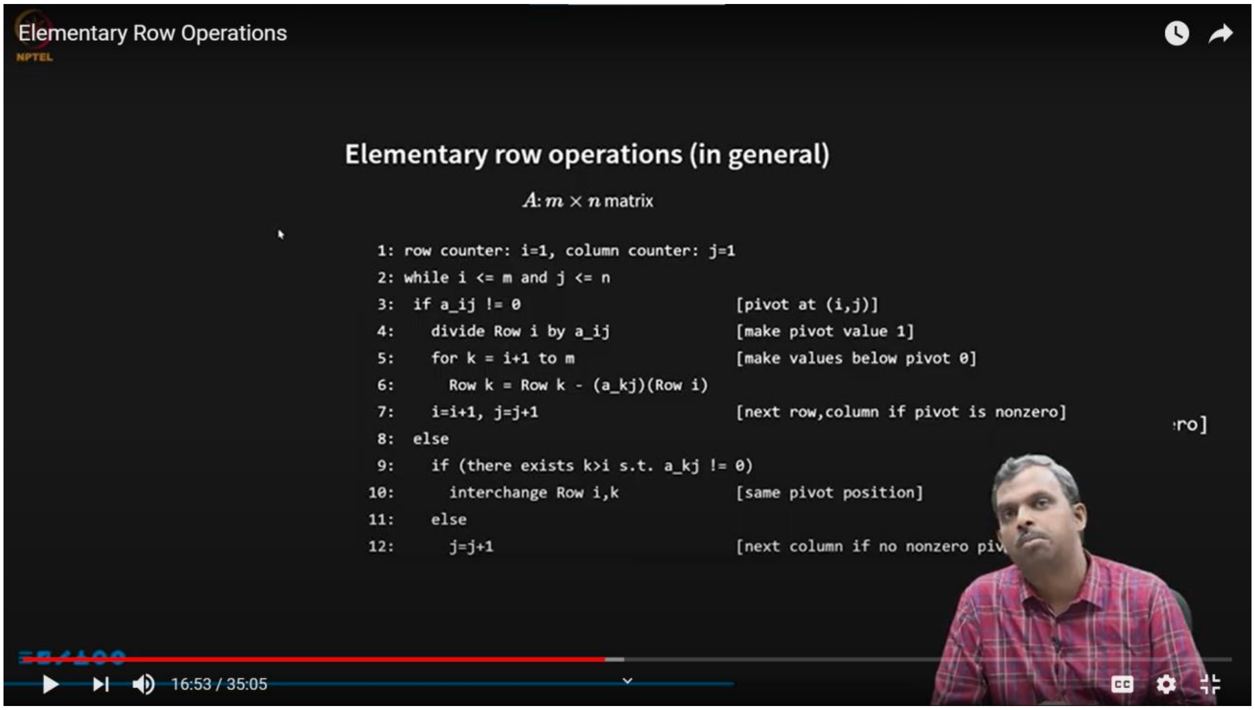

So that I have sort of codified into a little algorithm here which is what I am calling as the elementary row operations that you can do in general as far as this course is concerned. There are some minor variations here. If you see other linear algebra books and courses, they would do more than just row operations, some of them would do column operations and all that and then get a much simpler form, but I'm sort of resisting from doing column operations. Maybe we'll do it later on. For now, row operations are good enough for us, we'll live with it, okay? So here's the little procedure that you can walk through and you can do this. You can see how I've worked here, I have this counter to maintain which my pivot position (i, j) is. My pivot position always... Initially I start with $(1, 1)$ and then, you know, you shouldn't exceed the boundaries m and n. If you have a non-zero pivot already, you know what to do, right? You divide that row i by that value to make the pivot 1, and then everything below that I am going to make 0 by row operations, okay? And then once you have made everything 0, I can go to the next row, next column, okay? I can go pivot to the next row, next column. So that's why both of them get added. Now in case a_{ij} is not equal to zero, you found (i, j) is your pivot position and a_{ij} is zero, then there are two possibilities again. Something below i on that same column could be non-zero, okay? If there exists $k > i$ such that a_{kj} not equal to zero, then you interchange row i and k. So you made a non-zero pivot, okay? And then you continue back again. You do not increment (i, j), you have the same pivot. But now you have a non-zero pivot, you will come to step three. Non-zero pivot, you will proceed, okay? But supposing you cannot find anything non-zero below it, okay? (i, j) is also zero, everything below it is also zero. You simply go to the next column, okay? You do not increment the row, you go to the next column, okay? So that is the simple procedure. You can write down the code for it if you like and you can implement elementary row operations very easily. Today there are a lot of numerical tools out there which will do all this for you, you do not have to do it yourself. But it is good to know elementary row operations, okay?

So what is the major result if you think in terms of structure, okay? Supposing you do elementary row operations. After that, what is the general structure that you will have, okay? So that is an important question to answer, right? So for any matrix A, there are elementary row operations E_i such that if you do them in a proper sequence like I described in the previous algorithm, multiply by A then you will get a matrix which looks like this, okay? You could have a few zero columns, okay? This you know. So I have put a bunch of zeros here. It may happen that you do not have it at all, okay? So this may not be there, may be empty, okay? Keep that in mind. These, all these 0 columns that I put may be empty, okay? So they may not be empty, also? But in general they could be there, could be a lot of all zero columns in the beginning, right? So then you won't find a pivot, you will keep skipping columns till you find a non-zero column, right? So that's just sort of like to cover all cases. But usually you will not have these kind of things. So this could be empty also, you may not have all the zeros. So keep that in mind. Then you will find a non-zero pivot, okay?

And then you make everything below the pivot zero and then you move to $(2, 2)$. And there again you may have a bunch of zero columns. But this also may be empty, right? This also may be empty, keep that in mind. So this may be empty. I am just showing you the general case. This also may be empty, okay? Keep that in mind and then you will have a non-zero pivot, okay? So you will keep on going. And then that's how it will proceed, okay? So any matrix after you have done your row operations will look only like this. A whole bunch of zeros in the lower triangular part and it'll let you infer many things about the range and null directly just by looking at the structure. Linear independence can be quickly figured out. The dimension of the range can be quickly figured out. Dimension of the null can be quickly figured out in this form, okay? So that is the crucial idea, okay? So that is what I put here. The rank of A, which if you remember, rank is the dimension of the range of the transform, dimension of the column space. Rank of A is simply equal to the number of non-zero pivots you found, okay? You can see why that is true, right? So where all you have one, that will be equal to the rank. That will be the dimension of the range space, right? So that's easy to see, so dimension of null of A is nothing but $n - rank(A)$, okay? So once you find out the non-zero pivots, number of non-zero pivots control everything okay? And how do you find the basis for the $null(A)$, okay? So we found the dimension of the null of A. Supposing you want to find basis for the $null(A)$. All you have to do is to solve for $Ax = 0$ using A in this form. After you do this transformation, you can solve for $Ax = 0$. Solving for $Ax = 0$ also has a very simple procedure and I will describe that in the next few slides, okay?

But even before that I want to emphasize that when you are looking at solving linear equations $Ax = b$, you can do elementary row operations and reduce any general matrix to a matrix of this form. This is sort of a row reduced form. People do more reductions. By the way, they also do some column operations to have a much simpler form than this. But I'm going to stick to this form, this is enough for us at this point. If we need more reductions, we'll do later, okay? So this is the reduction. And the next step is to figure out more than just the rank and dimension of the null space, how to figure out the basis for the null space. How do you do that, how do you solve the $Ax = 0$ I will illustrate that quickly in the next few slides.

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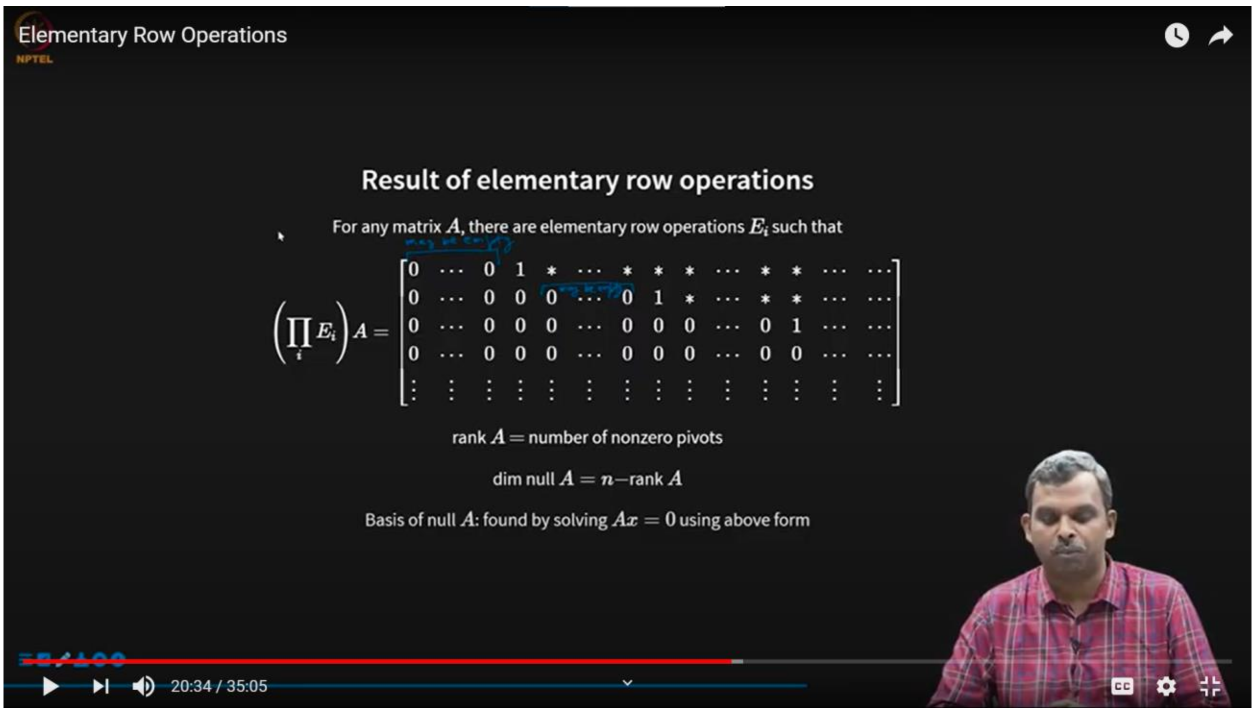

So now that we have done the elementary row operations, we are now going to see how to use the result of elementary row operations and actually infer something about the range and dimension. We already know how to do dimension of, you know, range of A and dimension of null of A quite easily. How do you do further, how do you go further than that? How do you find the basis for the null space is what we're going to see next, okay? So the first thing is finding the rank of A which is very easy once you have a form like this. Suppose somebody gives you a 4×7 matrix A which you have done elementary row operations for and reduced it to the form that is shown up here on top. You can count the number of non-zero pivots. You can see that you got 3 non-zero pivots. They are not very difficult to identify. You got your first non-zero pivot at the $(1, 1)$ place and then you went to $(2, 2)$ and you couldn't find a non-zero pivot. And then you went to $(2, 3)$ you could find a non-zero pivot. And you went to $(3, 3)$ you... I mean $(3, 4)$ I am sorry. And you could find a non-zero pivot and then after that no non-zero pivot. So you could find three non-zero pivots.

So the column space is rank three. You can see the column space is rank three, right? I mean those are the places, that's the three independent vectors in the, you know, columns and then from there you can infer the dimension of the null. That would end up being 4, okay?

So that's the first step. And the next step to find the basis for the null is to solve this equation, right? $Ax = 0$. And $Ax = 0$ you know after these elementary operations you can equivalently solve this matrix, this reduced form matrix times x equal to 0, okay? How do you go about solving this? Even for this there is a very simple process. You can maybe quickly infer something here. But let me give you a systematic way to go through this. So what you do is you first identify the variables corresponding to the non-zero pivot positions. What are the non-zero pivot columns? In the first column, third column and the fourth column you found non-zero pivots. So your first variable which actually multiplies the first column and the third variable which multiplies the third column and the fourth variable which multiplies the fourth column become your dependent variables. All the other variables x_2, x_5, x_6, x_7 become free variables and you see clearly the number of free variables is equal to the dimension of the null space. All these are not coincidences. They are, I mean, they have come because of the very reason why null space and dimension are all defined, okay? So you have, you identify your dependent variables and you identify your free variables and then you can solve for the basis of the null space or the set of all solutions for $Ax = 0$. How do you go about doing that?

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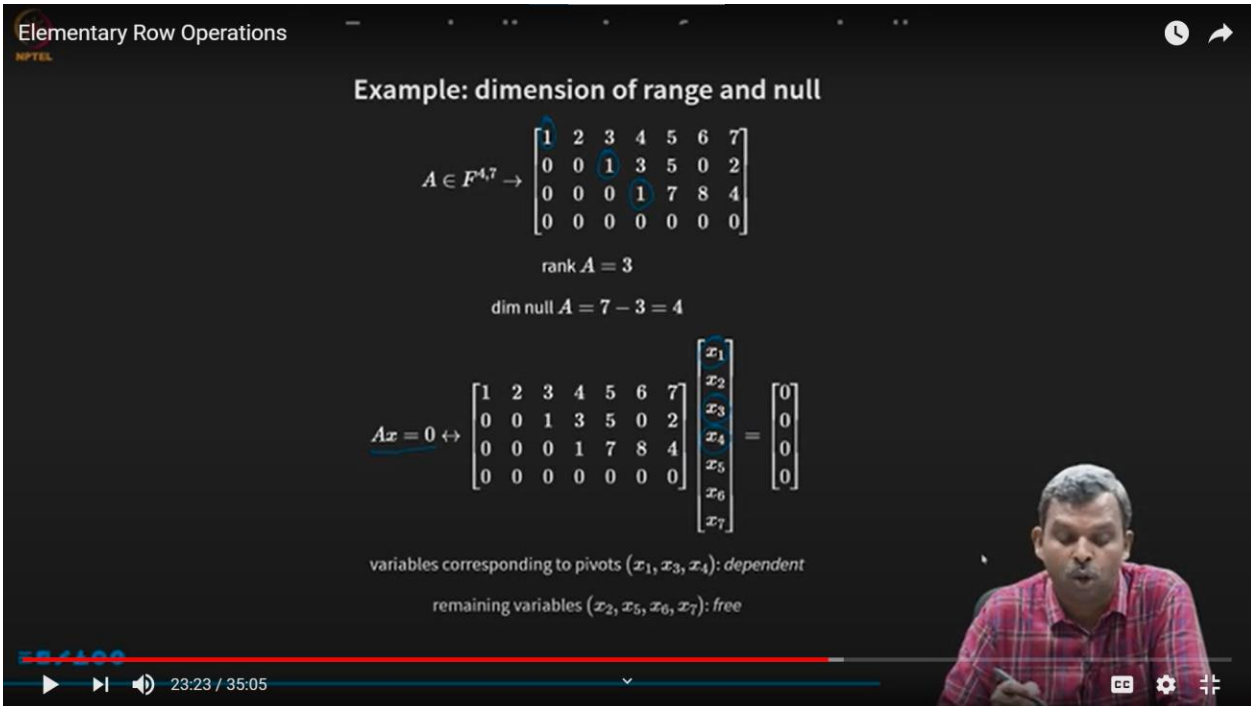

Here is how you do it, okay? So once again I have this reduced form for the matrix. I have identified my free variables. I know what my free variables are. I have to assume some values for the free variables. What values do I assume? You assume values like the standard basis. You have free variables, four of them in number. So you assume in the first case that the free variables will take values $(0, 0, 0, 1)$ and $(0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0)$. I mean technically you can assume these to be any four linearly independent length four vectors. I am simply taking them to be standard basis, that is very common, okay? So you take them like this, okay? Once you assume values for the free variables, it turns out the dependent variables can be solved for. So you solve for the dependent variables, so that $Ax = 0$, okay? So the equation is $Ax = 0$. The free variables are known to you, four free variables are given to you, you can solve for the dependent variables and your solution will be unique, okay? You can also imagine and think why that should be true. Once you put values for the free variables, right, you end up having the remaining variables give you like an injective map, right? So you will have a unique solution and that unique solution you have to find by solving... So let me show you maybe one example alone and then I will leave you to do the other ones, okay? We will take the first example where I put (x_2, x_5, x_6, x_7) as $(0, 0, 0, 1)$. Remember it's... Sometimes I will find it convenient to write the variables on top of the columns, you know, just to quickly identify. Otherwise we'll keep going back and forth, you know? Even if you write it as a vector on the right, it is easy to identify the variables like this, so I like to write it like this. And then if you put (x_2, x_5, x_6, x_7) as (0, 0, 0, 1) and you're solving for $Ax = 0$, right? So you're solving for $Ax = 0$, remember that that's very important to remember, okay? When you're solving for that, it's just easy to see, right? So you put x_2 $0, x_5, x_6, x_7$ and you want to solve for the dependent variables. You start with the lowest variable so to speak, you know, the lowest equation you start with, okay? Because it's upper triangular you start with the lowest equation and then you go back. What is your lowest equation, x_4, no? If you see $x_4 + 7x_5 + 8x_6 + 4x_7 = 0$, that's your equation. Now x_5 and x_6 are 0, x_7 alone is 1. So you basically have $x_4 + 4 = 0$ which gives you x_4 equals -4, okay? So you've got x_4 here. Then you go to the next equation above it. Once you solved x_4, you can go to x_3 and that equation is x_3, okay? So x_4 you already solved as -4, so -12. And then x_5 and x_6 are 0. Then x_7 is 1 plus 2 equals 0, and that gives you x_3 equals 10 which is what I have put here, okay? So x_3 is 10. And then you go to the first equation. There you get x_1, okay? And then x_2 is 0, x_3 is 10 so +30, okay? And x_4 is -4 so -12. Then x_5 and x_6 are 0. And then x_7 is 1 plus 7 equals 0 and that you can check. It will give you 30 - 12... Did I get that right? Maybe I made a small mistake here, let me just check that once again. I think there is a mistake here. So you have x_1 + 30 minus... Oh no, this is not 12 this is 16, right? Yeah you're right, so that's correct. So I didn't make a mistake. So x_1 + 30 - 16 because it's 4*4, - 4, so - 16 + 7 = 0 and that will give you x_1 equals -21. So that's what I got here, okay? So (x_1, x_3, x_4) is (-21, 10, -4). (x_2, x_5, x_6, x_7) is $(0, 0, 0, 1)$. You put them all together, you get your v_1. So this is one basis vector for the null space, okay? The reason why it becomes a basis vector and all will be clear to you. But this one vector which definitely belongs to the null space. There is no doubt about it, right? If you find four linearly independent vectors in the null space, you are done. You know the dimension is four, so I have found one vector in the null space, okay? Why it's linearly independent you will sort of quickly see. Because of this $(0, 0, 0, 1)$ it will become linearly independent, okay?

So likewise you proceed. For the next set of values $(0, 0, 1, 0)$ you proceed in the same way. I'm not going to show you the details for this, you can see how to proceed. Just put the values exactly like the same way I described before. Start from the bottom, keep solving. You will get this other vector v_2. So v_1 belongs to null space, v_2 belongs to null space. Likewise v_3 again put $(0, 1, 0, 0)$. Solve for the free variable, solve for the dependent variables. You know v_3, right? And then you put $(1, 0, 0, 0)$. This is the easiest to solve. Everything else becomes 0. Only the first equation matters. So you can put $(-2\ 1\ 0\ 0)$. So you get v_4. So now it's important to quickly realize that the basis for the null space will become v_1, v_2, v_3, v_4. Why? Because v_1, v_2, v_3, v_4 clearly belong to the null space $Av_1 = 0, Av_2 = 0$, we have solved for it like that. And additionally they are linearly independent. Why? Just look at the free positions, you will get like an identity matrix there, right? So $(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0)$. So they cannot have any linear dependency between them. You have found four vectors which are linearly independent in the null space and that gives you a basis. It's a very simple process. I'm sure you must have seen some version of this process before but this is the logic for why it works, okay? So that's how you solve for the null space, okay? So you see that the elementary row operations tell you the dimension of the range, dimension of the null. They also allow you to solve for the basis for the null space of course. I mean, it's a little bit of computation, but you can do it quite easily.

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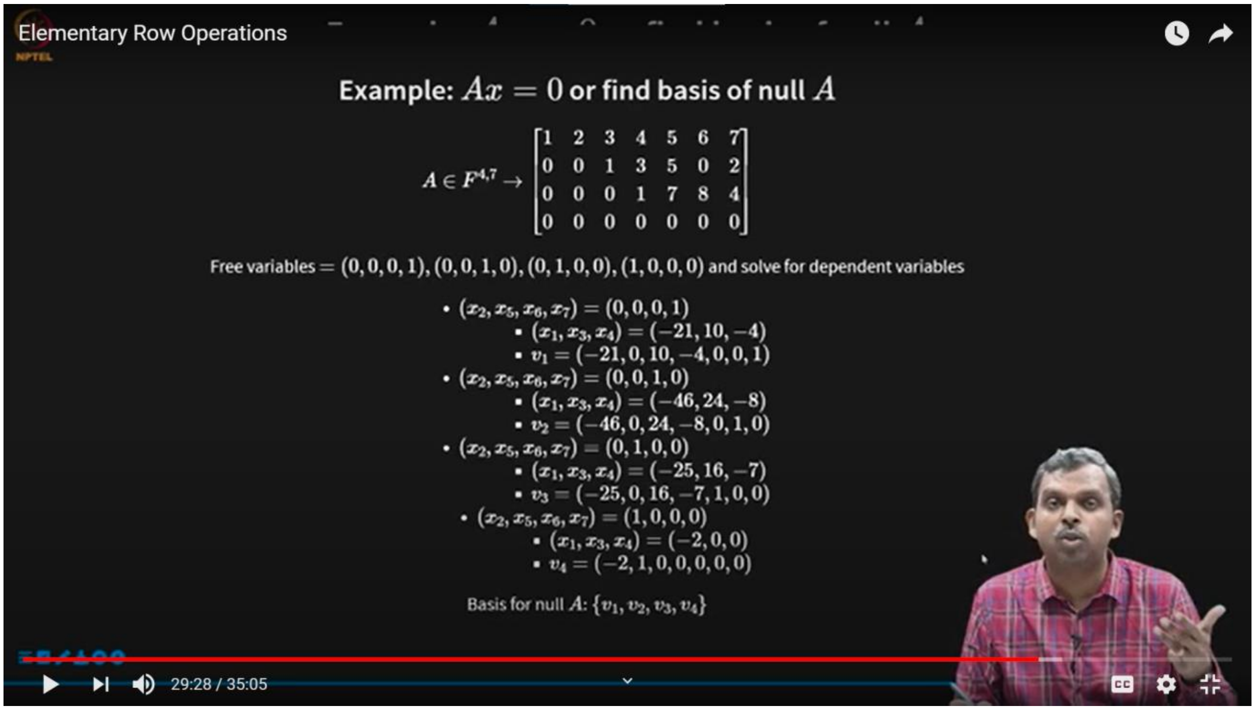

Okay. So now how do you solve for $Ax = b$, okay? I solved for the null space, we also saw that once you know the null space, once you know whether it's in the range or not, it's easy to figure out the solution. So let me just walk you through how you solve for $Ax = b$, okay? So the way you do it is the following, okay? So you simply set the free variables to be $(0,0,0,0)$, okay? You can set it to be anything else also. It does not matter what you set it to be. I will set it as $(0,0,0,0)$ because, you know, it gets rid of all my, you know, columns in the matrix. It becomes much simpler. So you simply set it as $(0,0,0,0)$ and solve for dependent variables so that $Ax = b$, okay? So in that equation, things will be much much clearer. It is very simple. First of all, it will be an upper triangular matrix and with some zeros, you can quickly identify whether or not solutions are going to exist or not. It is very easy to see that. So it just gets rid of all your, you know, free variables. You only have your dependent variables. So maybe I should write the simpler equations. Once you put $(0,0,0,0)$, you just have this equation, right? So you have [(1; 0; 0; 0) (3; 1; 0; 0) (4; 3; 1; 0)] and that times my dependent variables are (x_1, x_3, x_4) and that needs to be equal to $(1,2,3,0)$, okay? It will be in this form. It will be in your reduced form, right? And it will have all the ones coming one after the other. There will be none of these nasty zeros in the middle which confuse you, okay? So it will be just all ones together and you can quickly solve for this, okay?

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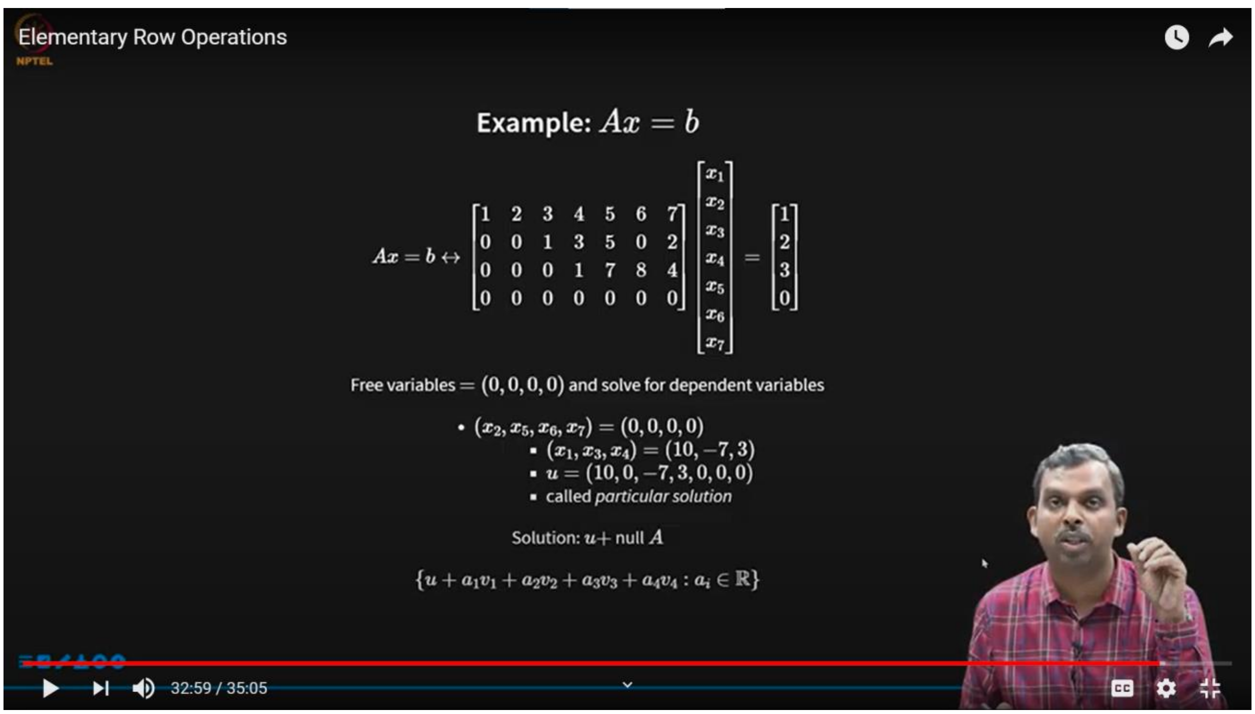

First of all you can identify that any zero row here also corresponds to a zero on the right hand side. If there is an all zero row here and it does not correspond to a zero on the right hand side, what can you conclude? You can conclude that there is no solution. But in this case, the zero here corresponds to a zero there. So I know that there is a solution. And then I can quickly read out what will be the solution for (x_1, x_3, x_4). I start with solving for x_4 first. x_4 ends up being 3, and then use that in the previous equation and get x_3. Use those two in the previous equation. Get x_1.

Very simple process. One can follow that and then solve for it, okay? So that's what I have done here and such a solution is called a particular solution, it's one vector in the solution, okay? So it is one value of v such that $Tv = w$, okay? One solution for x. But now look at this particular case. This particular matrix corresponds to a linear map which is not surjective, it is not injective also, right? So then what do I expect? I expect, if the value of b is in the range, I expect infinitely many solutions. Value of b was in the range, I saw before that the zero rows are corresponding to zero values on the right hand side. So all I have to check is find the null space and I will have a full solution, okay? So the entire solution is $u\ +\ null\ A$, the solution will have a picture like this. This u, particular solution plus any linear combination of the null space basis vectors. So this is the set of all solutions for this, okay? Hopefully this gave you a very clear view of how things will look and how you can find solutions for linear equations.

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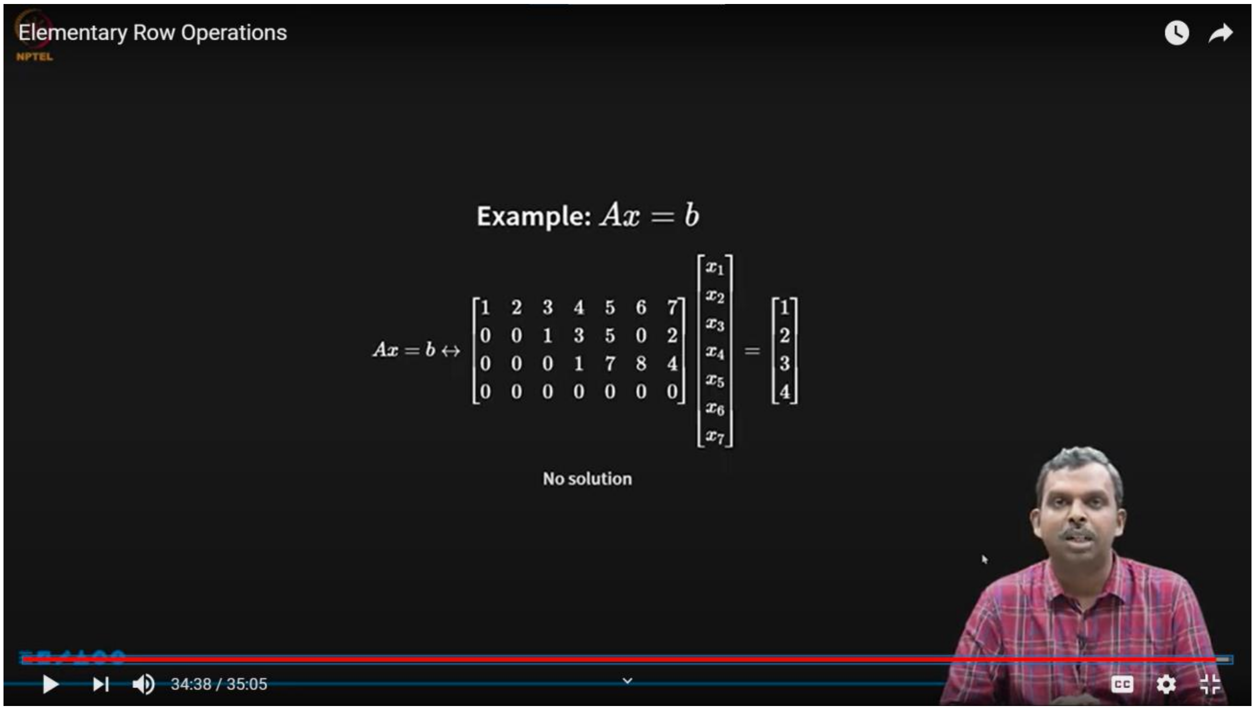

Now I also told you the other case. Where you could have no solution, okay, right? This is a linear map which is not surjective, not injective. So if you have a vector outside of the range, then you may not have a… So how do you identify? And you see that will also happen very easily, right? So here when I try to solve for the particular equation, what will happen? I will put my free variables as $(0, 0, 0, 0)$. Then I will get an equation where you will have an inconsistency. There will be an all zero row on the left hand side. It will correspond to a, you know, non-zero value on the right hand side, right? So look at this value and look at all these guys, okay? So there will be no solution corresponding to this, okay? So this will always happen because in this row reduced form, you see this will be the only way in which you can have no solution, right? So otherwise you will have a fully, you know, invertible sort of upper triangular form, okay? So from there you can conclude there is no solution, okay? So hopefully this tells you what to do given any linear equation $Ax = b$. You do a series of elementary row operations, reduce it to the form, the row reduced form, identify free variables, identify dependent variables, find your dimension for null, find the basis for the null space and then when you want to solve for $Ax = b$, find a particular solution and there you may have a solution, you may not have a solution... If you have a solution then the entire solution set is that solution plus the null space. So null space is trivial, then you have a unique solution otherwise you have infinitely many solutions. And if your b is not even in the range, then you don't have a solution at all, okay? So that's in simple terms how to solve linear equations using these elementary row operations in general, okay?

So I have set up a small quiz for you. You can go through and answer the questions in this quiz. It will give me very good feedback and it will also give you a good brush up of all that you have learnt here. And you can be sure of some of the concepts and get it reinforced, okay? Thank you very much.

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