Applied Linear Algebra

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Week 03
Translates of a subspace, Quotient Spaces

All right. So welcome once again. In the previous two lectures of this week we saw how to solve linear equations, okay? So first application may be that we are seeing of all the theory and ideas that we've learned so far. And hopefully it was reinforcing to learn the theory well and think of it in abstract terms. And also in solid terms, in terms of numbers, it gives you a very good grounding and foundation on what you are doing. And as you are doing it you can know whether you are doing it correctly or not, if you know both the theory and the practice of how things work, okay? So now let us move on a little bit ahead. What I am going to do in this lecture is going to be a little bit abstract. It's an abstract view of how linear maps work and how these linear equations work and how they look etc. It's very important to understand. And this notion has a lot of extension in abstract mathematics, a lot of abstract mathematics uses this idea of translates and quotients and the translates are called various other terms. They are sometimes called cosets of a set etc. And this idea of cosetting and quotienting is very very important and crucial. It gives you a nice structure result about the objects that you're studying, okay? If the study looks very scary to you, don't be too scared, you'll see it's actually very easy to follow. But it's a nice view of what a linear map does and you will see it gives you good intuition and what to expect, and it's very good for later use. So this picture of a linear map that you will develop, this structural picture of a linear map that we will develop in this lecture is very crucial for, you know, as we go further and study more advanced ideas. How to relate everything in a nice way, okay? So this is a crucial idea. Let me also warn you it's probably a little bit abstract but I'll try and make it as easy and simple as possible with some concrete examples.

Okay, a quick recap. Once again we are in vector spaces over real or complex fields. And we've seen that these linear maps play a crucial role and they are associated with $m \times n$ matrices. We saw null and range and Fundamental Theorem very useful in solving linear equations $Ax = b$. And elementary row operations make your work very very simple to help you identify the linear dependence, independence patterns in the vectors and quickly conclude something about the range and null and solutions for linear equations in general, okay? So that's what we saw so far. We'll dig a little deeper into linear maps and look at this structure of null and what's going on and, you know, the final structure we'll derive at the end of this lecture will give you nice insights into what is going on, okay? So let's proceed.

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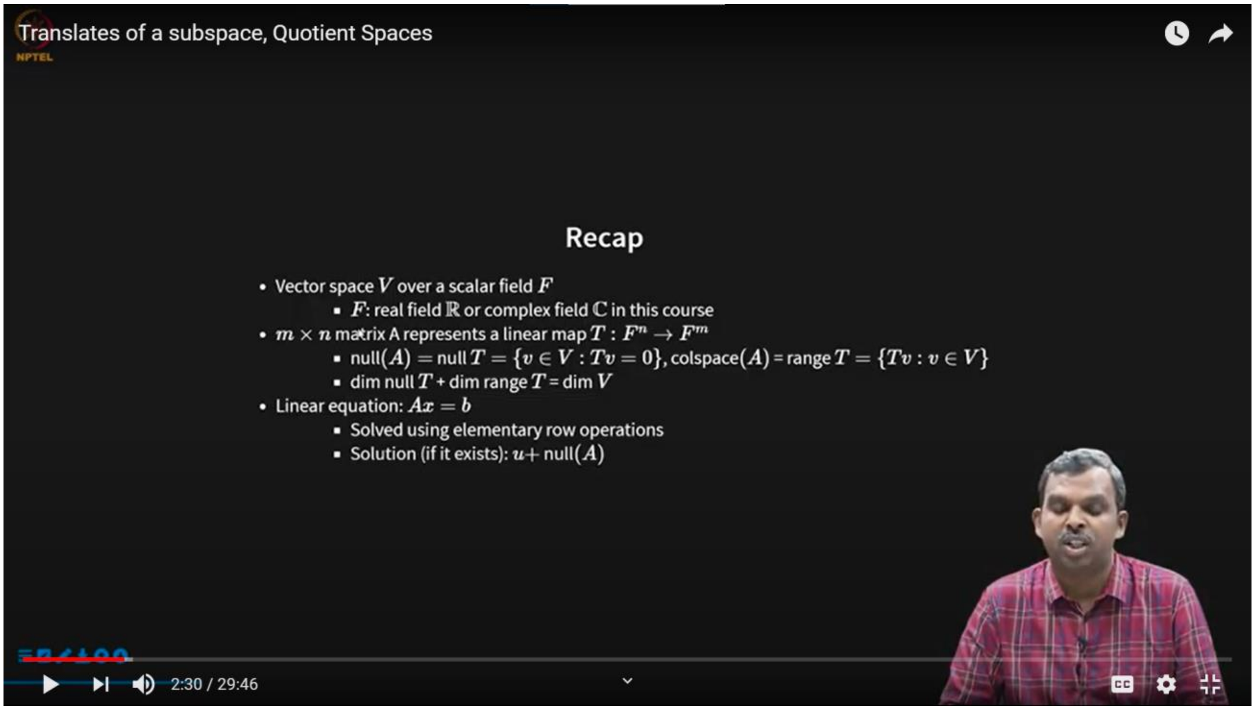

Okay. So what is a translate of a subspace? This notion is very important. So let's say we have a subspace, okay? You have U inside V which is a subspace, and maybe there is another vector, okay? This vector is what I have called as v in this definition. A vector v in the vector space. And a translate of the subspace is simply defined as $v + U$. So now what is $v + U$? It is a set, okay? U is a set. How do you add a vector to a set, okay? So if you are confused by that, it's easy to see the definition. $v + U$ is the set of all $v + u$ such that $u \in U$, right? So that's called a translate. And that's easy enough to see why the definition makes sense, right? So you have one vector and you have this subspace. How do you translate the subspace by this vector? You take every vector in the subspace, you add this vector to it, okay? So you get a translate, okay? So let me quickly show you a nice illustration of how this will look, right? So for instance, it will look… So let us take... Maybe I should go into this other, let me go into this white board and draw my $\mathbb{R}^2$, okay? So $\mathbb{R}^2$ is my favorite example as you have seen and gives you a lot of ideas, okay? So let us look at a subspace, okay? So this is my U in $\mathbb{R}^2$, okay? I am looking at $\mathbb{R}^2$ being the whole V, okay? This is my subspace U, and let me look at v equals let us say $(3, 0)$, okay? So I am translating this subspace by v. And what is v? It is 3 units in the x direction and 0 units in the y direction. So I am moving this subspace to the right by 3 units. And what will I get? I will get a picture that looks like this. It's parallel to this and this will be $(3, 0) + U$, isn't it? It's easy to see.

On the other hand if v were to be, let's say $(0, 2)$, then what am I doing? I am translating this subspace U by zero positions in the x direction and two positions in the y direction. Plus two. So that would look something like this, isn't it? So that would be $(0, +2)$. You can imagine I can do a combination now. I might take a v to be equal to, I don't know, $(-2, +3)$, okay? So I would like to go, you know, minus two positions in the x direction and then plus three in the y direction, okay? What would have happened? I would have gotten, 0 would go somewhere there and then from there it would draw. So it will go somewhere there, I think, right? So it's parallel to this and then it would go somewhere there, okay? So this would be $(-2, 3) + U$, okay? So as you do it, hopefully you are convinced that you always get lines that are parallel to this, you know, this subspace by translating. And you can take them anywhere, okay? So this is the picture to have in mind when you think of translate, okay? So it is not a bad picture. Except that when you go to larger dimensions maybe you can't visualize this translate quite like that, right? So when you have larger dimensions, 100 dimension and then a 30 dimension subspace, it's a huge thing and then... But still you can think of what is going on and keep this line picture in your mind. So I am simply translating the subspace by these vectors, okay? So that's the, that's the description of a translate.

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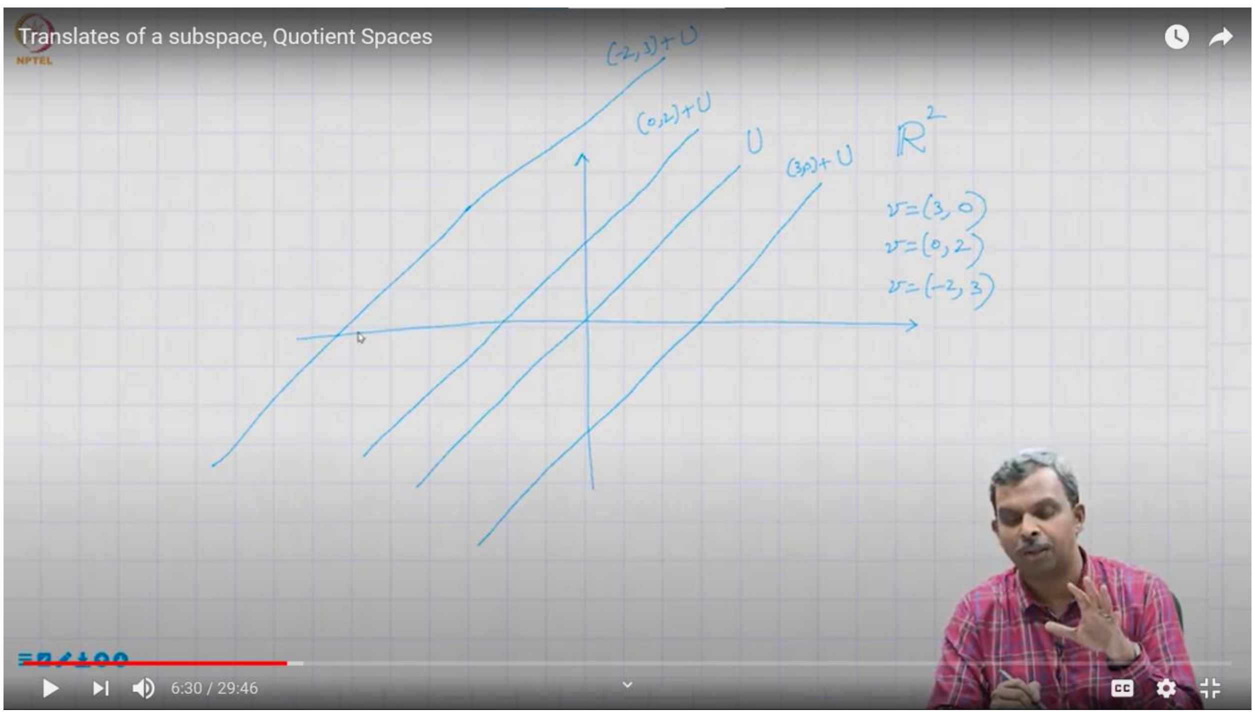

So there are two immediate possibilities. And I'll show you a lot more possibilities as we go along. But there are two immediate possibilities. This vector v that I picked could be inside the subspace itself, right? If this vector v is in the subspace itself then what will happen if I translate? I won't go anywhere, right? Because I am already in the subspace. I am adding subspace to subspace, I will only get subspace. So my $v + U$ will become U itself, okay? It's a nice thing to see. On the other hand, if v is not in the subspace then not only will $v + U$ be different from U, it will be disjoint from U. What is disjoint? Disjoint means there is no intersection at all. There is no point in $v + U$ which will also be in U, okay? They will be parallel in some sense, right? So they will not intersect at all, okay? So that is a nice property to know, okay? In fact you can say more. You can say something more interesting about translates, okay? Supposing there are two vectors v and w that are given to you from the vector space V and you look at the two translates $v + U$ and $w + U$. It turns out these two translates can either be exactly the same or they will be disjoint, okay? When you translate subspaces, you cannot have partial overlap, okay? That's ruled out. Not possible, okay? That makes something very interesting, right? Just because it's a subspace and you translate... If you translate other sets and all, you can do partial overlap very easily, right? But if you translate a subspace, you cannot have two different translates partially overlapping. They either fully overlap or they don't overlap at all, okay?

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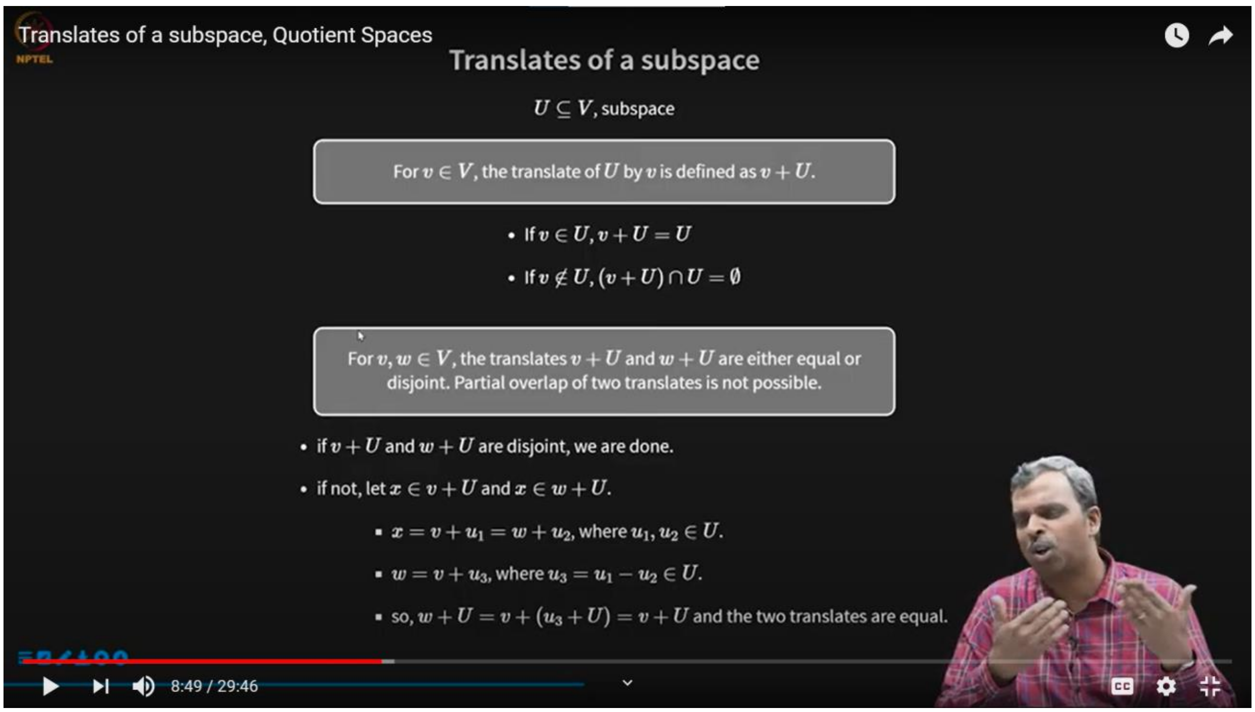

There's a little proof here given below. I'm not going to go into details. It gets a bit technical, but you can prove this. So any two translates... You can also imagine why, right? There's two parallel lines. If the shifts are actually different then you're not going to get the same thing. But you know, it can exactly overlap also, okay? So you may get the same translate by two different things. So maybe I should show you an example by going back to this little board that I had here. So notice while I wrote this as $(0, 2) + U$, okay? So maybe I cannot do this. Maybe I should show you one little picture here. So notice, so even in $\mathbb{R}^2$ if you notice, I start with the U, okay? And then I told you $(3, 0) + U$, right? So let us say you did $(3, 0) + U$. It was something like this, okay? This is $(3, 0) + U$, okay? So notice if you take any other point on this translate, so this is going to be $(3, 0)$. And what will this be when I do $(3, 0)$ to the right? And depending on what this line is, right? So it depends on the, you know, so let us say... Maybe I will take a simple form for this line. I will simply take $y = x$ to keep my life simpler, okay? So this is $y = x$. And then if I translate, right, by 3, what will this point be? This point is going to be $(0, -3)$. So now the same translate is also equal to $(0, -3) + U$. Think about it. So whether you shift it three places to the right... It's a $y = x$ line. Whether you shifted three units to the right or you shifted it three units below, you would get the same line at the end, right? It's not going to change because it's $y = x$ and similarly you can do for anything else also.

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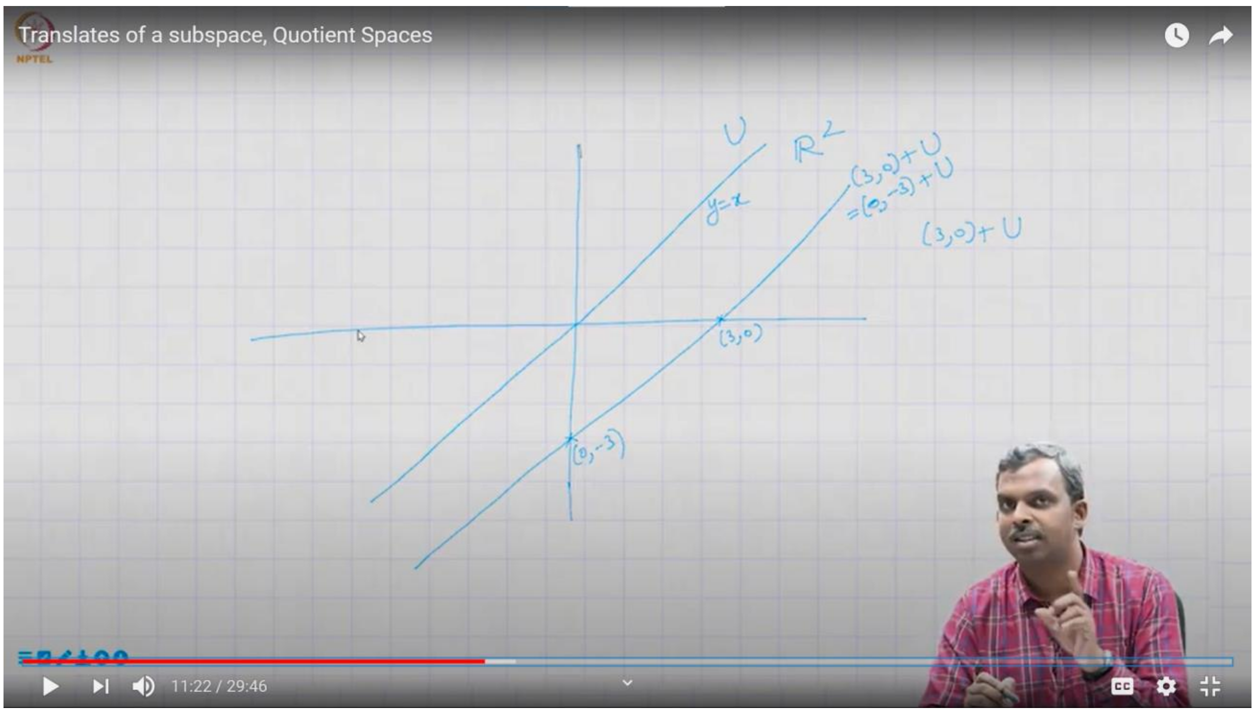

So while the translate is easily defined, you can get two different vectors of the same subspace giving you the same translate, that's possible, okay? So don't think that it's, you know, any two translates are always distinct just because the vectors here are distinct, okay? So two different translates can be exactly identical or they will not overlap at all. They will be disjoint. What is ruled out is partial overlap. You cannot have two different translates partially overlapping, that is not allowed at all. That cannot happen with subspaces, okay? So that is the interesting thing here, okay? So two different subspaces, two different translates may exactly coincide or they will be disjoint. Partial overlap is not allowed, okay? Hopefully that's clear to you. So these translates are beginning to look interesting, right? When you do translates with subspaces, they're not so, not very generic like sets and moving sets around. They have a particular structure to them, okay? So this structure is what we'll exploit to define this notion of a quotient space, okay? So you have a subspace U, and this notion of a quotient space. It's, you know, it has the strange notation V almost divided by U, that's where the quotient comes from and you'll see why this sort of division makes sense, okay? That's basically the set of all translates of U. You have a subspace U, you keep translating it by different vectors, you know there can be many... Many translates may exactly coincide whatever... So you collect all the translates together, you make a set out of them, you have your quotient space, okay? So it’s an interesting set to define. You can think about the line example for instance. In the line example, if I keep shifting it throughout, I would have covered the entire $\mathbb{R}^2$ for instance interestingly. But anyway, all the translates, all possible translates that you can do to the line, put them all together, you get the quotient space. The quotient space contains translates, not just points in $\mathbb{R}^2$, okay? So it's a different sort of a set, okay? Keep that in mind. Every element of the quotient space is a translate of a subspace, it’s not a point in V. You keep that in mind. I have U itself. I will have, you know, some vector plus U, some other vector plus U, those are the elements of my V/U, okay? So it turns out...

How do you find all translates? It looks like, seems like too many of them. Of course there are many of them. And how do you quickly find all the unique translates very nicely? There is a way to do it, okay? So you can find the unique translates using this, you know, the direct sum vector space W. So you find another subspace W. You know how to find this W, W subspace such that the entire vector space becomes $U \oplus W$. This is crucial. This is very important. Once you find that, you can write down the list of all unique translates, okay? So that is very important. And this direct sum is also very important, otherwise you cannot do that, okay? So I have given a little bit of a logic here to tell you that any translate $v + U$, okay, any vector v in the vector space V and if you look at the translate $v + U$, you can find a w so that $v + U$ will be $w + U$, okay? So this is not very difficult to figure out why, right? So you will have for any $v \in V$. Because of this direct sum, you can write it as $u + w$, right? So there is a w. So if you look at $v + U$, that will be the same as $u + w + U$. But this $u + U$ will sort of combine together, right? What is $u + U$? That’s the same as U. So this will be equal to $w + U$, okay? So that’s a quick little logic to tell you that any translate can be written as $w + U$, okay? So if you're going to look at all translates, you don't have to go for vectors outside of W, your translating vector can only be in W, you exhaust all the vectors in W which is this, you know, complementary subspace to U. $U \oplus W$ has to be equal to V. You find a W like that, okay? You know how to do it, right? You can do the basis extension method to find a W. Once you find the W, it's enough if you translate by the vectors in W, you would have covered all the translates. Any translate would have been covered. That's one thing that's very nice.

Another thing that's very nice is - if you have two different vectors in W and these two vectors are not equal, they should not be equal, then these two translates $w_1 + U$ and $w_2 + U$ will be disjoint, they will not be equal, okay? Two different vectors generally if you take from V, the translates can be coinciding, right? So we saw that it's possible. But if you take two different vectors from this W subspace which is sort of complementary to U, and translate by that, you will always get something disjoint. You cannot get the the identical variety here, okay? So that is also something very very nice, okay? So how do you prove it? You can think about how to prove it. Clearly, you know... So w_1 for instance is in the first one, okay? So w_1 belongs to this, w_2 belongs to this, okay? And you can show w_2 does not belong to this and w_1 or okay, whatever... One of these things. So w_2 will not belong to this and that's enough to show that they are disjoint, right? Because we know that any two translates will have to fully match or be disjoint and if I show that w_1 belongs here... That's very easy because U has 0. So w_1 is there. But you can also show w_2 does not belong here and w_2 belongs here. So these two are not identical which means they have to be disjoint, okay? So that's the way to prove it. I am not going into too many details here. You can write down a proof for it if you like. But this is, this result is important, okay? So we have this result that this quotient space which is the set of all translates of U can be written down in this precise way. You only pick the vectors from this subspace W and translate the subspace by those vectors alone. Then you will get the entire set of all translates, okay?

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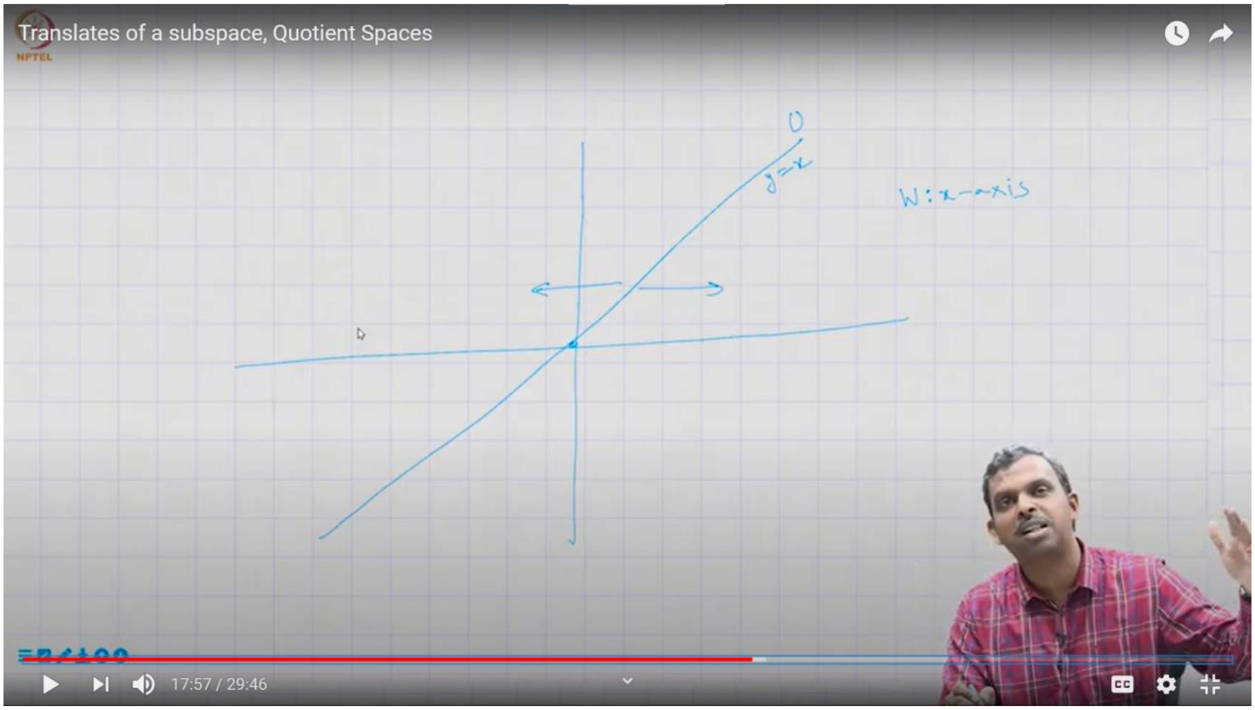

So maybe I should illustrate this once again with my famous $\mathbb{R}^2$ example. So let us look at the $\mathbb{R}^2$ example and look at the $y = x$ line, okay? So this is U which is $y = x$, you can find a w, right? W is an, x axis is a W, okay? There are so many W's. We know that there are many you can find. But x axis is a W, so it is enough if you translate U in the horizontal direction, no? Its $y = x$, you translate U horizontally, which is what translating by the x axis means, right? So you pick vectors from W, you will only translate horizontally, y is always zero, right? But horizontal translation is enough, you would have covered all the possible translates wherever you want to go, you know? It's enough if you, because it's a subspace it's enough if you translate horizontally, okay? So only horizontal translates are enough for you, okay? So think about why that is true. So the quotient space is given only by translates from W. You don't need to worry about other translates, okay?

So that gives you this wonderful result that the translates of U partition V. What is a partition? If you have a set, what is the partition of the set? You divide up the set into groups which don't intersect, okay, right? But together they should make up the whole set. Only then it's a partition, okay? So you have a set, you divide it up into multiple sets, together they should make up the whole set. But no two of them should intersect and that's what's happening in this quotient space, right? I know no two of them intersect but together they should make up the whole space. So how do I know they make up the whole space? Because every translate is included. Any vector v belongs to $v + U$, isn't it? Any vector that you pick in your vector space V will belong to a translate because if it doesn't, you simply translate by that vector, right? $v + U$, that has v for sure because U has zero, no? So $v + U$ has v. So set of all translates covers the whole vector space V. But if you look at this quotient space, they don't intersect, okay? There it’s disjoint. So they end up being a partition, okay? So this partition is very interesting and it gives you a structure result, right? A subspace partitions the entire vector space through its translates, okay? And the translates can be exactly identified by looking at this W space which, you know, direct sums to V taking all the vectors from that and translating by that, okay?

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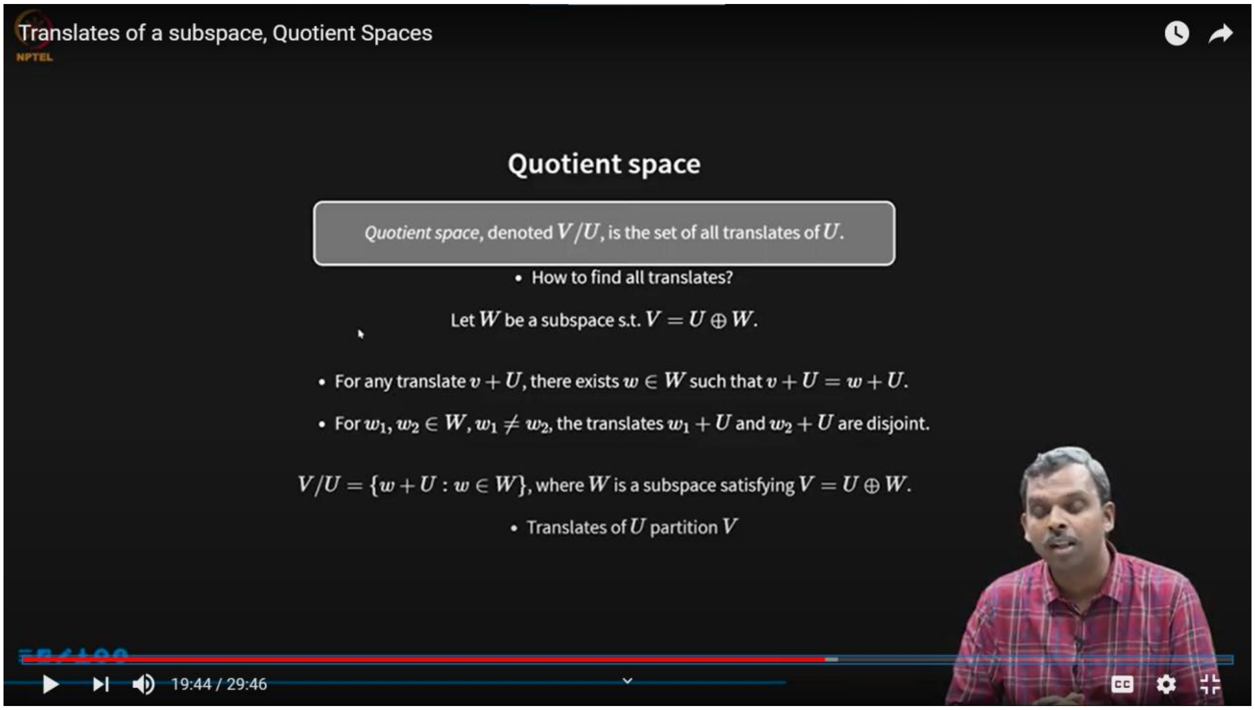

So nice enough picture that you have, Good structural result. How does it help us with linear maps and equations? Here is a picture for you to keep in mind, okay? So the important aspect of a linear map is the null space. We saw that the null played a very crucial role. Null is the subspace of V which maps the vectors and vectors in the null space get mapped to zero, okay? So now notice what happens when you take this null space and do a quotient of that null space. With that null space of V, you do $V/null\ \ T$, okay? You look at the quotient space defined by the subspace $null\ T$, okay? What will happen here? How do you do it precisely if you want? You pick up a, find a subspace W so that V becomes $null\ T \oplus W$, and then $V/null\ space$ is a set of all $w\ +\ null\ T$ so that $w \in\ W$, okay? So I know now this is a partition of V. Remember this is a partition of V. So let's draw a picture maybe a little bit later. But look at what is interesting here. If you look at T operating on $w\ +\ null\ T$, any vector in $w\ +\ null\ T$, what will happen? You will only get Tw. Why? Because the $null\ T$ part gives you zero, right? So T operating on a particular translate of $null\ T$, all the vectors in the translate when acted upon by T give you the same vector at the output, okay? So that is an important thing to see, okay? So this way the $null\ T$, null space of a linear map has translates. The translates cover the whole vector space V and every translate when operated upon by T goes to the same vector Tw, right? Nothing else. So that is an interesting picture to keep in mind. I will draw an exact picture for you soon enough but let us look at the properties first, okay?

And if you have w_1 not equal to w_2, then Tw_1 is not equal to Tw_2, that's also true, right? Because 2 different things don't get mapped to the same thing, okay? So you can look at why this is true also. If you want, you can write down a separate proof for this, it is not very hard. We have seen this before, okay? So this is very very interesting from a point of view of a linear map. Let me draw the picture here. Linear map T maps the quotient space to the range in a one-to-one manner. Let me draw a picture for you to illustrate what is going on here. So I have V. So this is a powerful picture to remember about a linear map T. I have V and there is a linear map which takes me to the range of T, okay? So this is range T and this linear map $T: V \to W$, ok? The entire thing comes to this range, okay? There are lots of points in the range, I just draw them like that. Several points in the range, okay? Now let us say there is a null which is... I will draw like this. It's okay, this let's say is $null\ T$. What am I going to do? I am going to do a quotienting of V with the null, okay? So I am going to define all these translates of the null space and notice what's going to happen here. The translates will look sort of like this. So they will partition V. So what is the meaning of partitioning? So they're going to start looking like this. So maybe you will have a translate which is on top like this. There will be another translate like this, translate like this, right? So I do not know how it will look. But they will partition V, okay? So this is an important thing to remember. What is partitioning? They will divide up V, carve up V into disjoint parts. So this is how the translates would look like. So these are all translates of $w_1\ +\ null\ T$. This could be, you know, some $w_2\ +\ null\ T$ etc. like that, okay? Remember there will be an infinite number of translates, okay? So every space here as infinite number. It was all real, right? So there will be infinite number of translates. All of that is true. But this is how the picture would look, okay? So keep this picture in mind.

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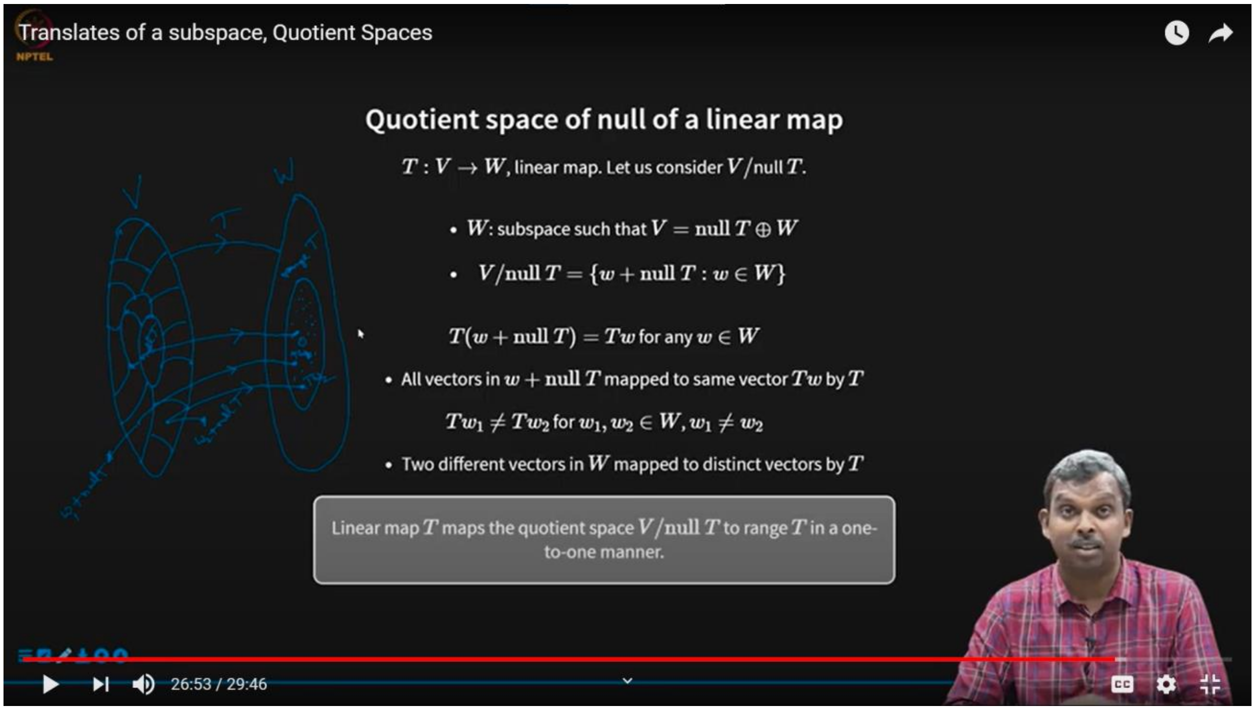

So now what is going to happen now? When T operates on any translate, okay, for instance there is this 0 here. When T operates on the null space, what happens? It takes every vector in the null space to 0, okay? To just one point in the range. Now there will be a point here which is Tw_1, okay? When T operates on $w_1 + null\ space$, every point in the translate $w_1 + null\ space$ of T gets mapped into Tw. When T operates on $w_2 + null\ space$, every point in $w_2 + null\ space$ gets mapped to Tw_2, okay? So if you look at the map T as mapping from the quotient space of $V/null\ T$ to the $range\ T$, you actually have a one-to-one map, okay? So this is what's going on when you define a linear map. When you have a linear map that's defined in whatever way you like, through a matrix or something, what immediately happens is a range gets defined, a null space gets defined, the translates of null space partition V and every translate goes one-to-one to a single point in the range of T, okay? So this is the structural result. This is the picture that you should have in mind when somebody says a linear map, okay? Every linear map will look like this, okay? You may have different cases. The null space might be trivial in which case, you know, the translates are all individual points only. There is nothing, no set there, right? Null space is trivial. So every translate is just a point. And then every point goes one-to-one to the range, right? Because it becomes an injective map, okay? And the null space could be a bigger set in which case the translates are actually infinite. Each of the translates have a lot of points, but all the points in every translate will go into only one point in the range, okay? So this picture of a linear map is very crucial as we go forward and study more intricate properties of linear maps. Linear map takes V to the range of T, okay? And takes null space to zero. Every translate of null space goes to a unique point in the range of T and you can find the points in the translates very easily using this, you know, complementary sort of subspace to T which directs sums to give you the whole V. And they are all tied up in this very nice fashion, okay? So this picture of a linear map, don't forget it, keep it in your mind. It will help you a lot in understanding so many more results that we'll see going forward, okay?

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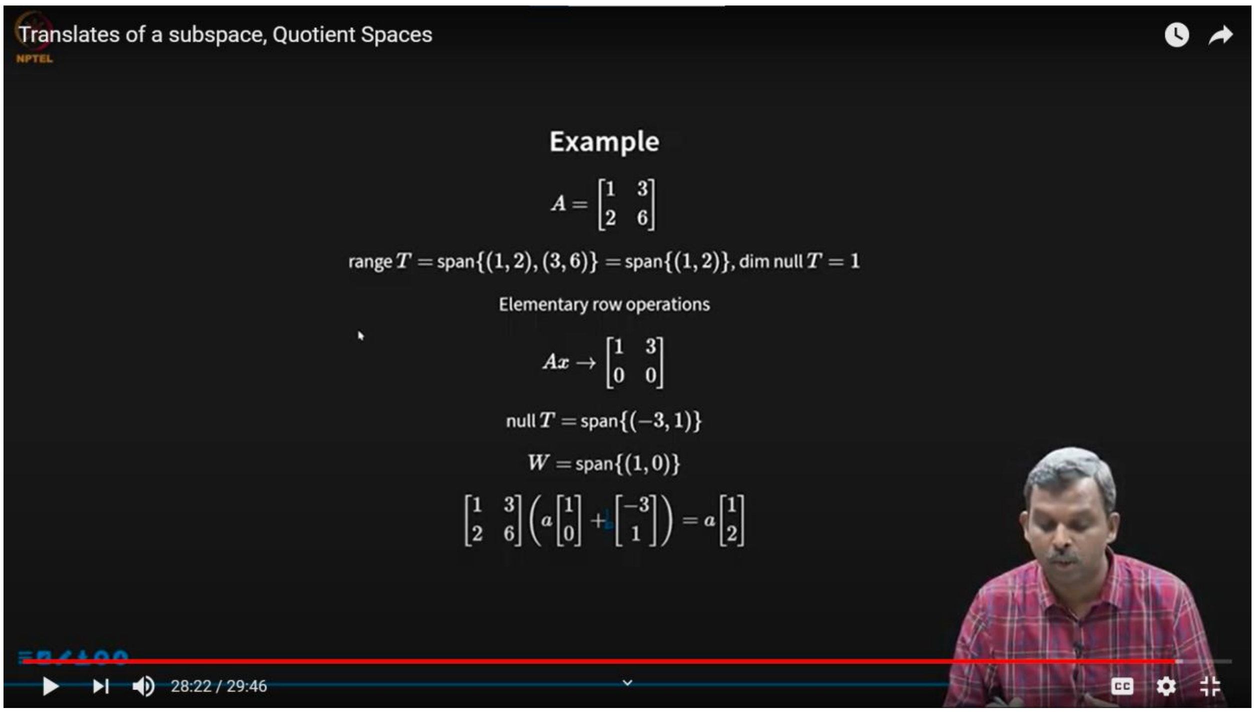

So I have a little bit of an illustration for you to show how this result actually works out in practice. So maybe you can think about it. Here is a very simple example. $[(1;\ 2)\ (3;\ 6)]$. I'll go through this example very fast. If you look at the range of T, span $(1, 2)$, dimension of $null\ T$ is 1. You can do elementary row operations to simplify this. So you quickly identify that the null is span of $(-3, 1)$. So W is span of $(1, 0)$. Just take the x-axis as W, okay? And then all that this result says is the matrix multiplied by any element in the span plus the null space, any element in w plus some vector in the null space will give you A times $(1, 2)$. So this is sort of the result which is the, you know, the translates are giving you. I mean this picture sort of depicts what is meant by this. So it's not actually a very complicated result. But at the same time you can see how this works out, right? So you translate the null space. Well I've put $(-3, 1)$ here just to show you. You can put a b here also, right? So you can put for instance, you can put a b here. So b can be an arbitrary thing here, to illustrate that, you know, whatever vector you add here from the null space, whatever translation you do, you always go to $a(1,2)$. So this is the unique map from W to the range that you have, okay? So this is a quick example which illustrates this, okay?

So let me summarize how this will help you in terms of solving linear equations. What is the connection between this structural result and linear equations? We have this wonderful structural result that the quotient space $V/null\ T$ is a one-to-one map under T, or a version of T if you think about it, to $range\ T$, okay? So $Ax = b$ has this very simple characterization. If $x \notin range\ A$, there is no solution. If $x \in range\ A$, you'll find one u such that $Au = b$, and the solution becomes $u + null\ A$, okay? So this is a good point to stop for this week.

So this week we've seen a nice application of all this theory that we built. How to solve linear equations, how to do elementary row operations and this crucial interesting structural result about how a linear map really works in terms of quotienting with the null space and having this one-to-one map between the translates of the null space and individual points in the range of the transform, okay? Hope this week was interesting. And we'll proceed further and look a bit closer at linear maps going forward in the next week. Thank you.

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