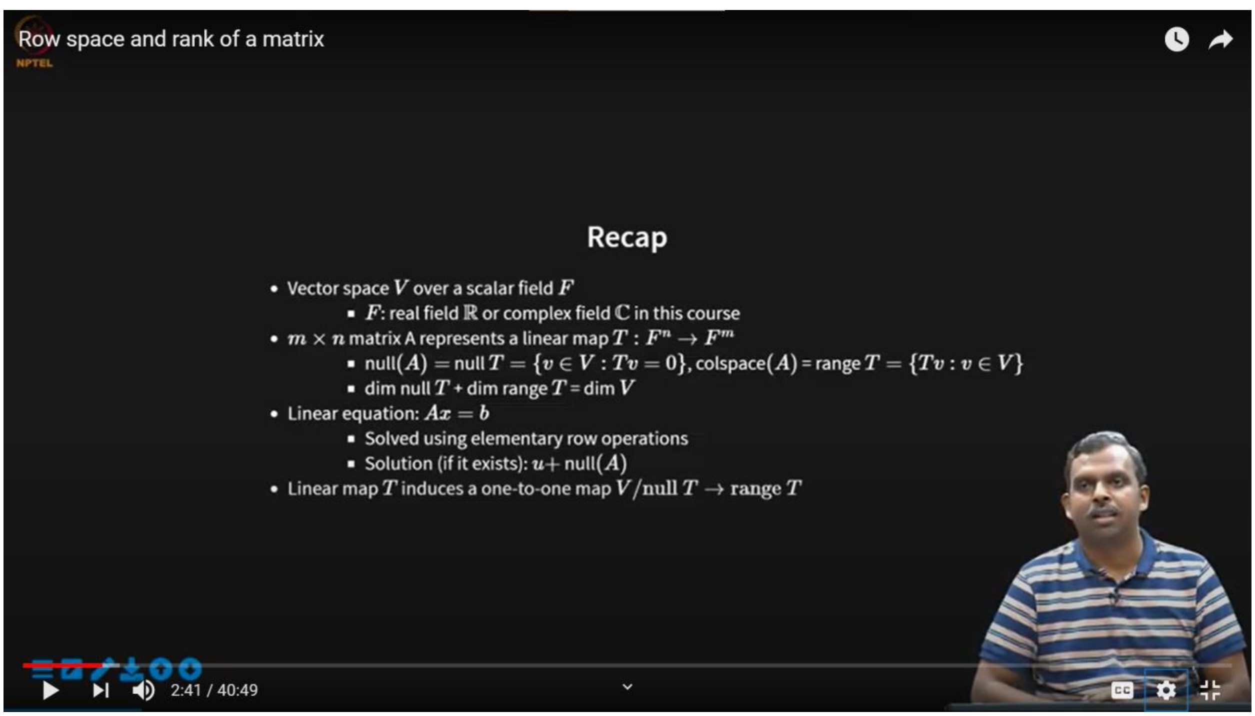
Applied Linear Algebra

Prof. Andrew Thangaraj
Department of Electrical Engineering
Indian Institute of Technology, Madras

Week 04
Row space and rank of a matrix

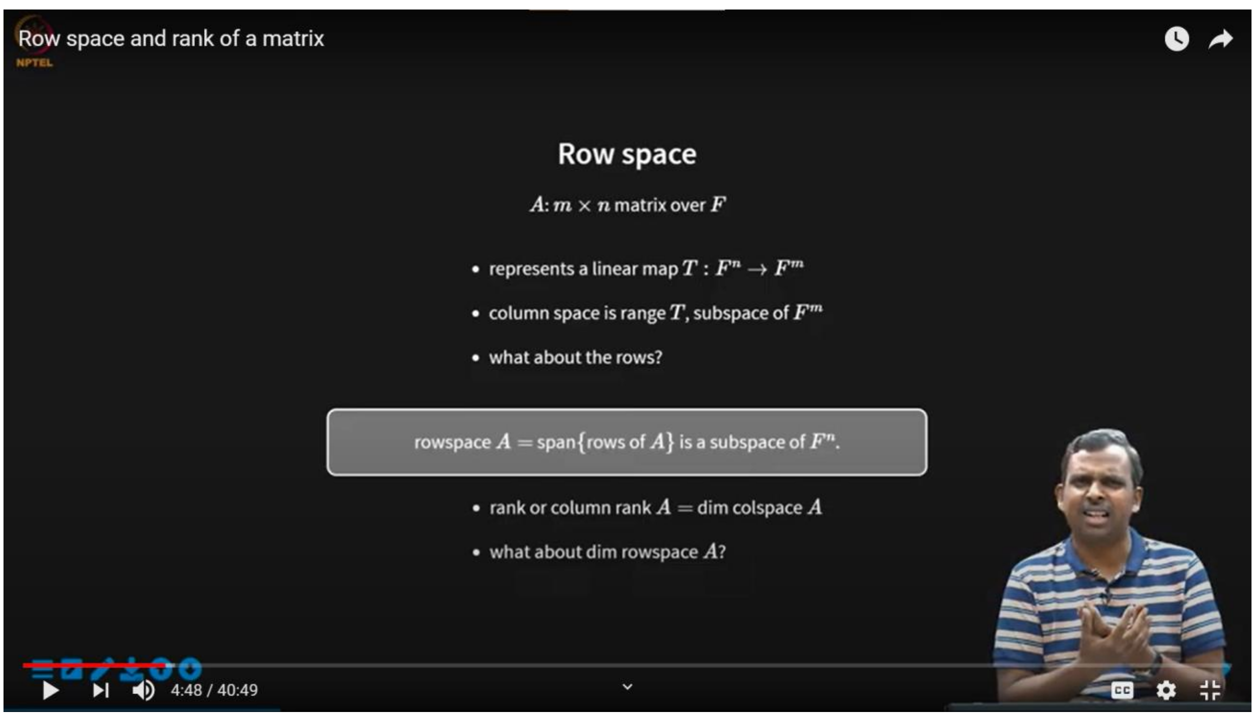
Hello. Welcome to week 4 of lectures in Applied Linear Algebra. This week we're going to sort of consolidate all that we learnt about, you know, linear maps, matrices, row space, column space and all that. I mean actually row space we didn't really learn, but we will look at what row space is. We've been looking at column space a lot. What is so sacred about the column? We might as well look at the row, right? So we'll look at things like that. Then we'll look at, I mean tie up as many things as possible together. I'll point out a few important properties that you should know. Few things that happen because you arrange numbers in the form of a matrix. Some interesting properties follow because of that. So it's sort of like a consolidation of some important properties for linear maps and their connection to matrices. So rank will play an important role as well. So we'll talk about rank quite a bit, discuss it in some more detail than what we did in the previous lectures. So let us get started.

(Refer Slide Time: 02:41)



Okay, quick recap. We are talking about vector spaces over the real field and complex field. We have this association between linear map $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$ and an $m \times n$ matrix A. We associate the null space of this matrix with the null of the linear map itself. We have the column space of the matrix A associated with the range of the linear map. And then we have this wonderful theorem which connects the dimensions of these two things. And then we looked at how linear equations can be solved using ideas from linear maps and, you know, null and all that. And we saw that the solution is a translate of the null space. And there is this nice little structural property for linear maps. How do you visualize a linear map? You have this partitioning of the vector space into translates of the null space and every partition, every translate goes to one point in the range. And that's how every point in the range is covered. It's a one-to-one map and that's how every linear map looks. So that's a nice picture to retain in your head, okay? So now let's look at this connection a little bit more, firm up some more properties. Rank and, you know, what the matrix is, what is row, what is column, you know? We will also look at this transpose operation which goes from row to column and what are the various results around these kinds of things. Now we will also extend these elementary row operations to elementary column operations and then get to a much more reduced form and look at what happens because of that, okay? So that is going to be our lecture, the first lecture for week 4, okay?

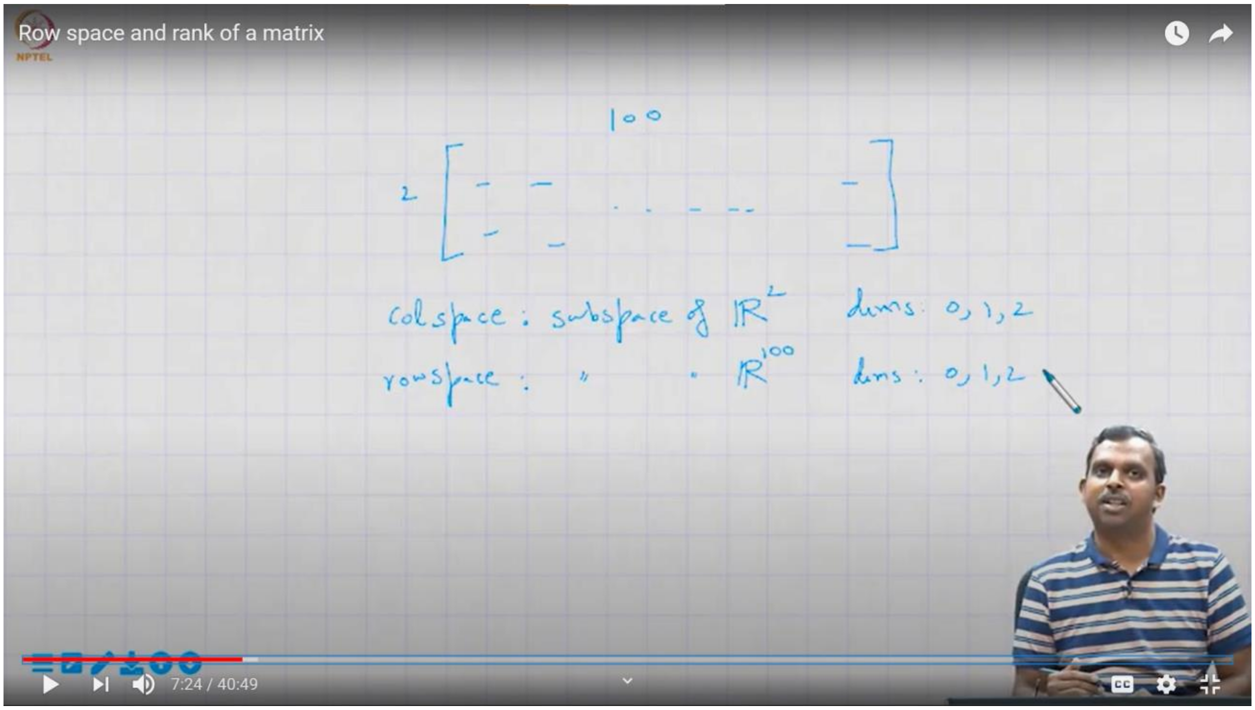
(Refer Slide Time: 04:48)



So first let me begin with row space. We have seen this matrix, $m \times n$ matrix over the field $\mathbb{F}$. We know that it represents a linear map from $\mathbb{F}^n \rightarrow \mathbb{F}^m$. We have looked at the column space and how the column space corresponded to the range of T. So that's the convention we took, right? We took a vector and multiplied the matrix with the vector on the right, so the column space naturally became the range of the linear map in the way we associated, you know, list of coordinates with the vector, okay? So that's what we did. What about the rows, okay? So it turns out the rows also are elements of $\mathbb{F}^n$, right? Every row in an $m \times n$ matrix comes from $\mathbb{F}^n$, okay? It is an n coordinate vector with elements from F. So naturally you can associate a vector with every row. So you can also think of the span of all the row vectors, right? So there are m row vectors and together they span a subspace of $\mathbb{F}^n$. And that subspace one can call as row space, okay? Just like columns span the column space, the rows span the row space, okay? So it's easy enough to imagine what it is, okay? So we saw, we defined the notion of column rank or the rank of the matrix itself. I referred to it as the rank itself, it became the dimension of the column space of A, okay? What about the row space, okay? What about its dimension? Of course you can call it the row rank, right? The dimension of the row space can be called the row rank. But is there a connection between the row rank and the column rank? Should there be any connection at all? It looks, on the face of it, it looks like, you know, the rows are one below the other, the columns are this way... But there is, like, a very intricate connection between the two, you know? I mean the entries in the rows and entries and the columns are exactly the same, except that you think of them differently, right? So is there going to be a connection between the dimension of the row space, dimension of the column space? First of all the row space and column space are not very directly connected, right? Row space is a subspace of $\mathbb{F}^n$, right? While column space is a subspace of $\mathbb{F}^m$, okay? There is no real... I mean the m and n can be very different. m can be 20, n can be 20000, right? So these two are not really, they don't look similar. But is there any connection between these two spaces? Particularly in terms of dimensions, is there, can you arrange it so that, you know, the dimensions can be different or not etc.? And that can be very nicely answered and that's the next result that I'm going to put out.

It turns out for any matrix $m \times n$, row rank will be equal to the column rank, okay? So you cannot put, I cannot create an $m \times n$ matrix for which the dimension of the row space will be different from the dimension of the column space, okay? So for the first... I mean if you're surprised by this result, there's many, very many ways to think about it. Here is one little way in which I think about it quite often. I think of, say, you know, this is an example I take to justify to my own head why this sort of makes sense. So let us say we take a 2×100 matrix, okay? So this is a matrix I take, I start putting entries in it, you know? I have so many entries, okay? So if you look at the column space, okay, so it ends up being a subspace of $\mathbb{R}^2$, okay? So what are the possible dimensions? Dimensions could be 0, 1 or 2, isn't it? Now notice what happens when I look at the row space. Even though the row space is a subspace of $\mathbb{R}^{100}$, okay? You start thinking that the dimension could be you know up to 50, 60 and all that, all that is not possible, right? Why? Because the number of rows, the number of vectors in my spanning set itself is only two, okay? So the dimension in spite of all this could only be 0, 1, 2, right? It can't be whatever I want, right? So it's, so you can see that even though the sizes are different, because of the way the number of, you know, number of columns… While number of columns becomes the dimension, the number of rows limits the dimension of the column space. Likewise, you know, the number of rows also ends up limiting the dimension of the row space, okay? So there is this, in this example at least you see that the least number is two and that imposes a constraint, right?

(Refer Slide Time: 07:24)



So even if you go the tall column wise, then the number of columns will start imposing some such constraint. So it's sort of natural to expect that the row space and column space dimension wise have to be tied up a little bit. And I guess perhaps the fact that they have to be equal is part of a strong tie up and it's not maybe what you expect naturally. Maybe, you know, you can say within each other or something, but they're exactly the same, right? You can't do anything beyond that. So this is a popular result called row rank equals column rank. You'll see a lot of pages on it if you search about it. Lots of ways to prove it. I'm going to give you a sort of a simple proof which may be a bit surprising. At least in this course we have not seen this, okay?

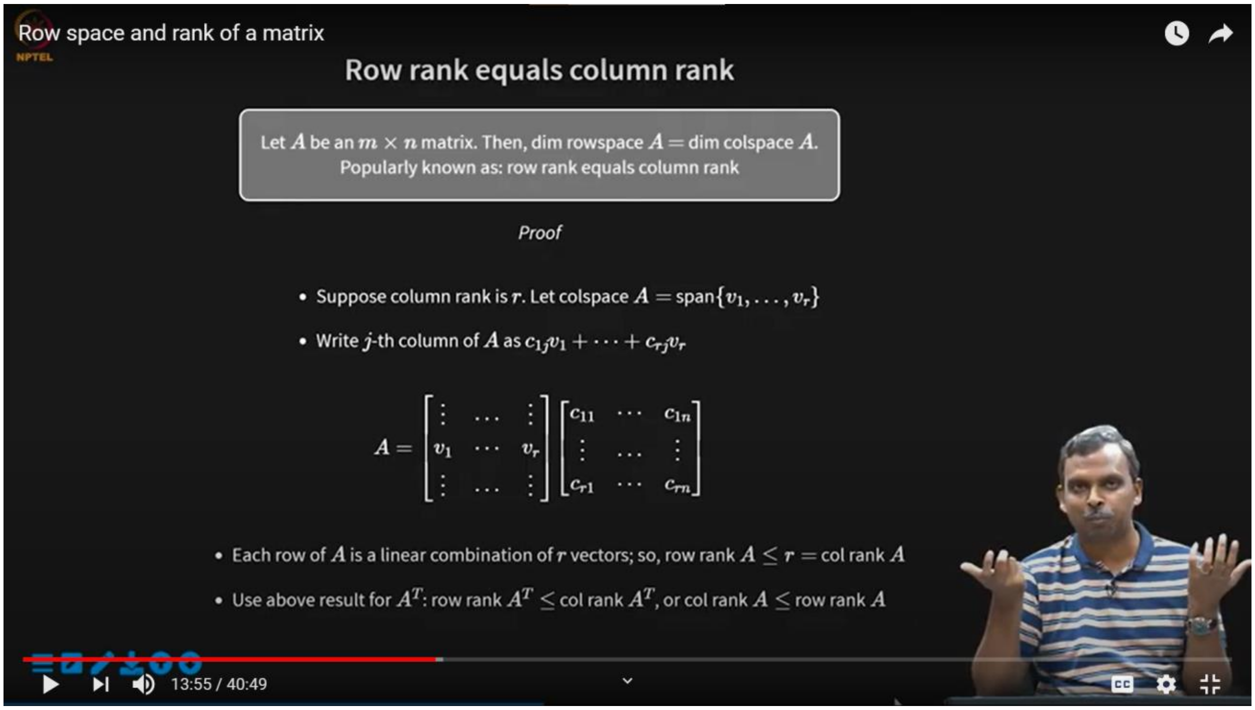
So what's the proof going to be? So I'll start by assuming... So the overall principle of the proof is - I want to show row rank equals column rank. So what I will show is that row rank is less than or equal to column rank, okay? All right. So that's what I'll show first. So how do you show row rank equals column rank? One approach is to show row rank is less than or equal to column rank and then also show column rank is less than or equal to row rank. If I do both, then I know that the two have to be equal. But in this case it turns out row rank less than or equal to column rank itself is enough, okay? See for an arbitrary matrix, $m \times n$ matrix, I'm showing row rank is less than or equal to column rank, okay? So it turns out that itself is enough to make it equal. You will see there is this little trickery you can do. You'll see when it comes. It may be a bit surprising, but it's an easy enough trick that you can see through. So all I have to do, it turns out, is to show that row rank is less than or equal to column rank, okay? So once I do that, it turns out I'm done, okay? So for an arbitrary matrix, if I show row rank is less than or equal to column rank, I have also shown that row rank has to be equal to column rank. You'll see why that comes out. Maybe it's a bit surprising but let me keep the suspense for a little while.

So let's begin by showing row rank is less than or equal to column rank, okay? So for that I'll assume the column rank is r, okay? Some number r, right? It should be between, you know, 0 and m, 0 and n also. So we will see later on how these things are related, but okay let us say column rank is r, okay? Which means the column space of A, right, has to be in the span of r linearly independent vectors v_1 to v_r, right? So that's the meaning of dimension, isn't it? So dimension of a space is the number of vectors in the basis. Number of vectors in the basis which is also a spanning set. So the column space A will be equal to span of r vectors, right? So in particular every column of A belongs to the column space of A, right? So I can write every column of A as a linear combination of v_1 to v_r. And that's the same thing as saying that rank is r, column space has dimension is r, everything is the same, right? So the jth column of A, I could write it as $c_{1j}v_1 + \cdots + c_{rj}v_r$, okay? So $c_{1j}, \ldots, c_{rj}$ and all are scalars coming from the field. So they scale each of these vectors and you make a linear combination, you get the jth column of this matrix A, okay? So I can do this for every column, okay? So now this tells me I can rewrite A in the following form. What form? Notice what I am doing here. I am writing A as the product of two matrices. What is the first matrix? First matrix has r columns and each of the columns are the vectors in the spanning set, okay? So v_1 is the first column, v_2 is the second column, so on till v_r that's the last column. That's how I make my first matrix. The second matrix is basically composed of these coefficients I got to form the columns of A. So notice what I am doing with the first column of the second matrix. It is c_{11} all the way down to c_{r1}. So when I multiply, when I multiply the first matrix with the second matrix, the first column of the product will exactly be the first column of A, the second column of the product will be the second column of A and so on, right? So I can go on up to the n^{th} column which would be $c_{1n}, \ldots, c_{rn}$. That multiplying v_1 to v_r will give me the n^{th} column of A. So this is a proper product. So this whole thing I've rewritten in this fashion, okay?

So once you write like this, you notice that every row of A now... So we've been looking at it column-wise. Now switch around and look at what's happening in this product row-wise, okay? So I know that every row of the product is a linear combination of the rows of the second matrix, right? In a product of two matrices, every column of the product is a linear combination of the columns of the first matrix, every row of the product is a linear combination of the rows of the second matrix, right? So now look at it that way. So every row of A becomes a linear combination of r vectors, okay? Isn't it? So every row of A is a linear combination of r vectors which means every vector in the row space is a linear combination of those same r vectors. Which means there is a spanning set of size r, okay? For the row space there is a spanning set of size r which means the dimension of the row space is less than or equal to r, row rank is less than or equal to r, okay?

So hopefully you follow the proof. It's a simple enough proof. So what I have shown here, right, is that the row rank of an arbitrary matrix A is less than or equal to column rank of A.

(Refer Slide Time: 13:55)



So now comes the suspense. The suspense is already out in the last bullet there. So once you show it for an arbitrary matrix, that's enough because you can now look at A^T. What is A^T? You must be familiar with this operation. You just make all the rows as columns and columns as rows, right? You flip it around and you'll get an nxm matrix. Now for that matrix, rows and columns are interchanged, right? So row space becomes column space, column space becomes row space. Now for that A^T if I use the same result that row rank is less than or equal to column rank, I'll end up getting column rank of A is less than or equal to row rank of A because you know row rank of A^T is less than or equal to column rank of A^T, okay? Using this simple little transpose trick, once you show row rank A is less than or equal to column rank A, in fact it turns out column rank A is also less than or equal to row rank A and both have to be exactly equal, okay? So this is the little trick here that, you know... So you can see why the transpose plays an important role. So the rows and columns are not really, you know, sacred in any one way. You can call the columns as rows and rows as columns. It is just a convention in some sense, right? So when you can flip it around, if you can show one is less than or equal to the other, that is good enough, okay? Both have to be equal. All right. So this is the little proof that, you know, row rank of a matrix has to be equal to the column rank. So the dimension of the row space, dimension of the column space is the same. So that is why we are justified when we say rank of a matrix. You do not have to say column rank of a matrix, row rank of a matrix. Both have to be equal, okay? So that's a nice result to know.

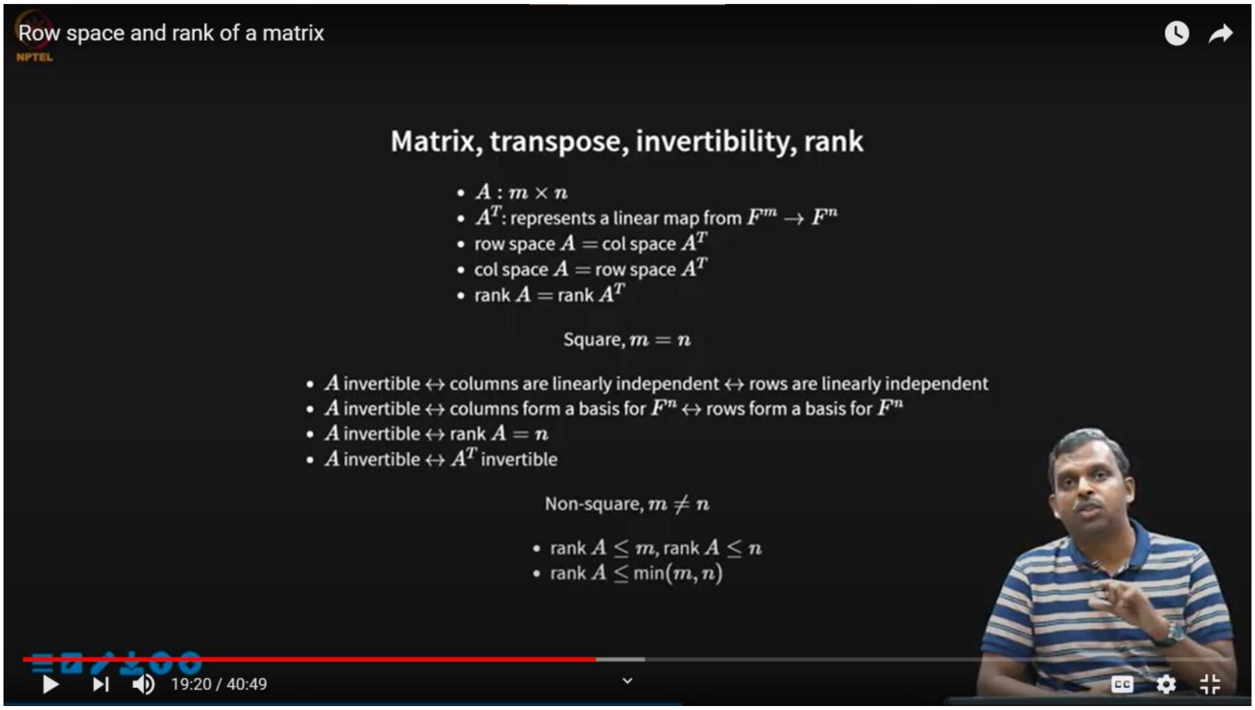
Okay, so let's just sort of assimilate all these things. A whole bunch of results in this area of matrix transpose, invertibility, rank... So it's good to know these results and know to be... I mean you should be able to recollect them very quickly so that it's useful to, you know, use it when you have to, in some applications or something you should be able to quickly recollect. So let's just put some results together, okay? So you have this $m \times n$ matrix A. A represents a linear transform from $\mathbb{F}^n$ to $\mathbb{F}^m$, right? Now A^T, if you switch the rows and columns okay, represents a linear map from $\mathbb{F}^m$ to $\mathbb{F}^n$, right? The other way around, okay? So is there a connection between these two linear maps? It turns out there is, but maybe not immediately apparent. We will think about it. I mean later on. So you see for instance, if I have to draw a little picture here... So I have $\mathbb{F}^n$ and then I have $\mathbb{F}^m$. A represents a transform from here to here, right? And A^T represents a linear map, a linear transform from $\mathbb{F}^m$ to $\mathbb{F}^n$. Is there a connection between these two? Is there a connection is a natural question, right? So maybe right now it’s not quite apparent what that connection is and later on we'll need some more study to figure out what that firm connection is. It turns out there's lots of interesting things you can say about A, A^T, how they work together.

So for instance, if you can imagine this... I take a vector. With A, I go to $\mathbb{F}^m$. And with A^T I can come back, right? So what has happened overall in the round trip, right? So you can think about those kinds of problems later on, these are slightly advanced questions. We'll ask these questions later on in the course and, you know, we'll study those things a little bit more in detail. But for now it's just a linear map. It represents some linear map from $\mathbb{F}^m$ to $\mathbb{F}^n$ and already there's lots of tight properties between these two things, right? The row space of A is equal to the column space of A^T, the column space of A is equal to the row space of A^T and the rank is the same for both A and A^T, right? The spaces are all the same, they have all the same dimensions... So, well, spaces are not all the same, you know what I mean. But the dimensions of the spaces are all the same so you have the same rank, okay? So that's good to know.

So what happens in particular when the matrix is square, okay? So when m is equal to n, you can sort of talk about invertibility in that regime, right? And what happens to invertibility? So you can see, if A is invertible, the column space column rank, rank has to be full, right? Rank has to be n which means the columns are linearly independent. And together they have to span the whole $\mathbb{F}^n$. So columns become the basis. Rank of A equals n. A is invertible, rows are linearly independent, rows become a basis, rows you know linearly independent... Notice what's happening. So think of this result. Supposing somebody asks you to construct a matrix who's a square matrix. The columns have to be linearly independent but the rows have to be linearly dependent. Can you do that, okay? A square matrix where the columns have to be linearly independent but the rows have to be linearly dependent. Is that possible? It's not possible, right? You know that's not possible because the rank has to be the same. So just by ensuring linear independence on the rows, on the columns for a square matrix, you also enforce the linear independence for the row, okay? So these are a sort of coupling that you see between these rows and columns. So when you put an entry, that entry goes into both the row and the column. So it's sort of, you know, putting constraints on one implies constraints on the other, okay? So this is nice to know. So A and A^T you see in the square case, they are tied up in the invertibility also, okay?

What happens when it is non-square? So here you can say some inequalities. You know that the rank of A has to be upper bounded by the number of rows as well as the number of columns, isn't it? So if you think about it, both the dimension of the row space and the column space enter the picture. So the number of rows as well as the number of columns is an upper bound on the rank. So the rank is always upper bounded by the least of the two dimensions, okay? So that's a good thing to remember. But one can't talk about invertibility and all that. We'll say a little bit more about null spaces and all that going forward. Now the same thing holds with rank A^T, right? Even when it's non-square, rank A or rank A^T, they're both the same so they are upper bounded by the least of the two dimensions, okay? So this is a good thing to remember, just some sort of various facts. I mean, most of them are clear but it's good to recall these things or remember these things sort of by heart to know what will happen, okay? So that's a brief thing about this.

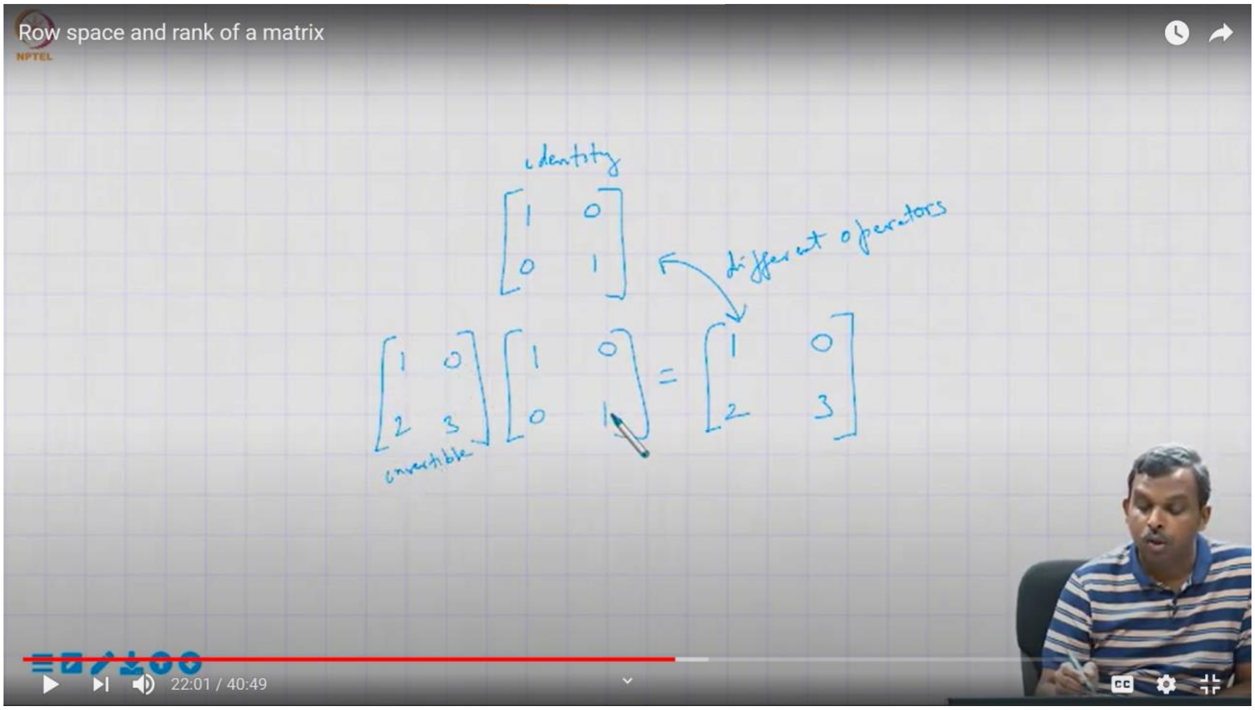
(Refer Slide Time: 19:20)



Okay. So here is another interesting little result, okay? So notice what is happening here. So you have a linear map $T: V \to W$, okay? Now supposing, okay... So it is good to draw a picture, I think a picture would help, okay? Supposing I have V here and then a W here, and then I have a linear map T going from here to here. And let us say I have an invertible map going from V to V, which I call S and an invertible map going from W to W which I call U, okay? Both of these are invertible, okay? S and U are invertable. So now notice what happens when you do the composition TS, okay?

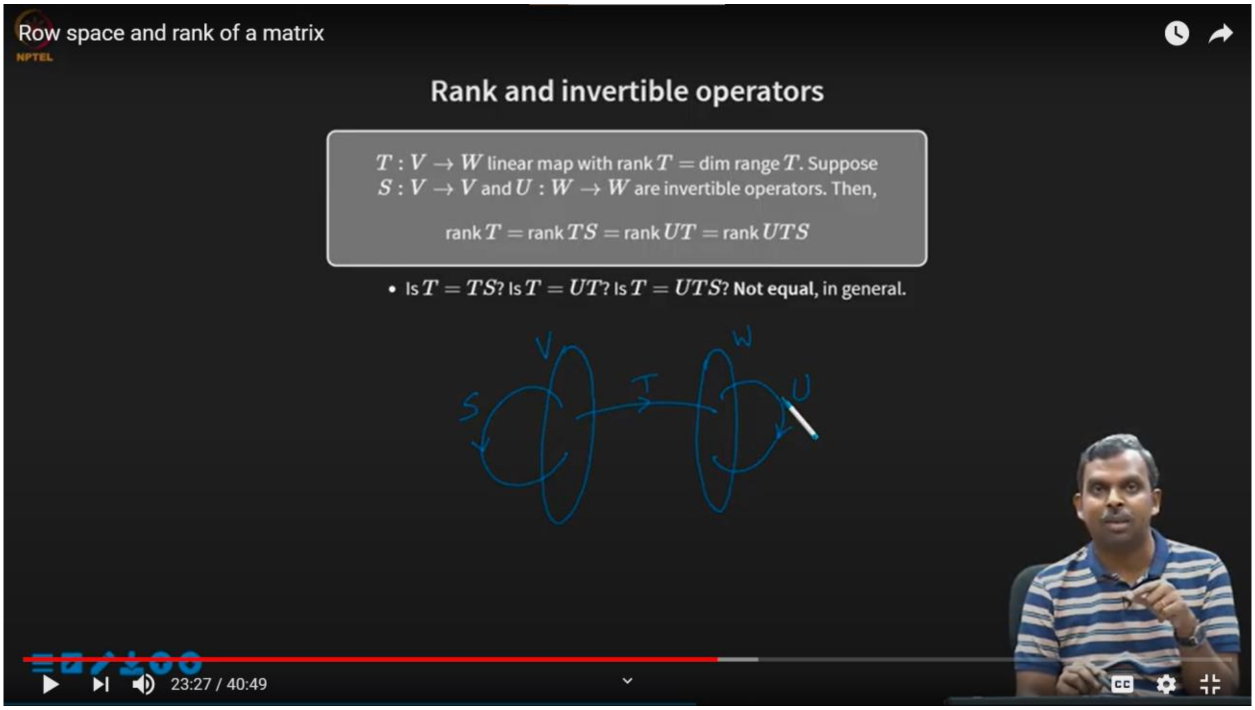
When you compose TS you take a vector, you apply S first. Notice when you apply S first, you're not really doing anything, any change, you know? I mean it's an invertible thing. Whatever you got here you can go back also, right? And then you apply T, okay? So nothing should really change, right? When I say nothing should really change, you should be careful with what changes. The operator changes, that's why I put the bullet below, okay? Is $T = TS$? No. The operator changes definitely, okay? Maybe I should give you an example, a little bit to emphasize what I mean by this, okay? So let us take a simple example. So let us say I put a 2×2 case, okay? So just the identity operator, okay? So this is just identity, right? Notice what happens when I multiply on the left with, say, an upper triangular or a lower triangular matrix $[(1;\ 2)\ (0;\ 3)]$. Let's say I do this, okay? This becomes, you know how to do this multiplication, right? So you get this. Now this is a different operator, right? So these two are different clearly, right? So it's sort of a trivial example, but just to drive home the point, when you multiply or when you compose with another invertible operator, this is invertible, you do not get the same operator. The operator is different. Operator has changed.

(Refer Slide Time: 22:01)



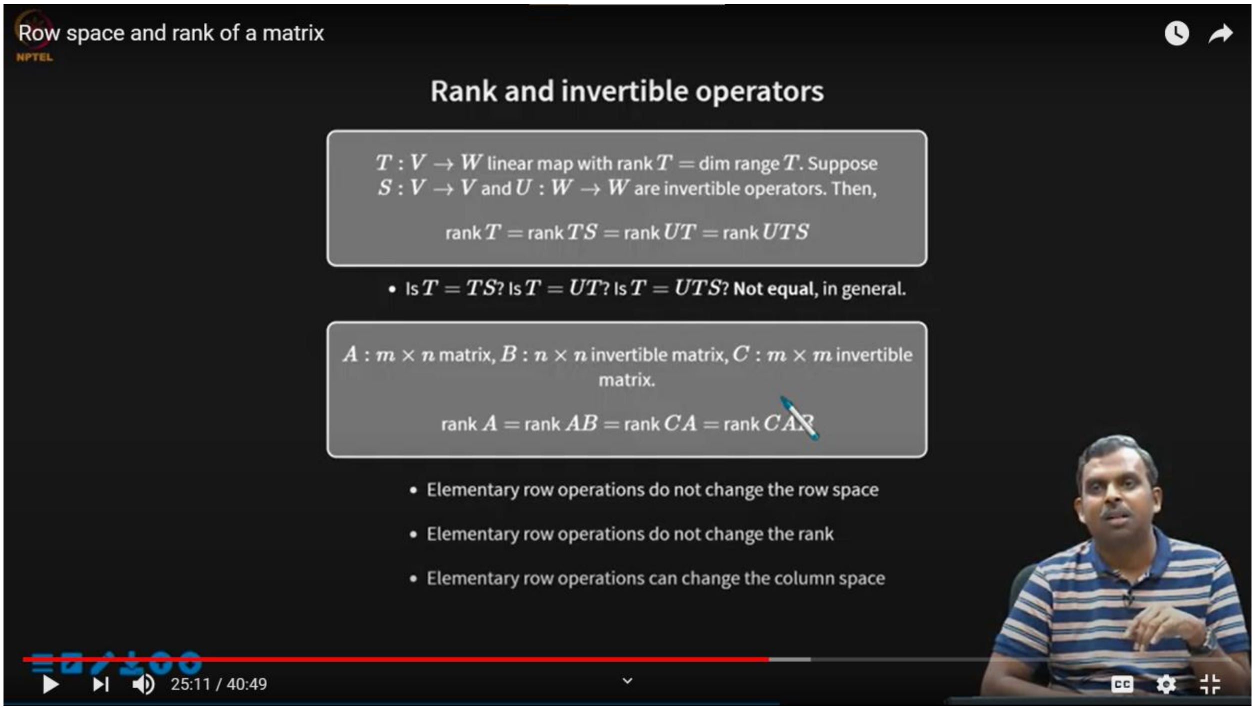
This one is identity, this is clearly not identity, right? So the operator will change when you compose with S. Or even UT, you know? You go to W and then you compose on the other side also. Even there the operator changes, okay? Or you could compose on both sides. UTS. The operator changes, okay? But because U and S are invertible, whatever you did with S you can sort of undo. So the dimensions won't change. Dimensions for the range, dimensions for the null and all won't change, right? How do you quickly prove this? Think about how you can quickly prove this, it is not very hard. If you look at TS, UT and all that, any vector in the range of T, right, will also be, you can also find, you know, the same vector or a corresponding version of it when you multiply with, compose with S and compose with U, right? You just do the inversion of that and then you can go back to this, okay? So think about what I meant by that. You can write down a quick proof for these things, vector to vector you can do a one-to-one map between the range of TS and the range of T, range of UT and range of T, range of UTS and range of T. Once you do that one-to-one map, you know that the range has the same dimension. It will have the same dimension with or without the multiplication, okay? And also notice the fundamental theorem doesn't change, right? So you're still in the same V and you can still apply the same thing. So dimension of the null space won't change, but the spaces themselves will change, the range will change, but the dimension will not change, okay? So keep this in mind. So this is something very useful to know, okay?

(Refer Slide Time: 23:27)



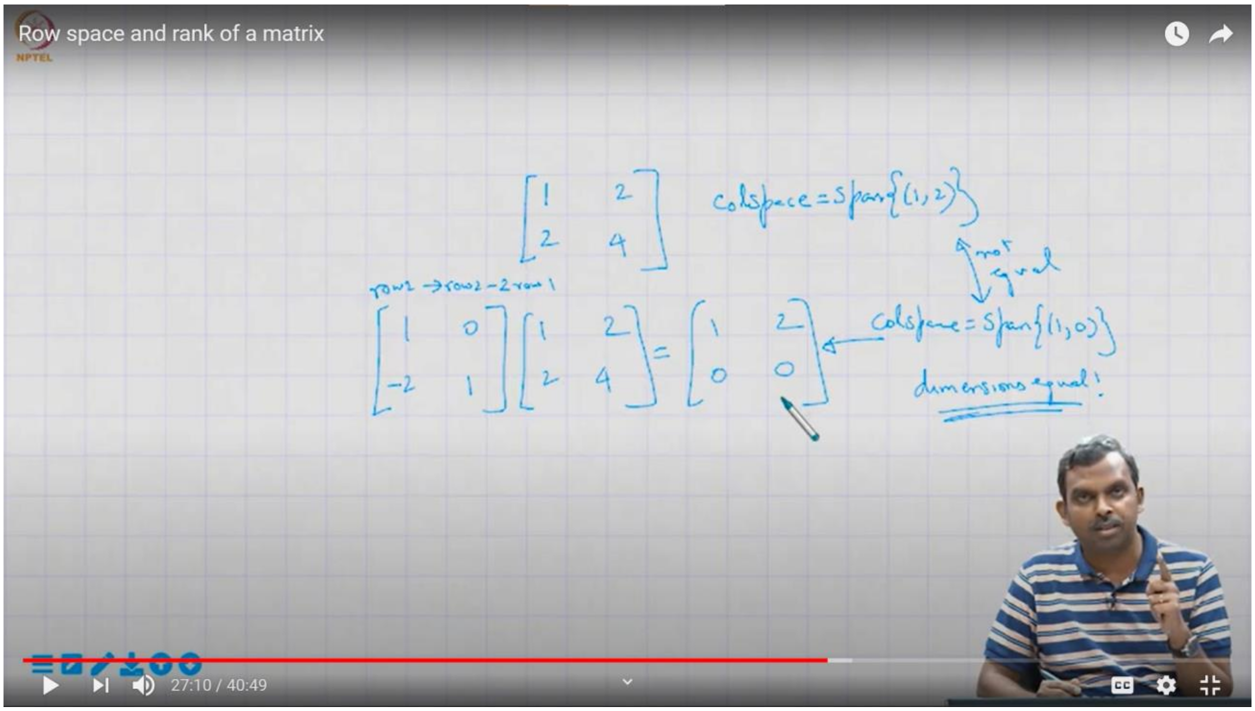
So now what is the corresponding result for matrices? In the matrices world, suddenly it might look all very different. So if you have A $m \times n$ matrix and B is an $n \times n$ invertible matrix and C is an $m \times m$ invertible matrix, then what we are saying from here is $rank\ A = rank\ AB = rank\ CA = rank\ of\ CAB$, okay? So because of the invertibility, rank doesn't change. Both of these are just analogous results, right? So you know that these are the same results. Whether you compose the linear map or multiply with the matrix, you are doing the same thing, okay? So these two are analogous results, okay? So this gives us a lot of interesting things to do, right? For instance we've been looking at a matrix and we've been doing elementary row operations, right? To change the matrix to a form that is suitable. So what we are showing with this result is elementary row operations do not change the row space, okay? So this may be not from this directly, but the second point - elementary row operations do not change the rank, okay? So think about why that is true. So why do elementary row operations not change the row space? Because they are invertible, okay? The row space is not changed when you do elementary row operations, that's sort of easy to see. But the rank also has not changed. We have seen why that is true here. But keep in mind - elementary row operations can change the column space, okay? The column space can be changed. I showed you a little example with the identity matrix. Maybe there it's not very clear, but you can see other examples where the column space changes. So maybe I should give you an example here for this. Why is it that elementary row operations can change the column spaces. It's easy to come up with an example. So you take an example where maybe you have, you know, [1 2; 2 4], okay? So what is the column space? Now the range of this linear operation, this is basically span of $(1, 2)$, isn't it? It's just the dimension 1. If you want, you can plot it in the x-y plane, it will be a certain line, right? A line that goes through.

(Refer Slide Time: 25:11)



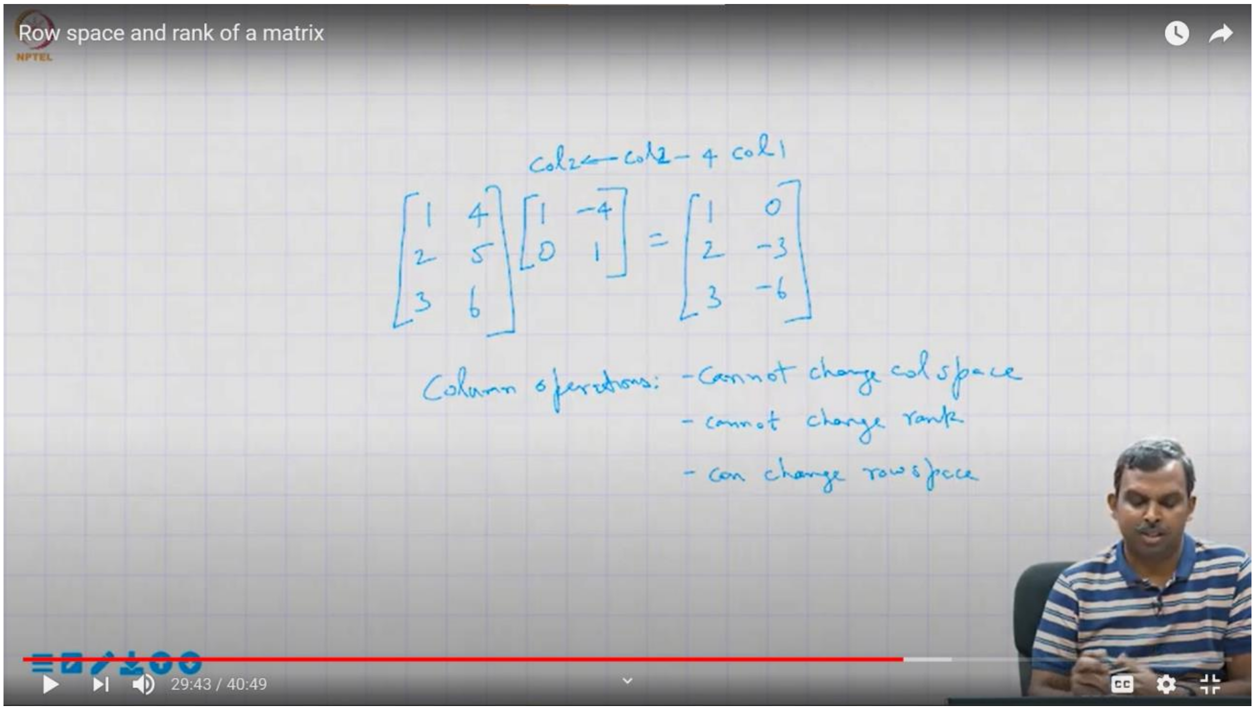
Supposing I do this elementary row operation. What is the elementary row operation I can do here? I can retain the first row, okay? I can do $row\ 2 = row\ 2 - 2(row\ 1)$, okay? So this is my standard elementary row operation. So $row\ 2 = row\ 2 - 2(row\ 1)$, okay? So that is the operation. $row\ 2 = row\ 2 - 2(row\ 1)$. If I do this, it is the same as multiplying like this. You know what I will get. [1 2; 0 0]. And what's the column space of this guy? It equals, you know, span of $(1, 0)$, okay? So clearly these two are not equal, okay? But the dimensions are the same, okay? So remember this. This is very important to keep in mind. So when I did an elementary row operation, I cannot change the row space. Row space will remain absolutely the same, I cannot change the row rank, I cannot change the column rank but I can change the column space by elementary row operations, okay? So something to remember. Keep that in mind. You can see clearly how it changes. A simple example, you can convince yourself that that is true, okay?

(Refer Slide Time: 27:10)



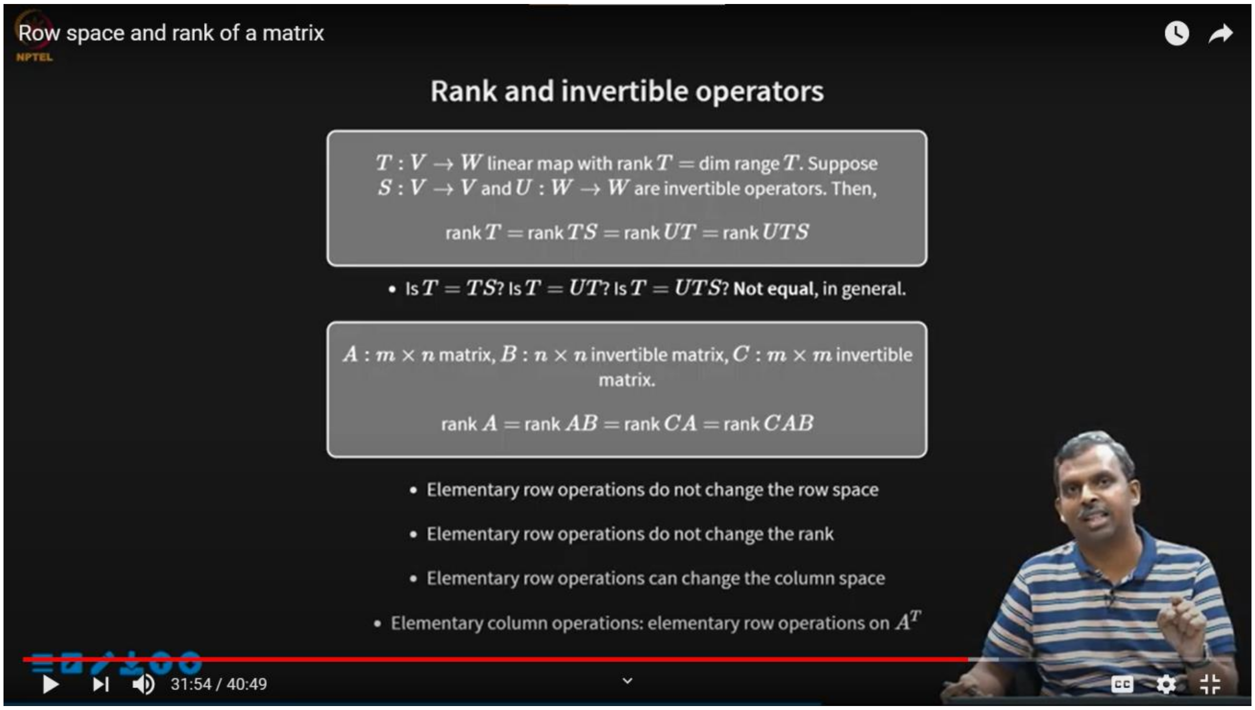
Now one can also do elementary column operations. I have so far not spoken about column operations. But now that we have A^T also in our midst, you can do elementary column operations. So how do you think of elementary column operations? Elementary row operations are multiplication by invertible matrices on the left. Elementary column operations will be multiplications by these elementary column operators on the right, okay? So you can do the same thing. When you do $column\ 2\ =\ column\ 2\ -\ 2(column\ 1)$, you're multiplying by a matrix on the right. In fact, the same matrix you put it on the right. Instead of operating on the rows, it will operate on the columns, okay? A very similar matrix. So maybe I should show you how that works, just a quick example. So supposing you have a matrix. Maybe I will take a slightly different example $[(1;\ 2;\ 3)\ (4;\ 5;\ 6)]$, okay? Supposing you want to do $column\ 2\ =\ column\ 2\ -\ 4(column\ 1)$, okay? You might do this. So this would be... So the first column I want to retain, okay? The second column I want to make -4(column 1) and 1 for column 2. Remember every column here multiplies the columns, right? So when I want to retain the first column, I put 1 0, when I want to multiply the first column by -4, I put a 1 here, I get this. So this would give you 1 2 3, here I would get -4. So this will become 0 -3 -6, okay? So this is what happens. So now what can column operations do? Column operations, okay, they cannot change column space, right? They cannot change rank, right? Row rank, column rank, whatever? But they can change row space, okay? This was maybe not a very good example to show you why they can change the row space, but they can change row space, okay? So just by the same argument I gave for the column space before, okay? So that is how column operations work. You can think of column operations as row operations on A^T. You know A and A^T are sort of cousins in so many different ways. But keep in mind that when you want to retain the column space, you should not do the column operations, it can change it, okay?

(Refer Slide Time: 29:43)



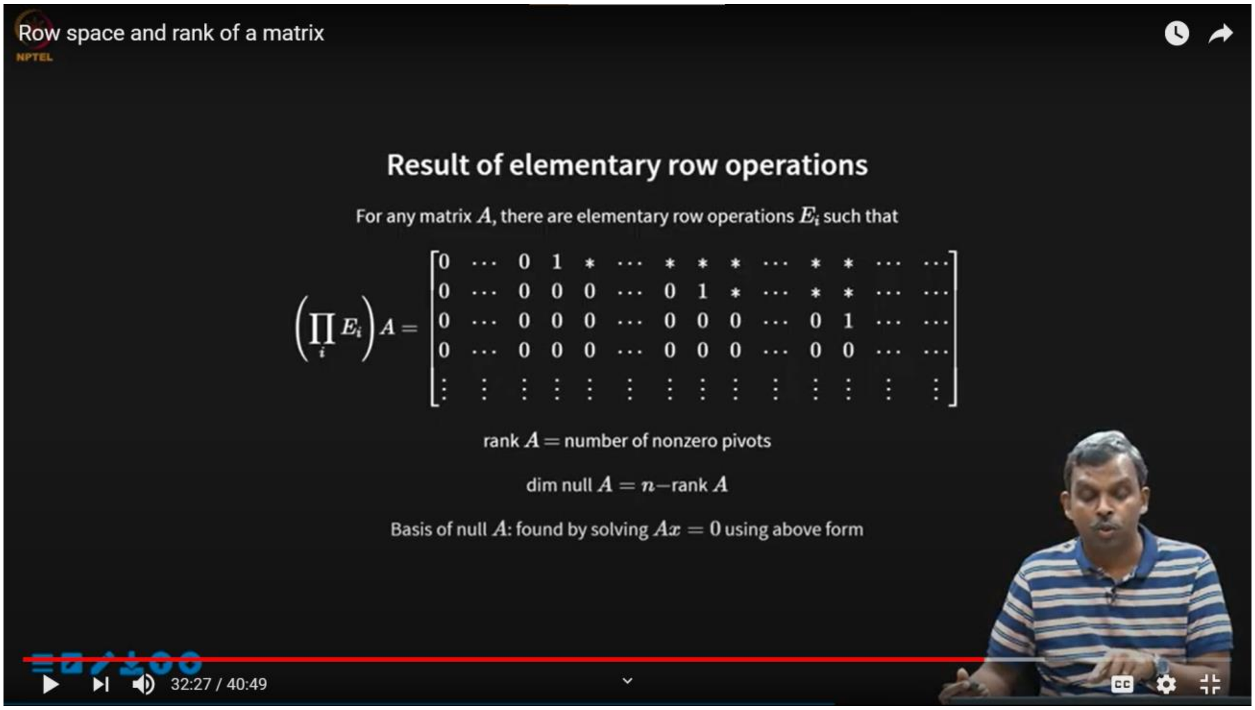
So now there are different types of column operations. Remember there was a column swap, right? Column swap doesn't quite change the column space, right? It just changes the ordering. But you know scaling and you know... So column okay? So I should be careful here. Column swap doesn't change the column space, column linear combinations does not change the column space, scaling of columns does not change column space. But you know, can the column swap change the row space? Absolutely it can, right? So think about how a column swap can change the row space, you know? I mean if you can have a column, you can have a matrix which is, you know, first row (0 1) and (1 0) and or, maybe $[0\ 1; 0\ 0]$, right? So that's a simple example to take. $[0\ 1; 0\ 0]$. If you swap the columns, it suddenly became (1 0), right? So the row space went from (0 1) to (1 0). So even a column swap can change the row space, okay? So the row space can get altered when you do column operations, okay? So these are things to remember. And now that we are talking about row operations and column operations, you can take a matrix, you can do invertible operations on the left, row operations to modify, make an upper triangular matrix. You can also do column swaps, you know, and change the way the matrix looks. And then you can also do row operations or in fact you can also do column operations if you like and make the matrix even simpler, okay?

(Refer Slide Time: 31:54)



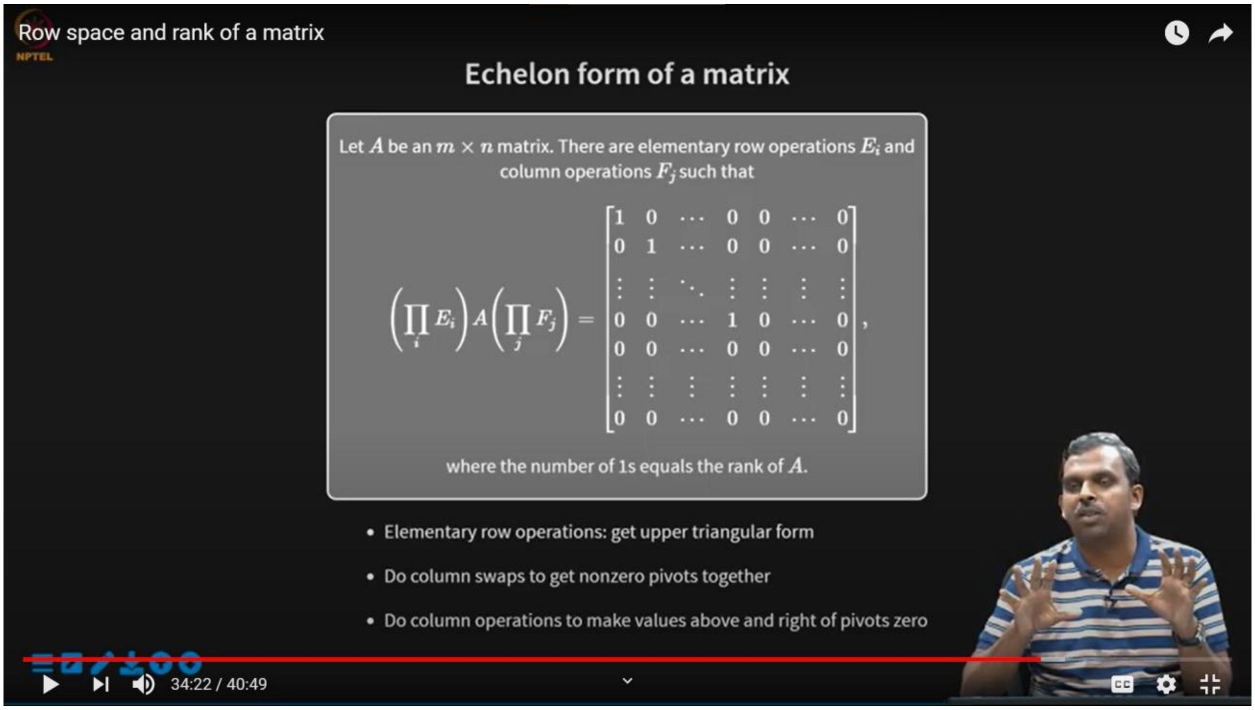
So let us go back and see how we used elementary row operations to simplify how the matrix looked, okay? So let me recall that result and then now I will add some column operations and you will see you can simplify the whole matrix even more, okay? So let's see how that is done, okay? So this result helps you in doing that. You know I can do both row and column operations and not affect the rank. If I only want to not affect the rank, I can do both, okay? So let's see how that is done. So first let us recall what we had from elementary row operations. When you are only doing elementary row operations you could get these all zero columns, right? Zero pivot rows. And that can happen between two non-zero pivot columns. You will get a non-zero pivot column and then you may get a bunch of zero pivot rows and then you will get a non-zero pivot, you know? You can have a structure like this, okay? So we saw this structure before. This is possible, okay? And this was very useful. The structure itself was very useful. Now if I can do column operations, you see I can further simplify this, okay? You can get a very surprisingly simplified form. So it turns out when you do only row operations, you can only do so much to the matrix, right? So you can get it to the upper triangular form and then there can be, you know, values above the pivot and to the right of the pivot which could be non-zero. In fact you can do a little bit more and make even values above the pivots go to zero. But let’s not worry about that too much.

(Refer Slide Time: 32:27)



But really, you know upper triangular form, you can get with row operations quite easily, right? So now when you allow column operations you can see whatever you did with the rows you can do with the columns also, right? So you can make everything else zero, okay? So it's quite easy to see. So maybe if I go back to the previous slide, you can see how I can make that happen, right? See all these stars which are above and to the right of 1, right? So below you do not have anything. Above and right of 1 if I can do column operations, I can use, you know, this star and then this 1 I can do this column minus, you know, whatever that value is times this column. I will make that zero. So likewise I can make everything zero, using the first row you can make everything zero. And then you come to the second row you make these guys zero, right? So you can do this and then it is easy to, you know, sort of get rid of all these stars to the right also, okay? And then you will be left with these 0s in the middle and you can do column swaps. So you swap and make all the pivots come to the beginning. So that is what I have listed out here. What is it that you have to do? Do row operations and then swap the columns to make the pivots come together. And then after that simply use column operations to make everything zero, okay? So that will give you a form as simple as this, right? So very simple form. The identity matrix of size equal to the rank of A on the top left, everything else is 0, okay? So this also tells you from a high level if you have an $m \times n$ matrix, arbitrary matrix and if you don't care about the actual row space and the column space, you only care about the dimension, you might as well pick a matrix like this, right? So you'll eventually get to a matrix like this. But usually of course we want to preserve the row space and column space because that's what makes the operators interesting. But still, you know, if you don't care, you can get down to as simple a form as this. Only the dimension matters in this echelon form, okay? So it's a good result to know.

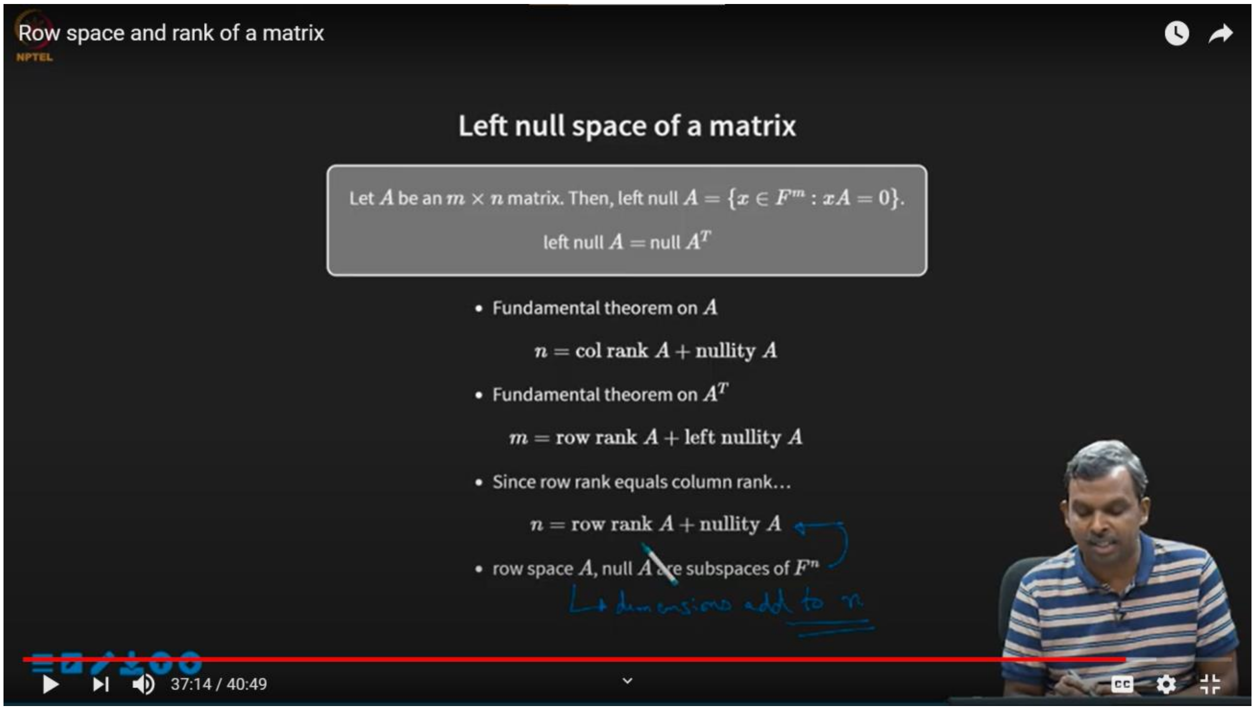
(Refer Slide Time: 34:22)



Okay. So the other space which is involved in the matrix... Okay, so we have talked about the column space, we've talked about the row space, we've talked about the null space which is multiplying a vector on the right, okay? You know also that you can multiply with the row vector on the left, right? When you multiply with the row vector on the left, can you get zero, okay? Can you get a left null space for the matrix? So that is the last space that is left out so far in the matrix and let us talk about that, okay? So if you have an $m \times n$ matrix, one can define the left null space as the set of all vectors in $\mathbb{F}^m$ such that $xA = 0$, okay? So that is called the left null of A and it is also the null of A^T, right? So if you think about it, if you think of A^T where the columns become the rows, this left null space of A becomes the actual null space of A^T. So it's sort of meaningful to talk about the left null also. Now notice you can do fundamental theorem for A, fundamental theorem on A^T, right? So you will get n equals column rank plus nullity, m equals row rank plus left nullity, okay? So now you know that, you know row rank and column rank are the same. So you have this n equals row rank plus nullity, right? So row and column rank are the same. So you have this nice little equation. So notice here. This particular equation is a bit interesting, okay? So all these results you can do with, you know, null space and left null space and all that. But notice row rank and column rank are the same. Nullity and left nullity are derived from the fundamental theorem. So depending on n and m, these two will be different, okay? So these two need not be the same. Row rank and column rank are the same, so these two can, this n and m are different.

Of course if n and m are the same, for square matrices, these two also become equal. But if it's not a square matrix, then these two can be different also. So something to note in mind, keep in mind.

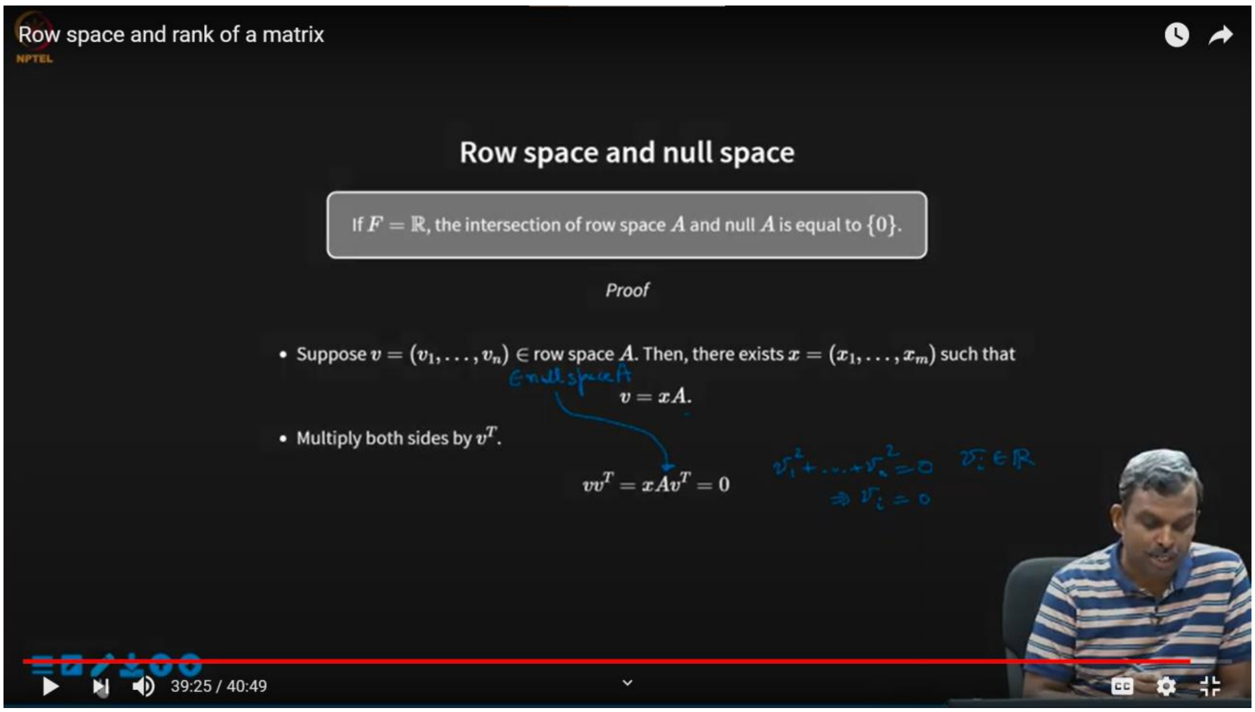
(Refer Slide Time: 37:14)



But notice this equation. This equation is a little bit interesting particularly because of this fact, right? Both the row space, okay, whose dimension adds with the dimension of the null space to give you n, both the row space and null space are subspaces of $\mathbb{F}^n$, okay? So you have $\mathbb{F}^n$ and their dimensions add to n, okay? So you have two subspaces of a vector space of dimension n and their dimensions are adding up to n, okay? So is there something interesting going on here? Is there a nice connection between, deeper connection between row space and null space? It turns out it's true. Maybe not in general but definitely for when the field is real numbers, it is very much true, okay? So it turns out row space and null space cannot have an intersection, okay? When the field is real numbers, okay? So this, when the field is real numbers, this is very important, if you relax that condition this will not be true anymore, okay? And for the real space, the proof is also very easy, okay? So how do you show that, you know, row space and null space have no intersection except for the trivial intersection? You assume that there is a vector which is in the row space of A, okay? And you sort of show, okay... So basically what am I going to assume here? So you assume that this v is in the row space and it's also in the null space, let's say, right? I didn't put that down very clearly. Okay? So this v is in the row space of A and the null space of A, okay? If this result were to be true, I should eventually show $v = 0$, right? How do I show $v = 0$? If it is in the row space then there exists some vector x, okay, such that $v = xA$, okay? So that is quite easy to see. Now I can multiply this both sides with v^T. So notice what happens here. I will get

vv^T. But it will become xAv^T. But since v is also the null space, this will become equal to zero, okay? So you have $vv^T = 0$ and v is real, okay? So what does this mean? Now this will mean you know $v_1^2 + \cdots + v_n^2 = 0$ and then $v_i \in \mathbb{R}$, okay? So that implies $v_i = 0$, right? So you cannot have anything else, okay? So of course even if this real is replaced by complex this result is not true, right? So if you got v_i^2, you know the sum of squares of a bunch of numbers, complex numbers can be equal to 0, okay? So this is not true even if this real is replaced with complex. But for real spaces this is true, okay? So you can do something slightly different with complex spaces. You can think about what's happening there, all that's very interesting, we'll take it up later on.

(Refer Slide Time: 39:25)



So in general there seems to be some mysterious sort of thing going on with this A^T, what is this etc. I mean I deliberately introduced it at this point for you to think about it a little bit. We'll later on come back and look at A^T and this kind of a connection between row and null space, dig a little deeper here and get some more ideas in, later on in the second half of the course, okay? For now these are the results that one can derive just based on, you know, just the dimension of the row space by doing elementary operations and by looking at, you know, fundamental theorem of A and A^T and connecting it, okay? So overall any matrix A, there are four fundamental subspaces that people associate with a matrix A, okay? So this is sort of important. If you want me to summarize this lecture, four fundamental subspaces - the column space, the null space, row space and left null space - they are all intimately connected. Particularly for the real number field they have some even deeper connections. Even in other things there are some connections that you can make later on, okay? So dimensions are connected by fundamental theorem. And also row rank and column rank are the same, okay? So these are all interesting connections that you have between these subspaces. And we'll keep exploring the connections in the later part of the course. Thank you very much.

(Refer Slide Time: 40:03)