Applied Linear Algebra

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Week 04
Coordinates and linear maps under a change of basis

Hello and welcome to this lecture. We are going to talk about change of bases. Now we look at two things and what happens to them under change of basis. One is the coordinates of a vector. So you remember when you think of an abstract vector and you fix a basis for the vector space, you can get a list of coordinates. I mean we are thinking of finite dimensional vector spaces, so a list of finite coordinates that will come once you fix a basis, right? Now when I change the basis... We're always keeping the abstract vector as some fixed quantity, right? So that vector itself remains the same. The coordinates will change as you change the basis. So the same vector has different sets of, lists of coordinates depending on the basis that you pick, okay? So how does that happen? Given one coordinates with respect to one basis and given another basis, how do you go from these coordinates to the next coordinates? Is there, like, a very systematic easy process? That we'll describe. The same thing with linear maps also, right? So you have an abstract linear map once again that I have defined from V to W which works on a vector v takes you to $T(v)$. Now once I specify a basis for V and the basis for W, you know how to write down a matrix for this linear map. And what does that matrix do? You take coordinates for a vector v in that basis, multiply on the right, multiply this matrix on the right, you will get the coordinates for the vector $T(v)$ in W, okay? So that's what happens with this matrix convention. So now what I can do is - the linear map remains the same, but the basis in V and the basis in W are going to be changed. Then what's going to happen? What's going to happen to the matrix? What's going to happen to the operation, because it's the same linear map, but it's being thought about in another set of bases, okay? So these are the two things that we look at in this lecture. So let's get started.

A quick recap as usual before we start. We've been looking at a whole bunch of ideas and the last couple of lectures we saw this notion of different subspaces associated with the matrix. The four fundamental subspaces - column space, row space, null space, left null space and all of these things are very important. So when you see a matrix, you should think of all these guys and they tell you a lot about the linear map that it represents. And the determinant of a square matrix, we saw that function in the previous lecture. We saw that this function has many interesting properties. It tells you so many nice things about the linear map once again. Particularly invertibility and nice things and it, also it's preserved under the matrix algebra in a very nice way. Transpose doesn't change it, multiplication has a simple relationship in determinant. So it's a nice function to know and be familiar with, okay? So that's the recap. So let's get into these coordinates and change of basis, okay?

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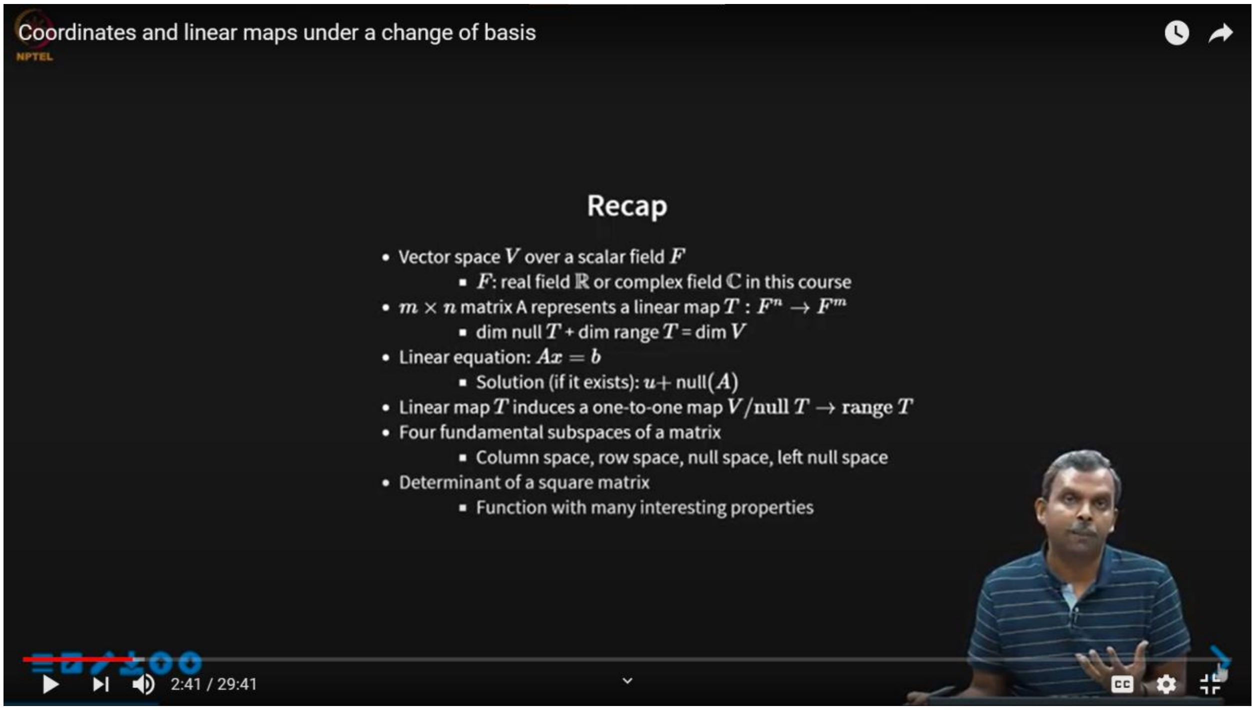

So we know this already. You have a basis for a vector space V, and any vector v can be written as a linear combination of the basis vectors. And those coefficients you collect them together and you call them as coordinates with respect to a basis. So whenever you write a vector v, usually people even write v equals the list of coordinates. I'm writing this, you know, arrow notation here. That's because, you know, in this particular lecture I'm talking about change of basis and all that so let's not say v equal to one thing here, in the same way equal to something else there, it just looks a bit bad. So I'm putting this arrow. But usually you know you can write v equals if the basis and all is clear in the context, okay? So that's the, this is something we know. We've seen this quite often. Now what am I doing? I'm going to change the bases, okay? So when I change the bases, what will happen? I have to list out the next basis which is $v_1', v_2', \dots, v_n'$, I'm calling that basis as B'. Now the starting point is the original basis vectors. $v_1, \dots, v_n$, each of them can definitely be written as a linear combination of the new basis vectors, right? So that we know for sure. So let's do that, okay? So you write v_i is $b_{i1}v_1' + \dots + b_{in}v_n'$. So I do that. So I have these coordinates for v_i in terms of the v's okay? v_1', v_2', ..., okay? So once you have that and you have v written as a linear combination of the $v_1, v_2, \dots, v_n$, you can plug it in and infer the coordinates of v in the new basis, okay? So that's what we'll do. We'll take these v_is and then plug them in into the formula here, okay? So you do that, then you know you will get this nice looking equation. So this v is represented by $(b_{11} \dots b_{1n})a_1$, plus so on till $(b_{n1} \dots b_{nn})a_n$. So if you want me to write a little bit more detail, this guy is v_1, okay? This guy is v_n. In what coordinates? v_1 with respect to B', okay? So likewise here v_n with respect to B', okay? So those are these vectors here. And once you write that, this equation is sort of true. So you put a_1 here and then write the v_1 coordinates here, so on. Now this product of, you know, vector, column vector multiplied by a scalar on the right and added up can be conveniently represented as a matrix product. So you get this matrix times the coordinate system, okay? The original coordinates, okay? So once again notice this. This is coordinates of v with respect to B, okay? So this is sort of like the matrix representing change of basis, right? So you get a matrix representing change of basis. How do you find this matrix? Every column, the ith column is v_i, the ith basis vector of the original basis represented in the new basis, okay? How do you express this? So that is what each column is. Once you write that down, you get a matrix and that matrix sort of gives you how to, gives you a method to transform the original coordinates to the new coordinates. After you do the multiplication you get the new coordinates. This whole thing, this coordinates of v with respect to B', okay? Hopefully that is clear to you. So that's what happens when you do coordinate transformation. So coordinate transformation under change of basis seems quite simple, right? So you have an original list of coordinates. When somebody says change of basis, you go find this matrix. You go find this matrix and you multiply the original coordinates on the right, you get the new coordinates, okay? So it seems like a simple enough thing that you can do, okay?

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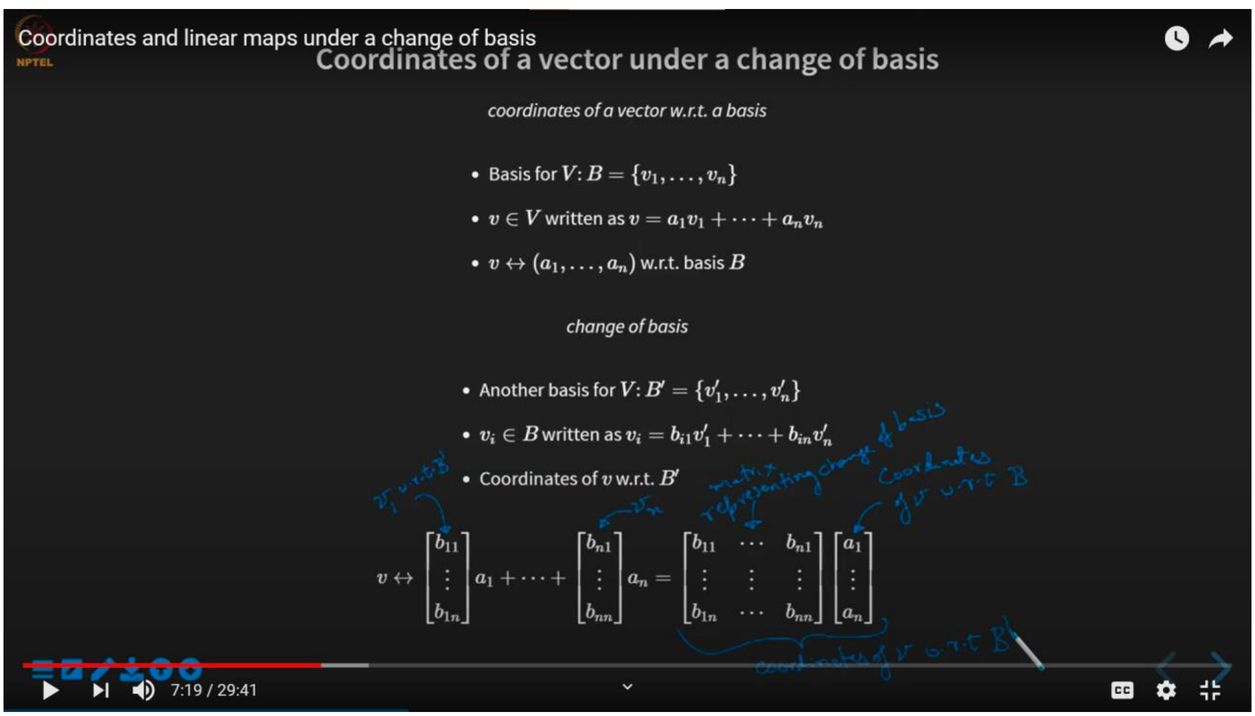

So remember this matrix here that you have, it will be an invertible matrix, right? So you can see that this $v_1', \dots, v_n'$ are linearly independent. You take the coordinate systems, coordinates corresponding to each of those, they will also be linearly independent. So the columns are linearly independent, so this becomes an invertible matrix. And the opposite is also true. Suppose somebody gives you an invertible matrix, you can think of that as a matrix representing basis transformation, right? So why? Because you can imagine a basis with the columns of that matrix being the new basis, okay? So you take any invertible matrix, the columns are linearly independent and together they span the space. So you can imagine a new basis being just defined by this matrix, right? So that's, so this matrix not only represents, you know the change of coordinates, it also represents the new basis itself, okay? So that's a nice thing to remember. So this thing, so you should keep in mind. So when somebody gives you a matrix corresponding to a coordinate transformation, the basis itself is hidden there, the new basis itself is basically the columns of the matrix, okay? So these are all nice things to remember.

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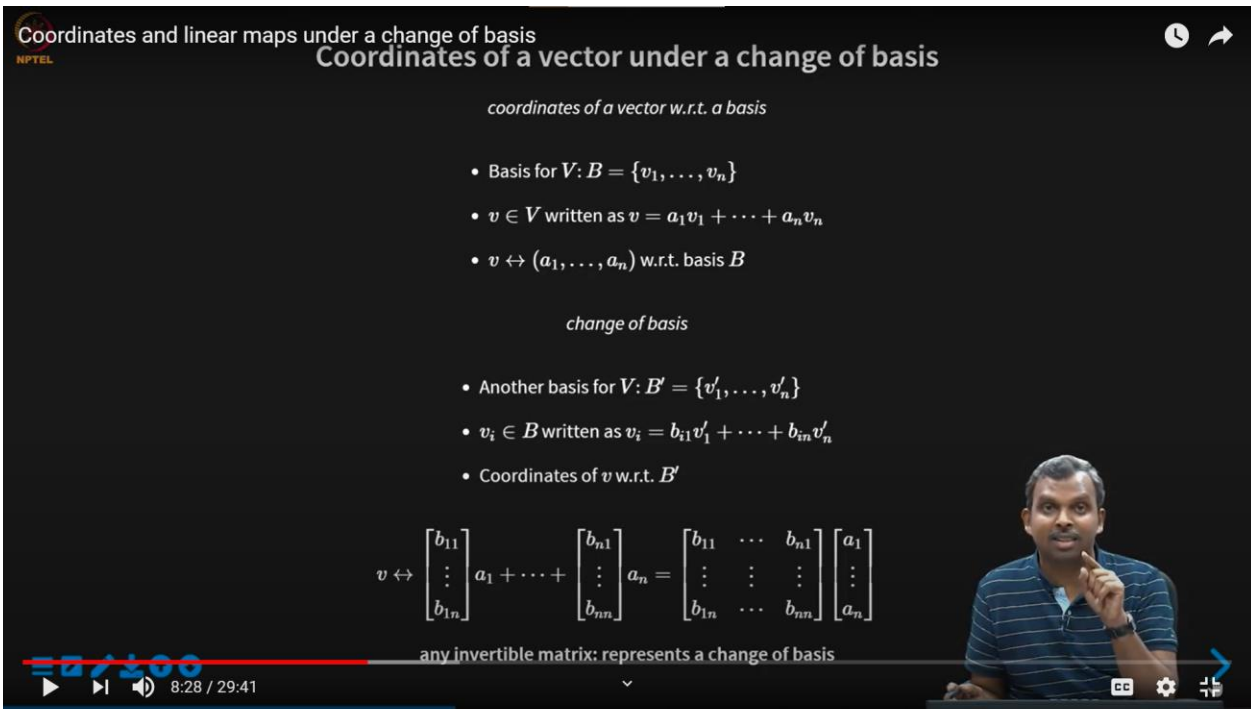

Okay. So how do you go back, okay? So you changed from B to B'. How do you go back from B' to B? Is it related in any way in going from B to B'? It turns out there is a very clear and simple relationship. So we know that going from B to B' is multiplication by this matrix. Now suppose somebody gives you a v' which is in the basis B'. You know that there will be a matrix corresponding to B' to B. What will that be? That will be, the first column of this will be v_1 written in terms of the v's, right? So be careful again. So this matrix here, the original matrix is going from, you know, v_1 written in terms of v's, so this will be v_1' written in terms of vs, okay? So I should be a bit careful here. This one is v_1' with respect to B, okay? So like that. So each column, the i^{th} column will be the v_i' written in terms of the basis B, okay? So you write that, okay? So now what is interesting about these two matrices? You can see that these two matrices have to be connected. One is going from B to B', the other is going from B' back to B, okay? So what will happen is that this matrix will be the inverse of that other matrix. Why? Because you take any vector v, any coordinate v, you apply this first matrix to go from B to B' and then apply this second matrix to go from B' to B, you better get back the same original list of coordinates, right? You can't get some other coordinates. So these two matrices better be inverses of each other. You should get the identity when you multiply them together, okay? So if you want, you can write down a clear proof for this, it's not very difficult to imagine how you will write a proof. So these two matrices have to be inverses of each other, okay? So when you multiply them, you should get identity, right? So that's how we define the inverse of a linear map. So these two have to be inverses, okay? So good thing to remember this. When you use a matrix to go from one basis to another, the inverse of the matrix will take you back, okay? So that's something to remember, okay? That's good.

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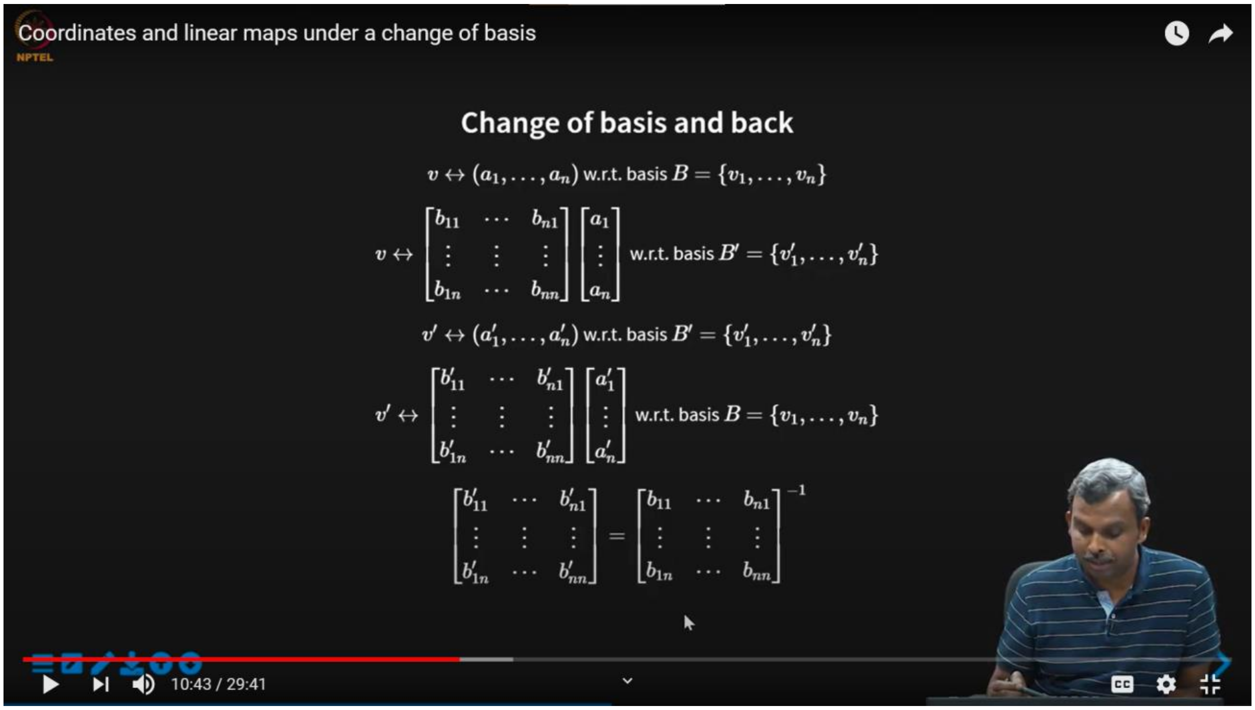

Okay. So now we're ready. Hopefully we've seen how to change coordinates, all that is clear to you. I'm not giving examples for coordinates, I think it's easy enough. We have given that example before. But hopefully you have enough practice to write down examples of your own. If you want examples for, you know, coordinate transformation and how going from one to the other, will be based on very simple, you know, inverses and matrix multiplication, so I'm skipping examples for that. I'll give you examples for linear maps though. So let's talk about linear maps. You have a linear map $T: V \to W$. Once you fix a basis for V like I fixed here, some basis for V and a basis for

W, see I'm going to pick the dimension of V to be n and dimension of W to be m like before, okay? So I'm using this notation B_v and B_w for a basis for V and a basis for W. Once I fix that, I can make a matrix for my linear map with respect to B_v and B_w, okay? So you know how this is done, right? So you take each of the basis vectors from V, do a transform with the linear transformation $T(v_1), \dots, T(v_n)$ and then you go take $T(v_1)$ which actually belongs to W, write it as a linear combination of B_w, of vectors in B_w, okay? What coordinates you get, you put in the first column, okay? So that is $T(v_1)$, that's the first column of this matrix, okay? So likewise you do the second column, third column... Last column will be for $T(v_n)$, okay? So you get an mxn matrix which looks like that, okay? So that's the matrix, okay? So that's what I've written down here. The ith column is coordinates of $T(v_i)$. And that would be a_{ij}, okay? So that's the way to do it, right? So how do you compute coordinates for $T(v)$, right? So given a vector v, you first express it in the original basis B_v, okay? So as a linear combination of v_1. And then you apply T to it. You know that this T is a linear map, so it preserves linear combinations. So you get $b_1 T(v_1) + \dots + b_n T(v_n)$, okay? So that is your $T(v)$ here, okay? So v to $T(v)$, how do you write it in terms of coordinates? You take the matrix that you got representing this linear map, multiply on the right with the coordinates of v, okay? The coordinates, whatever coordinates of v you got. Overall finally you will get the coordinates of $T(v)$ in the basis B_w, okay? So this is the, this is the product, all right? So I'm not...

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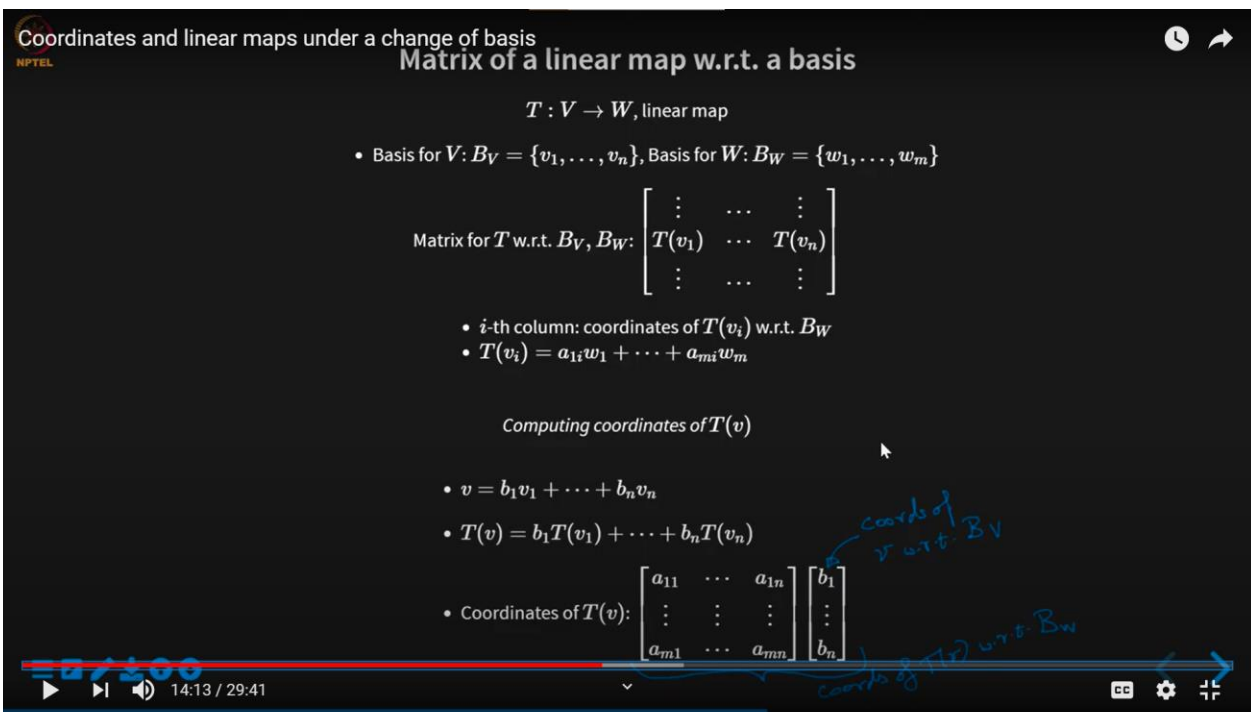

Maybe I should write this in some detail. This is coordinates of v with respect to B_v. This is the matrix, right? So this a_{ij} is the matrix with respect to B_v, B_w. This whole thing is coordinates of

$T(v)$ with respect to B_w, okay? So this is the simple little result that you get. So this we have seen before how to represent, how to find the matrix corresponding to a linear map and how do you use the coordinates to compute, you know, input and output. Given input coordinates, what will be the output coordinates and so on. Okay. So let's take a few examples. The first example I always like to take is the simplest possible linear map which is the identity map, right? So identity map, simply given a vector, outputs that same vector, okay? So now notice I can do some interesting things here. The first thing is: I need a basis for the input and basis for the output, okay? But remember this is V to V, okay? So identity map means V and W are the same, right? So maybe I can pick the same basis for the input and the output, okay? Which is a good thing to do, we'll see later on that it's crucial sometimes, okay? So you pick the same basis. If you pick the same basis, then it's very easy to write down the matrix corresponding to the identity map. You will get the identity matrix. It is easy to see, right? So because $I(v_i)$ is simply v_i with respect to B_v. So you write that, you will get $[1\ 0\ 0\ 0;\ 0\ 1\ 0\ 0;\ \ldots]$, okay? So it is very easy to see how you get the identity matrix, right? Under this map.

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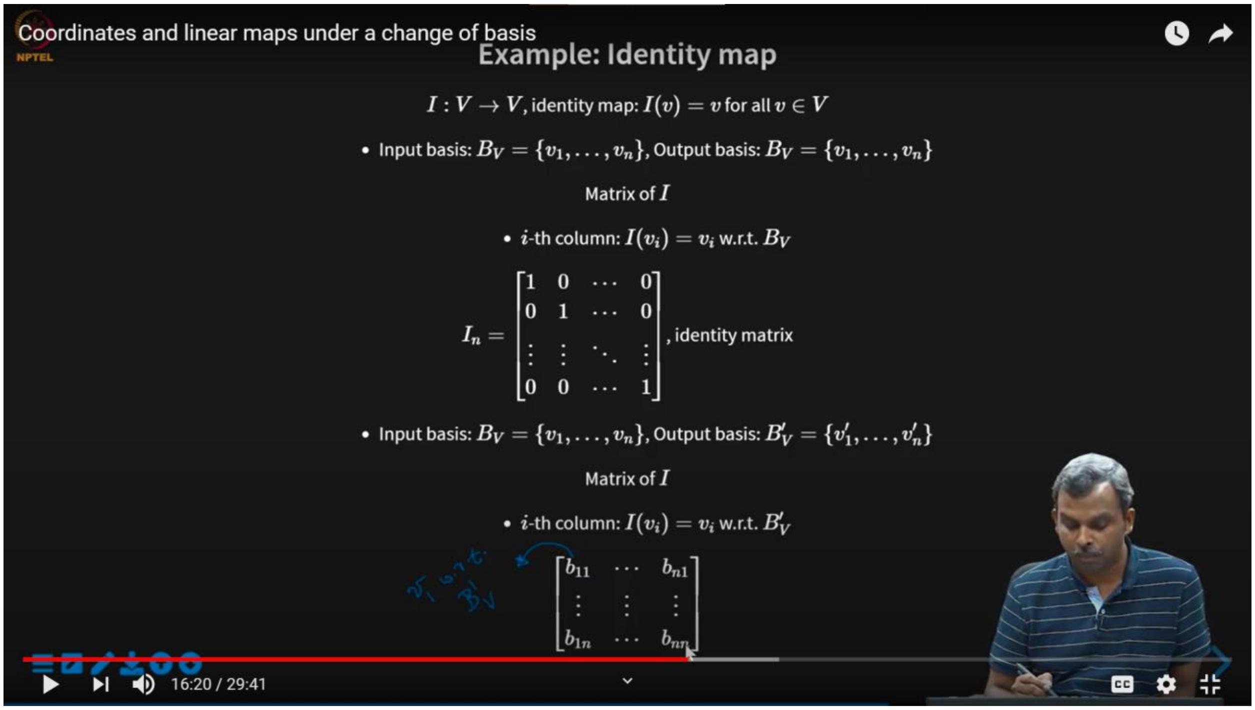

Now what I can do, which you probably don't want to do for the identity map, but let's try it. See what happens. For the output, instead of using the same basis B_v, maybe I want to use another basis B_v'. I can do this, right? So nothing stops me from doing that, okay? So now what will happen is, what will happen to the i^th column of the matrix now? I will have $I(v_i)$ being v_i because that's the, you know, identity map. But I have to write v_i with respect to B', right? The output, I do not write with respect to B_v, I write it with respect to B_v', okay? So that is what I do. I get a new matrix for I, okay? Where $I(v_i)$ which is v_i, I write with respect to $B_v{}'$, okay? So once again to remind you, this is v_1 with respect to $B_v{}'$, okay? If it were v_1 with respect to B_v, I will get $[1\ 0\ 0\ 0; \ldots]$ again. But if it is with respect to another basis, I am going to get some other coordinate, right? So that is what I will get here, okay? So remember that. Likewise I find column after column.

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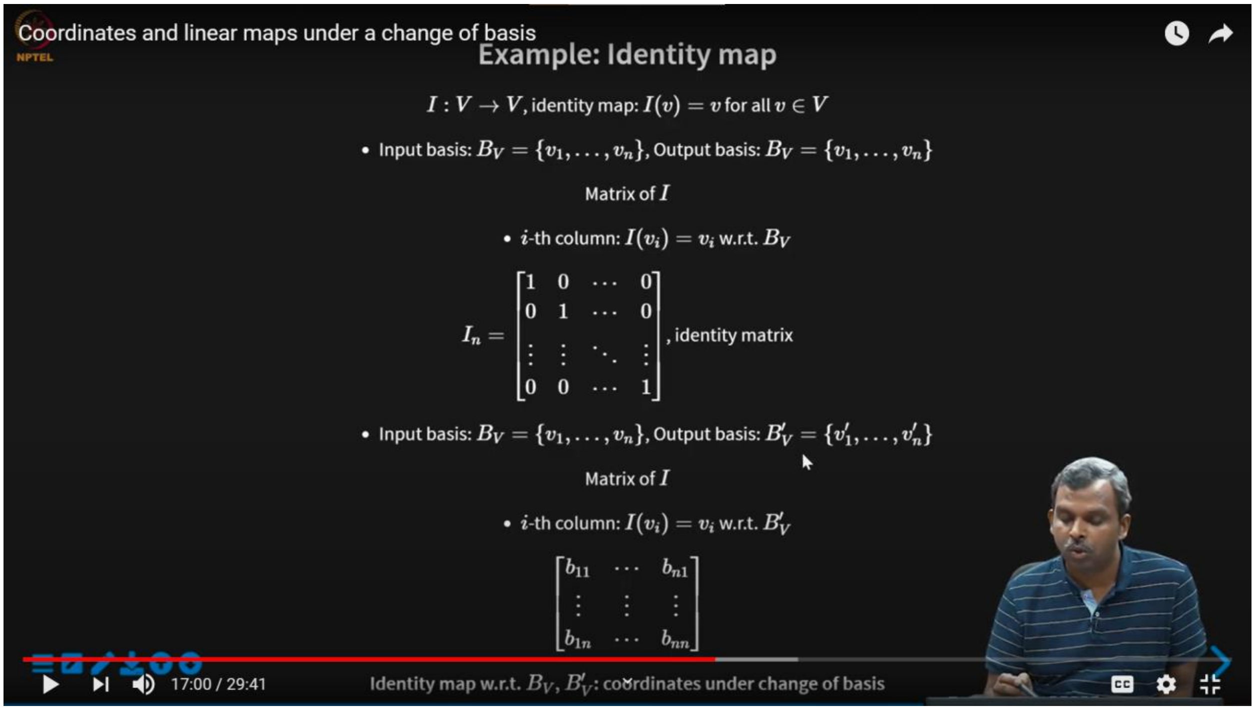

So now notice what has happened here. It's interesting what has happened here. What will you get here? The matrix you get here is exactly the same matrix you got when you looked at change of coordinates with respect to change of basis, right? If you take coordinates for v and you want to change the coordinates to B', right? Coordinates for v with respect to B_v you want to change it to $B_v{}'$, then you got the same matrix, you had to write the exact same thing again. So what you are doing for this coordinate change with respect to change of basis is nothing but changing the basis input and output basis in the identity map. Instead of keeping output bases equal to input bases, you changed it to some other basis and then you saw what happened, okay? So that's the idea behind what's going on. So this is a nice way to think about change of coordinates. So change of coordinates is actually a linear operation. It's the identity map. When you change the output basis to something else, okay, you can also do the change of basis and back, okay? So you see that, you know, you can keep the input basis as B' and output basis as B_v, right? So now you know if you do that, you will get the inverse, right? So you got, if S was the first matrix, you will get S^{-1} here. So this is the same as what we saw before, okay? That's good.

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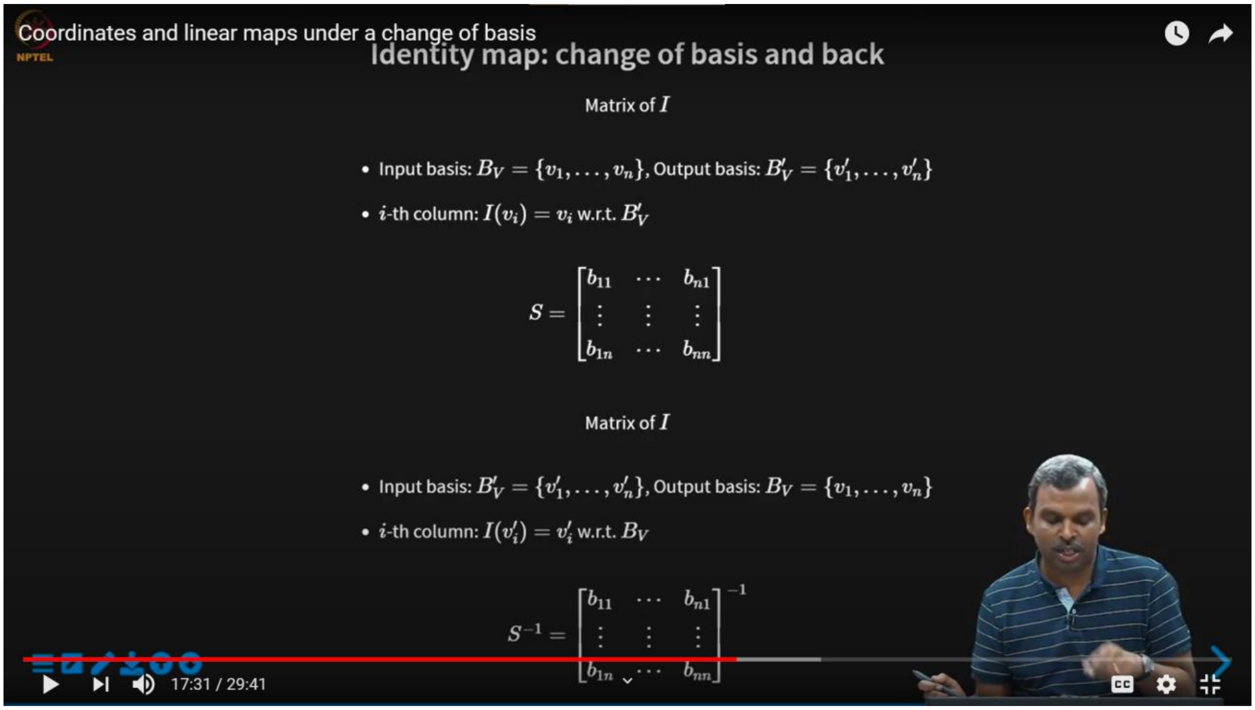

So now let's look at an arbitrary linear map, okay? $T: V \rightarrow W$, I have a basis for V, basis for W. I found a matrix, okay? What happens when you change basis? Notice what's going to happen. Let's say instead of B_v I'm going to pick another basis B_v' here. Instead of B_w I'm going to pick another basis B_w', okay? So that's what I'm going to do and now I ask the question: what's going to be the matrix of the linear map T with respect to this new basis, okay? This is a simple view, you can view this as some sort of a composition, okay? So this view is very interesting, is very simple and at the same time immediately tells you how to go from one matrix to another with respect to one basis to another matrix with respect to another basis. So notice what I am doing here. This change of basis operation I'm going to view it as some sort of a composition for the overall transformation, okay? I know the transformation from B_v to B_w. But my vector input is being given in B_v', right? So that's what this means, right? When I want a change of basis, I want to specify my input as a linear combination of B_v' and I want you to give me the output as a linear combination of B_w', that's what I want when I say change of basis from B_v' to B_w'. So my input coming into my T is not from the basis B_v but from the basis B_v'. So what do I do? I first hit it with the identity map with input basis B_v' and B_v. I know how to do this. Basically I do a change of coordinates from B_v' to B_v . And then use the same T and the same matrix for T which I know from B_v to B_w, okay? I use that matrix, I go from B_v to B_w. But I don't want the output in B_w, I want it in B_w', what do I do? I hit the identity map again which goes from B_w to B_w'. I change basis in W from B_w to B_w'. So this is what I need to do to do a change of basis for a linear map. Linear map T is specified as a matrix. Given a particular basis B_v and B_w, I want to now change basis to B_v', B_w' because my inputs maybe are coming from another basis, right? So how do I do that? I first do this identity map which takes me from B' to B and then hit it with the same matrix as before. It takes me from, you know, in the representation B_v to the representation B_w. And then you once again do this identity map with basis being different. So when I say identity map, it refers to change of coordinates, right? So that's what happens. So you do that, you get this.

So in terms of matrices, I have the matrix for T sitting in the middle. To its right, I'll have the matrix for the identity map in V, okay, basis B_v' to B_v. And then on the left I will again have a matrix for the identity map in W which will be from B_w to B_w'. Remember both of these will be invertible. This will be an invertible map. This will be an invertible map because it represents the identity map, right? Just that the bases are different, but it's going to be an invertible map, okay? So that's it, that's all is there to a change of basis for a linear map, right? So instead of just thinking with that matrix, I know the basis has changed. So I put in my identity maps in front and right and left to convert to the basis I have and then go back to the basis that I want, okay? Very simple idea, okay?

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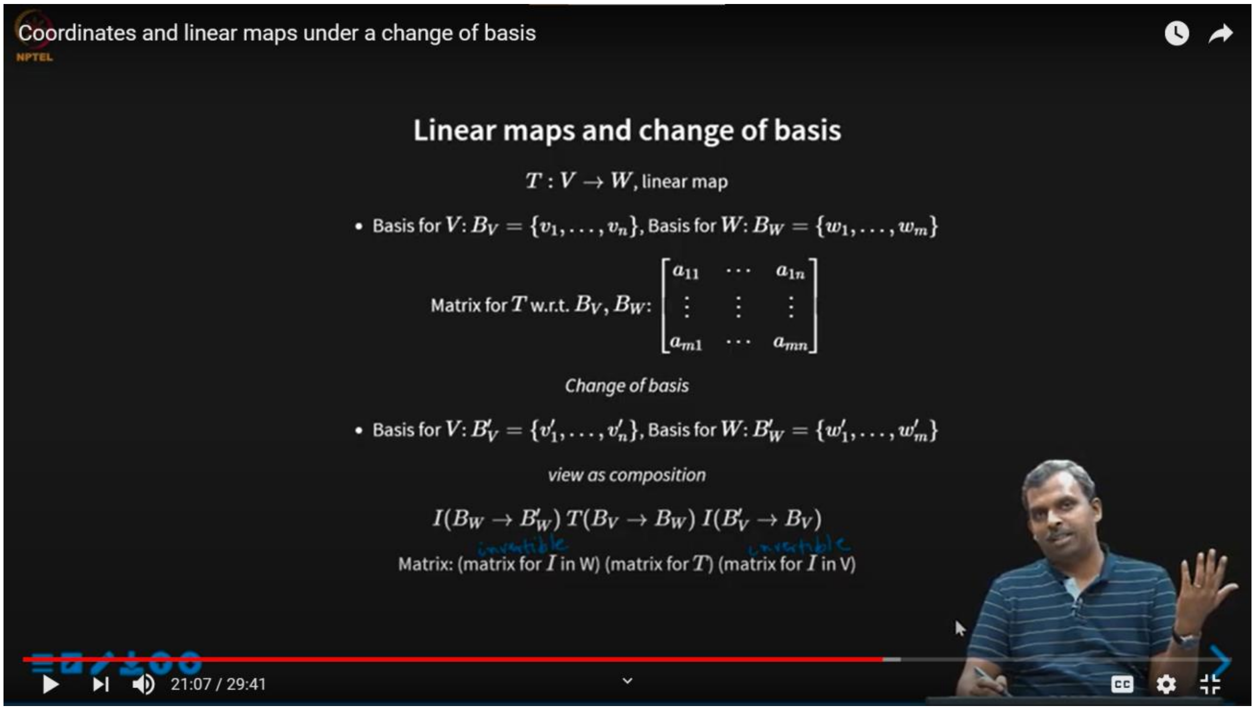

So now what happens for operators, okay? It's not really too special or terribly different. But notice for operators quite often you may want to keep the input and output basis as the same. You do not have, you know, B_v and B_w, you maybe have the same basis, okay? And then you can compute the matrix for T and maybe I want to go to a change of basis. But again input and output are the same. I may keep the input and output as the same, right? Maybe it can be different also, we look at that case later on. But you may want to keep the input and output basis as the same for linear operators and this is a very popular choice, there are good reasons for why you want to do this. But let's say

I want to do this. Then what will happen? Notice, you know, what happens when you do this, right? You will have the matrix for T which will be SAS^{-1}. S is what? And S^{-1} is what? S will take you from B_v to $B_v{}'$, okay? And S^{-1} will take you from $B_v{}'$ to B_v, okay? So that's the map you will get. Because notice what you have to do at the output, you have to go from $B_v{}'$ to B_v, okay? So you will get a S^{-1} map there. And then you hit it with the same matrix A which is from B_v to B_v. And then on the left hand side you will have the identity map from B_v to $B_v{}'$, okay? So that is what we have as S. And so here you will have the S^{-1}. So you have this nice little, nice looking transformation of matrices when you do a change of basis from B_v to $B_v{}'$ in an operator, with an operator. The same V to V. Then the original matrix gets transformed as SAS^{-1}, okay? S represents the transformation from B_v to $B_v{}'$. The coordinate change from B_v to $B_v{}'$ which is an identity map with input and output bases different, same thing with S^{-1} which is in the other direction, okay? So this is a nice looking transformation. In fact it has a name. It's called a similarity transform. You can pick S as any invertible matrix. We know that also SAS^{-1} represents change of bases to columns of S, okay? So remember this is the columns of S and S should come on the left, that's the new basis that you're going to from the original basis, okay? So that's important to understand, okay? Okay so I'll stop with this. Hopefully you can see how this works, okay?

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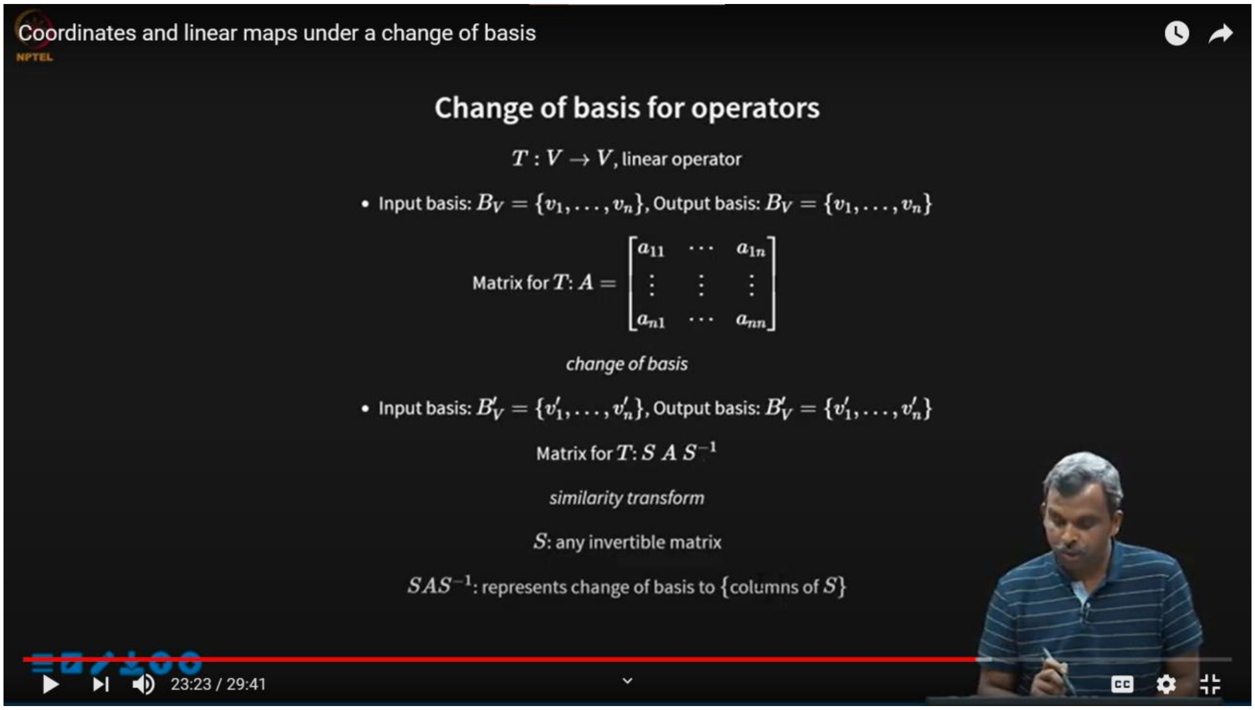

So maybe we should do an example or two, maybe we should do an example here to see what I mean. I'll do $\mathbb{R}^2$ examples which is easy enough, okay? So let us look at a matrix A which represents some linear transform. We'll take our familiar matrix or famous matrix that I always write down. So this, say it represents some linear map with respect to the standard basis, okay? So we'll say standard basis for input, standard basis for output, okay? So let's say B_v, so let me write it down. B_v is $\{(1,0),(0,1)\}$, okay? That is good. So now I might want to go to another basis which is say B_v'. Maybe that is $\{(1,2),(3,4)\}$, okay? Is this a basis? Yeah it's a basis, you can check that it's a basis. What do I have to do? I have to find this S which is going to do the coordinate transform from B_v to B_v', okay? B_v'. So a lot of people have a lot of confusion about change of basis, because, I mean, they say okay B' I've written coordinates here, what do these coordinates represent, okay? These coordinates also need a basis, right? So usually the hidden assumption is, when nothing is given, it's always standard basis. So this B_v' has to be represented in B_v only. Only then you can do the transformation properly, okay? Remember this is all with respect to B_v, okay? So these coordinates are in the original basis that you specified. If this itself is different, you have to change this also. You have to assume that these coordinates are with respect to B_v, okay? So that's sort of a tacit assumption. It's not explicitly mentioned usually, but that's what it is, okay?

So my S is what? S going to be... What happens? So I have to write $(1,0)$ in terms of $(1,2)$ and $(3,4)$. So what is the first column? Remember it's $(1,0)$ written in terms of $(1,2)$ and $(3,4)$. So you have to sort of do like an inverse calculation. So let us try and do that. If I want $(1,0)$ in terms of... I can do I think -2 and 1. I think this is true. $-2(1,2) + 1(3,4) = (1,0)$, right? So that's good, okay? So I've already got that. This is my S, (-2, 1). This is $(1,2)$, this is $(1,0)$ in B_v', right? This is my S. The next is $(0,1)$. I have to get $(0,1)$. So for that I'll have to do I think $3(1,2)$. If I do 3 times, I am going to get 0 here, that will be... So maybe $\frac{3}{2}(1,2) - \frac{3}{2}(3,4)$, okay? So I think… No, minus. Maybe $\frac{1}{2}$, okay? Yeah I think this works. This would give you $(0,1)$, okay? So this is how you figure it out. So this would be $\frac{3}{2}$ and $-\frac{1}{2}$. So what did I write here? I wrote $(0,1)$ in B_v', okay? So this gave me my S. This is the transformation from standard basis coordinates B_v to B_v', okay? So you can also do S^{-1}. Now S^{-1} you can do in two ways, okay? You can directly find the inverse of S if you know some, you know, standard method or you can work it out in the way we did it, right? So what will be S^{-1}? S^{-1} will be, you know, it is a transformation from B_v' to B_v. So I will have to write $(1,2)$ in terms of these coordinates and that is very easy to write. So you will just write $(1,2)$, okay? I hope I got this right. Okay. And then you will write $(3,4)$, okay? So this is very easy to write down, right? So this is basically $(1,2)$ in B_v and this is $(3,4)$ in B_v. So it's very easy to write down S^{-1}, okay? S^{-1} is very easy. S is just the inverse, you can check that these two are inverses. If you take this and multiply with this, I think you should get it, okay? So it should work out quite okay, I think it should be fine. It looks okay to me. So this is the inverse. So what about the matrix? If you go from here to, change of basis B_v to B_v', you will have to do the, you know, SAS^{-1}, okay? So S takes you from B_v to B_v', and then from the standard basis to the new basis and then you know... Sorry, S^{-1} takes you from the new basis to the standard basis. You multiply with your A, you get the answer in standard basis and then you go from standard basis to the new basis, that is SAS^{-1}. I'm not going to do this product, you can figure it out, okay? So these are the steps that you use to calculate, you know, change of basis for linear maps, okay? So hopefully this was clear to you. Hopefully this example was also very useful, okay? So we will stop with this lecture at this point and then move on to the next lecture where we will see, why change of basis is interesting to do in the context of linear maps. Thank you.

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