Applied Linear Algebra

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Week 01
Vector Spaces: Introduction

Okay, so having talked about the course and the introduction, we are ready to jump into the material itself, okay? So the first thing you have to understand before we start talking about vectors is the scalar, right? What is a scalar? What is a number, right? So that is the way to think of this. So let's start thinking about what a number is, right? All of us have learned about numbers. We know the numbers really well. 1, 2, 3, 4, -1, -1, maybe 1.2, π, $\sqrt{2}$, all sorts of fancy numbers that we know. But in case somebody were to ask you what is a number, right? It's sort of difficult to give a clear answer right? So there are various ways to answer this question. Even in mathematics and physics there are multiple ways in which people approach an answer to this question on what is a number. The first and most obvious answer is that number helps you keep count, right? Or and if you want to abstract that a little bit more you can say number represents the amount or quantity of something. Ok, so something you want to quantify, you put a number against it. And then numbers are ordered, so you know if it is bigger, more quantity etc. subtract etc. all those things are interesting, okay?

So that's the physical way of looking at a number. But in this course and in most of mathematics people think of numbers in a different way, okay? So you want to think of numbers in like an algebraic way. What is, what do you mean by algebraic way? You want to think of a number as something with which you can do algebraic operations, okay? What are the algebraic operations that you can do with a number? You can add numbers, you can subtract numbers, you can multiply numbers, you can divide numbers, right? So you can do all these operations. More than the number itself, instead of worrying so much about what the number represents or what the number quantifies or what the number is, we will simply care about what you can do with the number. Can you add it, can you subtract it, can you multiply, can you divide? So that's, that's sort of like an abstract notion of what a number is. I don't care what it is, I only care what I can do with it, right? So that's the algebraic abstract way of defining numbers. And in fact, if you have a set of numbers where you can do all four operations in a meaningful way and some, with some rules satisfied etc. Like what are the four operations - addition, subtraction, multiplication, division. If you can do all that, then such numbers are said to form a field. Okay, there is of course a precise mathematical definition for a field. I am not going to make that in this class. You can look it up if you like. It’s not so important in this class. We will define other things a little bit more precisely, but the scalar I will leave a little bit vague, okay? So I will assume you know what i mean when I say add, subtract, multiply, divide okay?

The most important examples for fields that you study and you should be well aware of is first the rational number field which will be denoting Q. In fact we won't use the rational number field that much in this class, but still it's an important field to know. I mean, you can imagine why rational numbers are a field, right? You're given any rational number, you can... Two rational numbers, you can add, you can subtract, you can multiply them, you can divide one by the other. Of course division by zero is not possible, that's okay. That's acceptable in the definition of a field. The most important fields or set of scalars for us in this course will be the next two examples I have there. The field of real numbers and the field of complex numbers, okay?

So these two are the most important things for us to remember. So this is the real field. Maybe I should change the colour a little bit and make it red so that you can see the highlight. This is the real number field, and the complex number field. I've put down notation there. R and C. We will use this again and again and again. You'll see this notation show up a lot.

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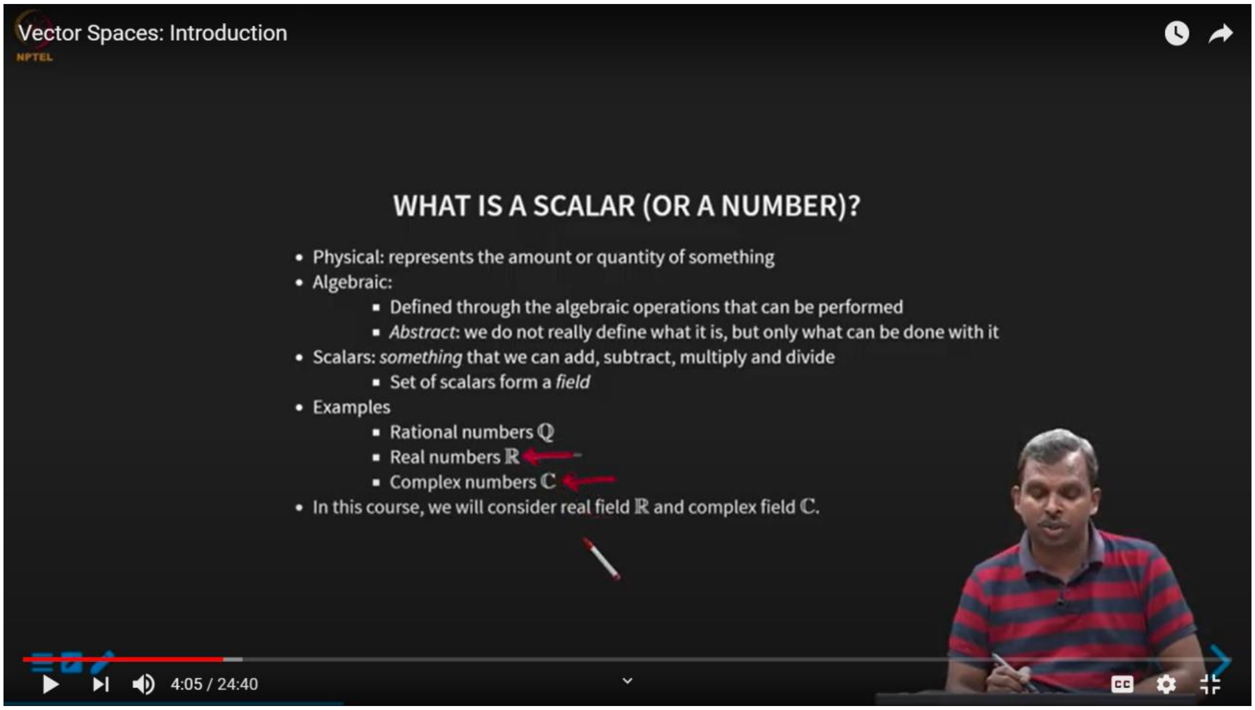

So I will not define these numbers precisely for you. I will not go over detail as far as these numbers are concerned. Let me just quickly show, ask for a few questions and assume that you have familiarity with it. So for instance let's deal with complex numbers. Typically we write complex numbers as $a + bi$. Where this i is square root of $\sqrt{-1}$, okay? And you know how to multiply complex numbers, right? $(a + bi) * (c + di)$. You know how to multiply these two and you can also divide, right? So you know how to divide complex numbers. Ok. I'll assume a basic familiarity of that type, I'll assume you know things like polar representation, this, that and the other... So some basic comfort with what a complex number is, okay? So it's not very difficult. You must have seen this in previous courses in mathematics. So this we'll assume. The same thing we will assume with real numbers also, okay? So these two things we'll assume in this class, okay?

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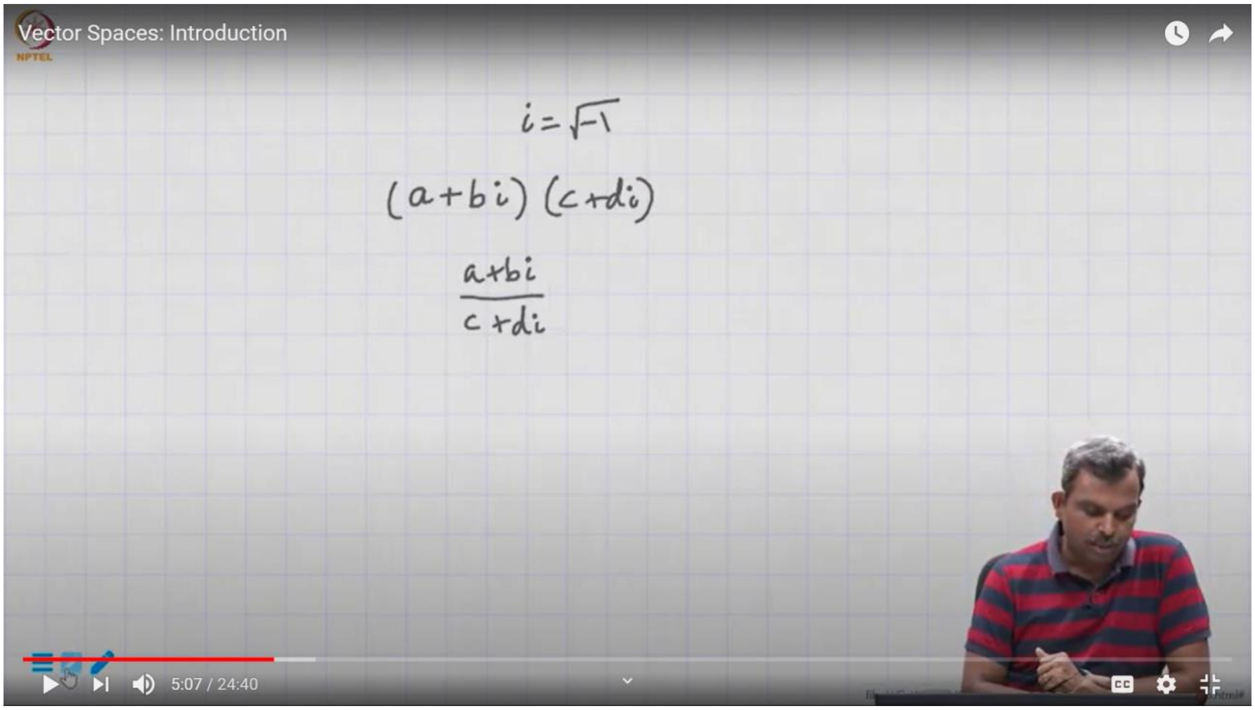

So these two fields are very very important to us, okay? So hopefully with this one slide, I've given you an inkling of what scalars are, what numbers are, how do we think of numbers in this course. We are ready to move to vectors, okay? So what is a vector. Number I said represents some quantity of something physically, but I don't really care too much about what it physically represents. I only care about the operations I can do with it okay? So when we go to vectors also, we'll do something very similar. Instead of worrying so much about what the vector represents physically, we'll worry about what are the operations we can do with the vectors, what are the algebraic operations we can do with the vectors and what are the properties we can derive for it. And that's how we mathematically study vectors in a sort of an abstract way as opposed to a physical way in which you study vectors. Okay?

Okay, so here's a little quiz. I will not answer this quiz for you. So I have asked which of the following are vectors. When you access this slide, slide-deck on a browser, you will be able to make these choices. In fact even I can make these choices if I want to. So here are a few pictures for you, and I've asked you what are, what are these? Are these vectors? So everything that you think is a vector, you can pick, select here. And here is a typical picture of a vector, right? An arrow with some direction and magnitude, right? So, and here is another way in which maybe you think of vectors - a list of numbers. What about this one? Okay, what about a matrix? Is a matrix a vector? Can you think of it as a vector? Does it make sense? Okay, what about this one? Here is an image, it’s a very popular image. It's called the Lena image and image processing people use it quite a bit to do image processing algorithms. Is that image a vector?

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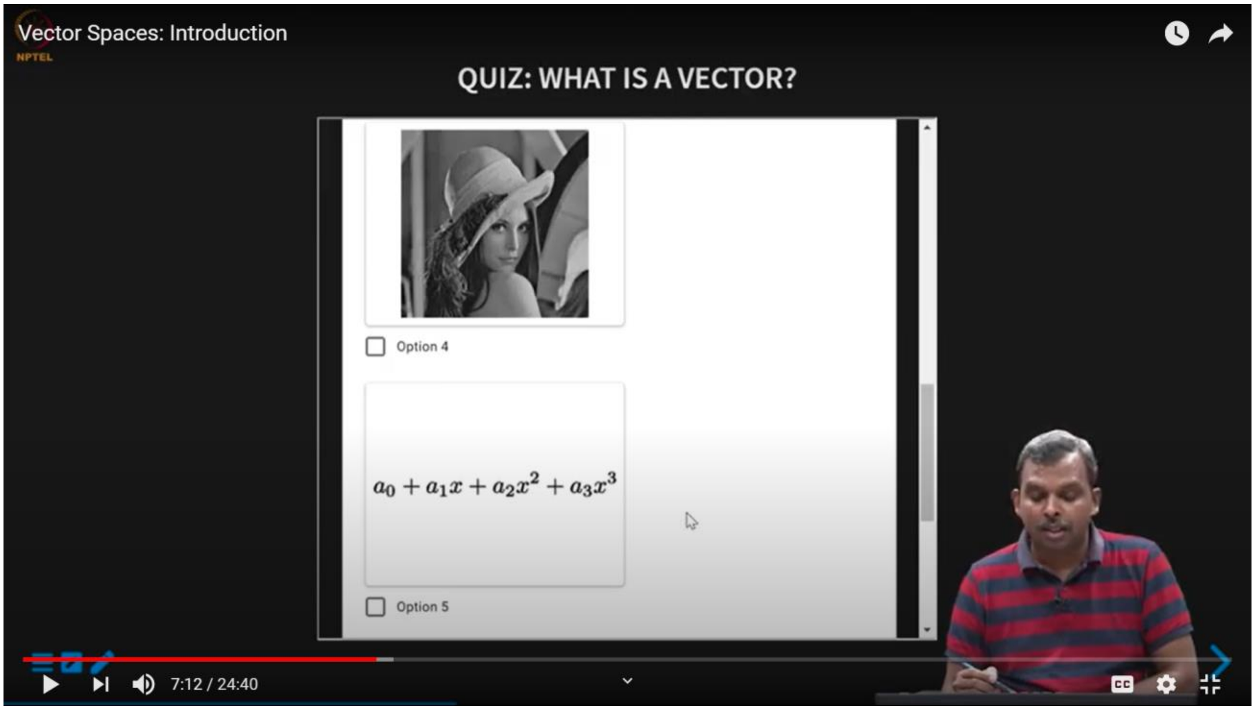

Okay so maybe, I mean, I don't know how you're thinking about it. Is it a vector? It's an important question to ask, right? What about this one? $a_0 + a_1x + a_2x^2 + a_3x^3$. When you think of it you think, you say immediately polynomial, right? The word vector doesn't really come into mind but is it a vector? The way we think of it in an algebraic way, can it be made to behave like a vector? Can we think of it that way?

So those are interesting questions for you to ask. Here is another another object. Okay. $\sin(x) * \cos(x) \quad x \in (0, \pi)$, right? So it’s a function, normally you think of it as a function. Can I think of it as a vector? Does it have algebraic properties which are similar to a vector? So these are all interesting questions. I would like you to pick these options and do a submit, and after you submit you can also check what your other friends have said, what other people have said responding to this question. I'll be curious to see what you will say when you... After the end of this lecture maybe, about what these are, which of these are vectors. But go ahead make your choice, submit. Let me see what your answers are okay?

Okay, so let me quickly go through sets and operations. I want to be very brief with this slide. When you think of a set, set is a collection of elements, okay? And that is of interest to us of course, that's why we are studying sets. And there are operations you can perform on elements of this, of a set, right? So typical operation is what's called a binary operation, and the most popular notation for a binary operation is plus, ok? And any binary operation, the first one you consider on a set can be thought of as an addition. Just calling it as an addition. Okay? What is addition? What is a binary operation? Given two elements from a set a and b, I think of $a + b$. And to $a + b$, I associate another element of that same set, okay? So that's addition. Addition could be commutative, that is, a + b could be the same as b + a. Most of the additions we consider in this course will be commutative, okay? So that's a binary operation.

I assume basic familiarity with it, I am giving you a bunch of examples here. If you look at integers, rational numbers, real numbers, complex numbers, you can, you know how to add them, right? So you have the addition operation well defined even for polynomials. There is addition, I want to point that out. Functions can also be added, you can do $\sin x + \cos x$. That's also another function, okay? So you can do addition. Matrices can be added, okay? So you notice here even though these objects are all very different - polynomial, function, image, matrix, you think of them as different types of objects representing different physical things maybe. But algebraically as far as operations are concerned, you see a unity here, right? So they can all be added together, you can add two of them, okay? So maybe when you think of them as vectors, you can gain something when you abstract these things out.

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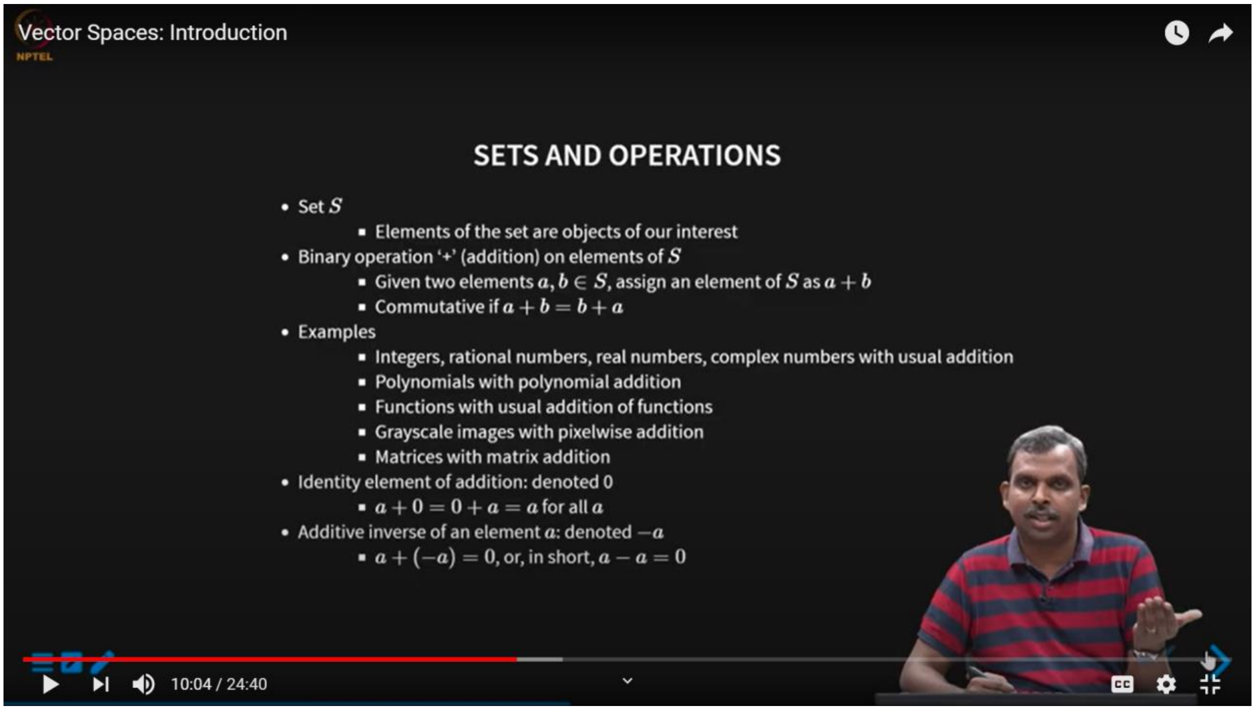

A couple of things about these operations. One is this notion of an identity element. Every operation you can associate an identity element. For addition usually the identity element is called zero, okay? It's denoted zero. And what does it satisfy? Anything you add zero to you should get the same thing, right? So you should not get any change, that's the property. That's put down there for the identity element. And also there should be some... quite often we think of an additive inverse, right? So for every +1, you have a -1. +10, -10 like that, right? So for every object you have its additive inverse, so that when you add to the inverse you get zero, right? So that's also something, so this level of sets and operations I'll expect you to have some basic comfort over. We won't need too much of it but just remember that this is the sort of abstractness that we'll be dealing with when we deal with vectors. Okay, so instead of talking about specific vectors, we will think of an abstract vector and will think of additions in this way, okay?

Okay so having said all that we are ready to define a vector space V over a field F, okay? So whenever we think of a vector space… Okay vector space is the set which can, which has all our vectors. The collection of all our vectors together are in the vector space okay? And associated with the vector space is a field okay? So this - the field and the vector space go very closely together, you can't separate the two. So you have to define a vector space over a field F, okay? So the usual field we consider in this course will be the real field and the complex field, but it turns out there are many more fields and you can define vector spaces over any field okay? But that's not too much of a concern for us, we'll restrict our fields to real numbers and complex numbers. We'll think of vector spaces over the real field and over the complex field. Okay the elements of the field F are called scalars, the elements of the set V are called vectors, okay? That is the parlance.

Now, two operations are very very important when you define vector space. Whenever you define a vector space, you have to clearly and completely specify both these operations, only then you have a vector space. Just because you have a set V and a field F, you don't have a vector space. You should say how to do vector addition. That's the first operation that you have to clearly specify - the vector addition is usually denoted plus and it is defined on the set V. Okay so given any two vectors, right, u and v, I should be able to add them and give another vector u plus v. Ok so let me just write it down here... So given u and v, u + v is another vector. And when I want to define a vector space or when I want to claim that something is a vector space, I have to clearly define what this plus operation is okay? So that is the vector addition. So I have to be able to add two vectors, ok, in whatever way I specify.

The next important operation i have to specify is what is called a scalar multiplication okay? it is usually denoted dot okay? And it is defined with two elements - the first element is a scalar, the second element is a vector. Ok so you have a scalar a and a vector v, and you can do a scalar product $a * v$. So usually when people write a dot v, this dot is painful to keep putting all the time so we'll drop the dot and simply say av, okay? So to every vector v and every scalar a, you should associate another vector which is av all right? So when you want to make some set V a vector space over a field F, it's your job to define an addition. Given two vectors, how I can add to get a third vector. Again it's your job to define a scalar multiplication. Given a scalar from the field and a vector, how do I do the dot product of these two to get another vector okay? I should get another vector, that's important. Not only that, these two operations should satisfy some basic conditions. Only when these requirements are satisfied, you have a vector space, otherwise you don't have a vector space. It's not interesting enough if these operations do not satisfy these conditions.

What are these conditions? The first condition is the V, set V with the vector addition operation should be what is called an Abelian group. Ok what is an Abelian group? It sounds very fancy. Basically all that means is this addition operator has an identity element, there is a zero vector okay? There is the zero element which when added to anything will not do anything to it, right? So you take a 0 vector, add it to anything you should not get anything else. You should get that same object. Likewise there should be an additive inverse - for every vector there should be a minus of that vector which will add to it to give you 0, the additive identity. So it is your job to define addition, to make sure it is Abelian, it has an identity, it has an inverse and on top of that the addition should be associative. What is associative? Associative is $u + (v + w)$ is the same as $(u + v) + w$. Ok, so if you have, if you add three things, then it does not matter in which order you add them, right? So you can add the first two and then the third, or the second two and then the first. You should get the same answer okay? Commutative we defined already: u + v should be v + u, okay? So these are the properties of addition.

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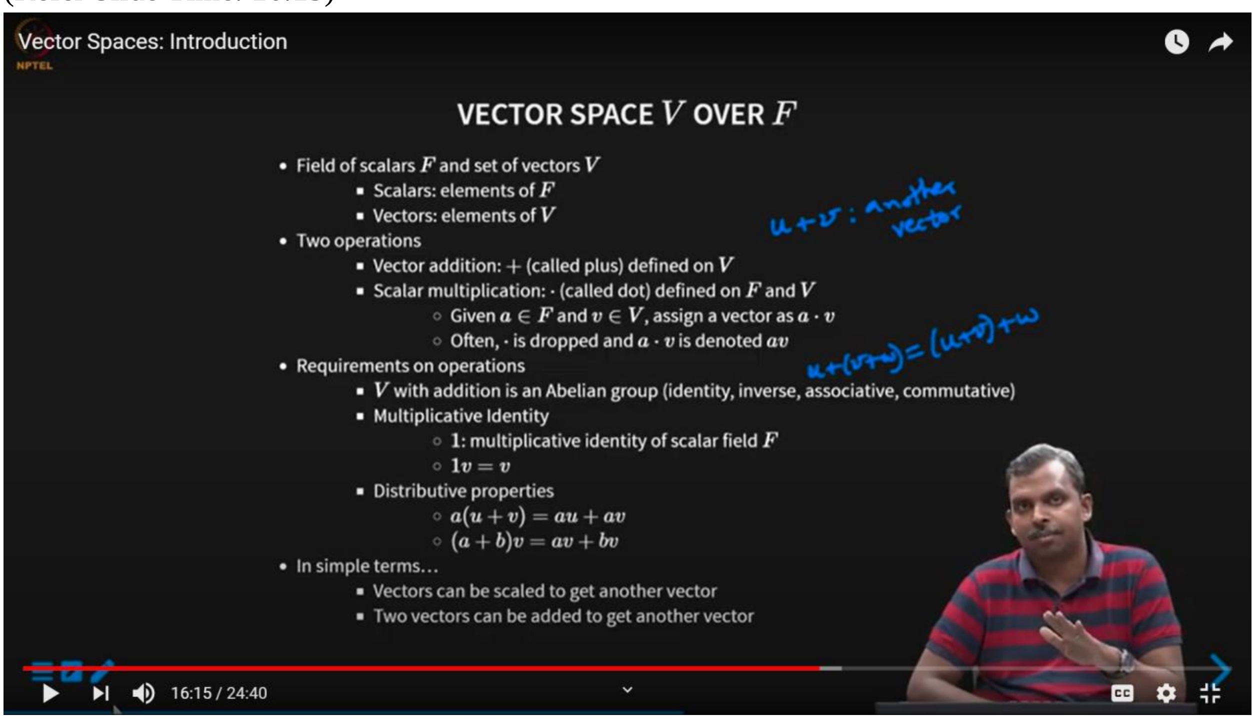

Okay now there are further requirements, okay? The scalar multiplication should satisfy this multiplicative identity property. What is that? If you take the multiplicative identity of the scalar field, which is one, okay? In the real number for instance you have the real number 1. You also have the complex number 1 right? 1 + i0 right? So that one- that's the multiplicative identity. Why is it a multiplicative identity? You take that one and multiply with any number, you get that number itself, right? You don't get anything else. So that's the multiplicative identity. If you take the multiplicative identity and do a scalar product, scalar multiplication with v okay? $1v$, you should get v itself okay? You should not get anything else. Your scalar multiplication cannot say $1v$ will be some $2v$ or something like that. Should be 1v, should be v okay? So that's important to keep in mind.

Now further properties, important properties are these distributive properties. What is this distributive property? If you have a scalar a and then you multiply it with the sum of the two vectors u plus v, you should get the same vector as multiplying a with u first then multiplying a with v first and then adding the two. It should be the same. Likewise you have another distribution property $(a + b) * v$ should be $av + bv$.

So like I said, you have a set of vectors V, and a field F and you want to make this set of vectors actually a vector space, a proper vector space, you have to define these two operations - the vector addition operation and the scalar multiplication operation. And they have to satisfy all these conditions. And only when they do that, you have a vector space. Otherwise you just have a set and a field, and some partially complete operations okay? So in simple terms if you want to drop all the specific things and you just want to understand it from a high level, then the basic idea is that vectors, so when you call something vectors, you should be able to scale them to get other vectors, you should be able to take two vectors and add them and get a third vector ok? As long as you can do that, it is a vector, it is a vector space all right?

So that's all it is, okay? Hopefully that definition was clear to you. I am going to give you a few examples and then show how these operations are defined for those examples, and how that makes it a vector space, all right? So that is going to be what we will do next, ok? So here is the first and the most important example for vector spaces, this is this F^n. You have a field F which is say R or C. I can define R^n okay? The n-dimensional vector space as it is called, over the real field R. R^n. What is that? This is basically... You take n real numbers, put them one below the other, put this like nice bracket, square bracket around it and call it a vector, ok? That's just my... One level, it is just a definition.

But most of you are probably familiar with this vector space right? And you've always been studying this vector space in one way or the other. And this is very very… I mean sort of a particular specific example, stereotypical example of a vector space… Many vector spaces will actually be like this only, right? So we will think about it more as we go along. But for this vector space, why is this a vector space over F? Because I can define an addition. How do I add? u plus v will be just component wise addition right? $u_1 + v_1$, $u_2 + v_2$, so on. What do I do scalar multiplication? a times v is easy to define, it's just av_1, av_2, okay?

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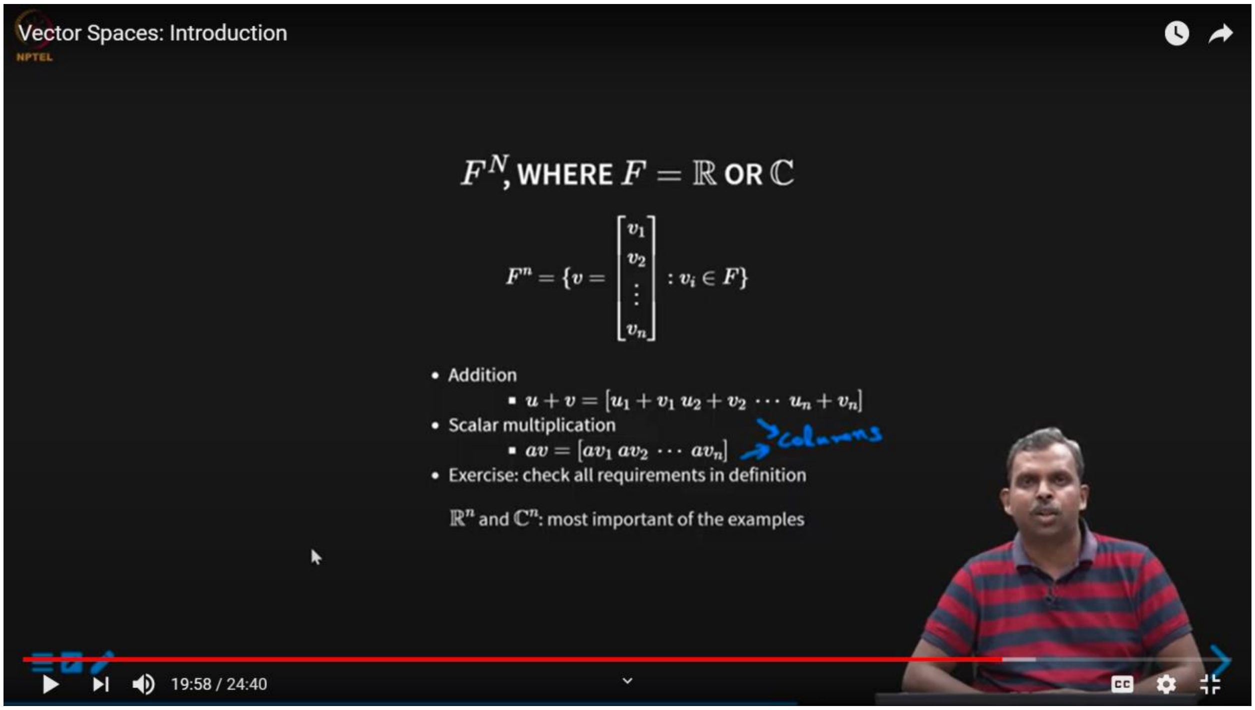

So there is a little bit of a thing to point out here. Think of these two as columns, okay? Think of these two as... put them as columns. To save space, I have written them as rows, you have to think of them as columns, okay? So one of the first exercises I'm giving you as part of this course is - given these definitions, go back to the previous slide, look at all those requirements and see if you can prove all of them, okay? You can check that all of them are satisfied, okay? So this R^n and C^n are the most important of all these examples okay? So many of the examples… We will primarily look at only these examples, there is a particular reason for it. We will see that later also okay? So this is the easiest, the simplest example, an example hopefully you are familiar with, you have done enough operations with this vector space.

The next example is polynomials, okay? So let's say I look at the set of all polynomials of that form okay? What is the form? $v_0 + v_1x + v_2x^2 \ldots + v_nx^n$. And each of these v_i's are scalars coming from this field F which could be R or C right? It could be real numbers or complex numbers. Now it turns out this set is also a vector space. I can think of it as a vector space. I can think of these polynomials as vectors. Why is that? I mean before you... don't get scandalized by why is this polynomial a vector and get confused in all these things. What do I care about? I don't care what that object is, right? You could call that object anything you want, as long as I can add them, as long as I can scale them I can think of them as a vector. That's all is my condition right. Can I add two polynomials? Yeah eminently, you can easily add them to get another polynomial which is in the same set. Can I multiply a polynomial with a scalar? Yeah of course, you can multiply each coefficient. And all the rules will be satisfied, okay?

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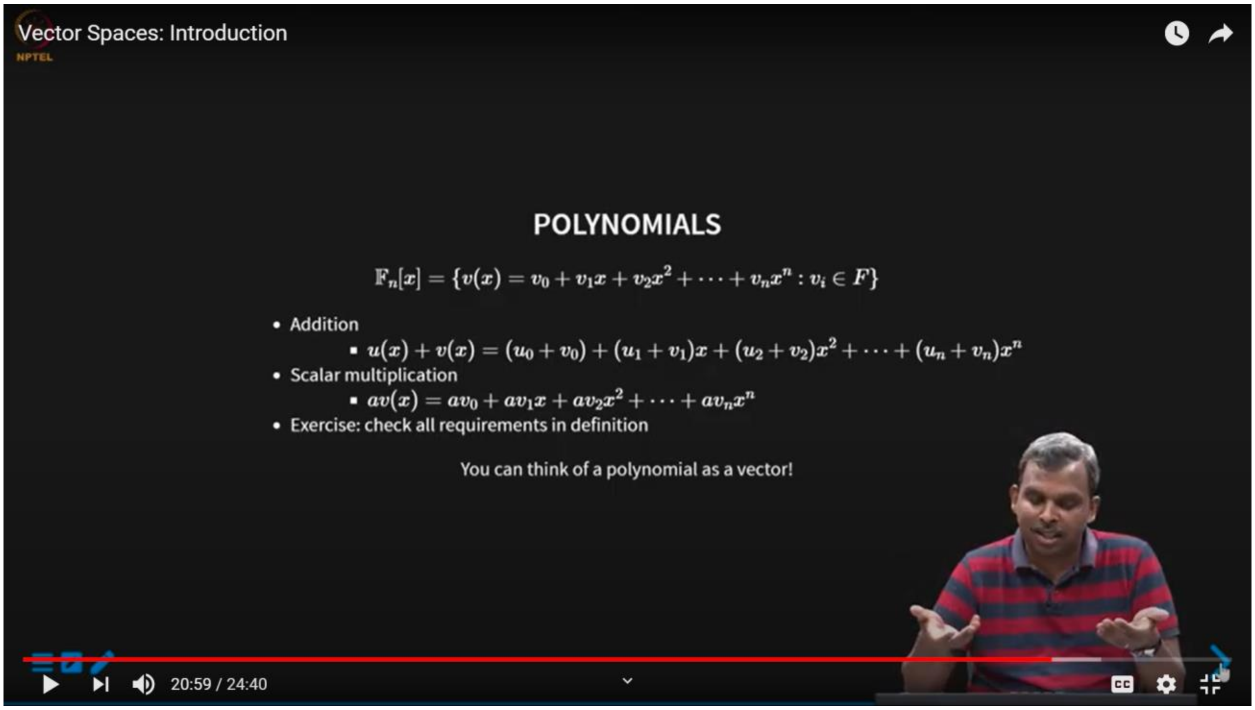

You can go back and check that. That's another exercise for you. You can check that this is a valid vector space, okay? So you can think of a polynomial as a vector, right? A vector is not something special, it's just something with which you can scale and add, right? So as long as you can do that, you have vector spaces.

Here's another example? Matrices, okay? So normally you think of matrices as being something very different from vectors, but it turns out matrices also satisfy the properties of vectors. You look at the set of all matrices, you can add two matrices, you can multiply a matrix with a scalar. You can go ahead and check that all these requirements are satisfied, okay? So you can think of a matrix as a vector. In fact the image I showed you before, that's also a vector, right? Image is a vector right? If you think of a grayscale image, it's just the intensity in each position, it's just a matrix. So it's also a vector okay? So all of these things can be thought of as vectors in some way.

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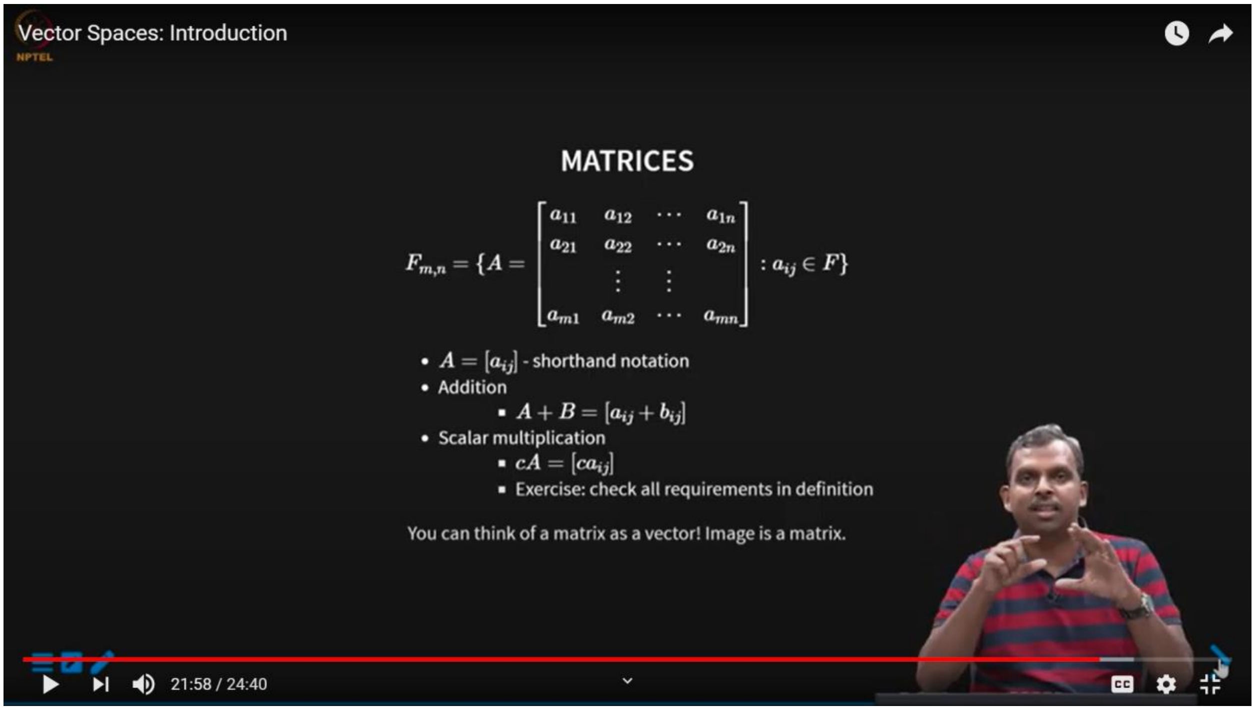

So here's the other final example that we had which was these functions. Functions from the real line to the real line. And things like $\sin x$, $\tan x$, x, $\bar{x}$, e^x, whatever, $\sqrt{x}$... I mean anything, log x, anything you can think of.

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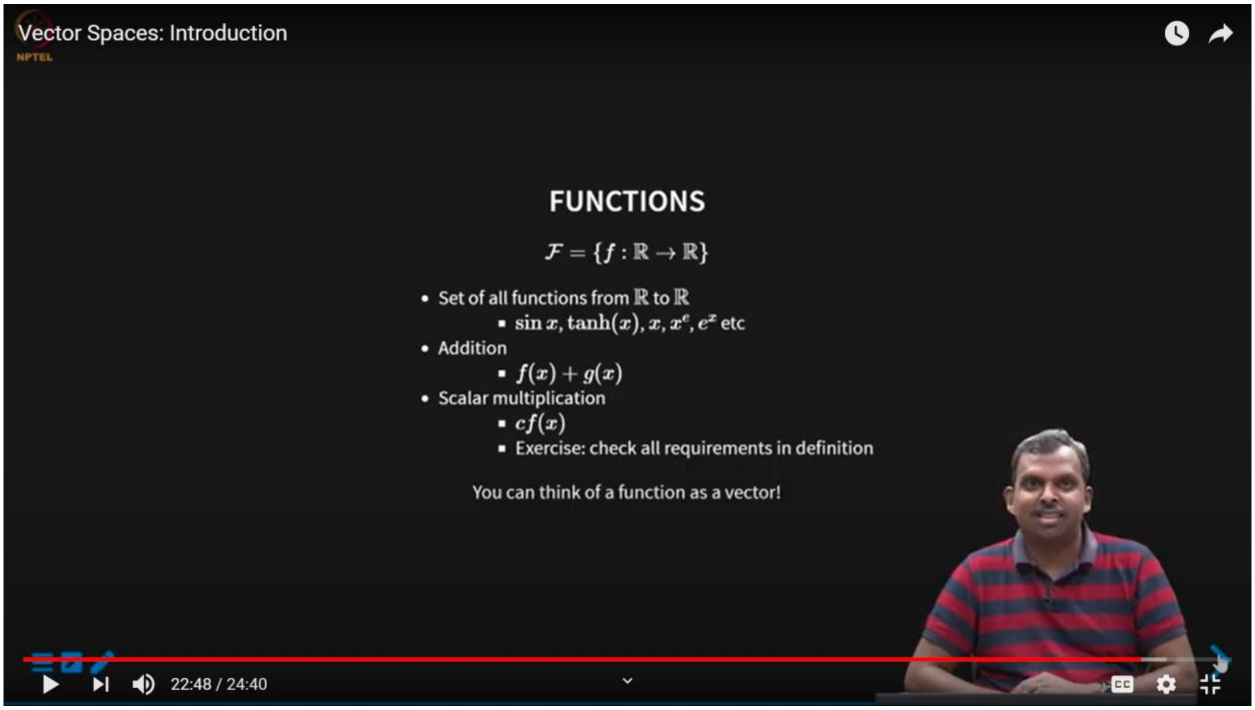

So these are also like vectors because you can add them, right? You can give any two functions, you can add them and given a function you can scale it with the constant. Both these things you can do. You can go ahead and check that these definitions satisfy all the requirements that you have in the definition of vector spaces. These are exercises for you. So you can think of a function also as a vector. So at the end of this lecture you can also go back to your... that little question I asked you - which of the following is a vector, and try and answer that question given that you now know the definition of a vector space and what makes a vector space, okay?

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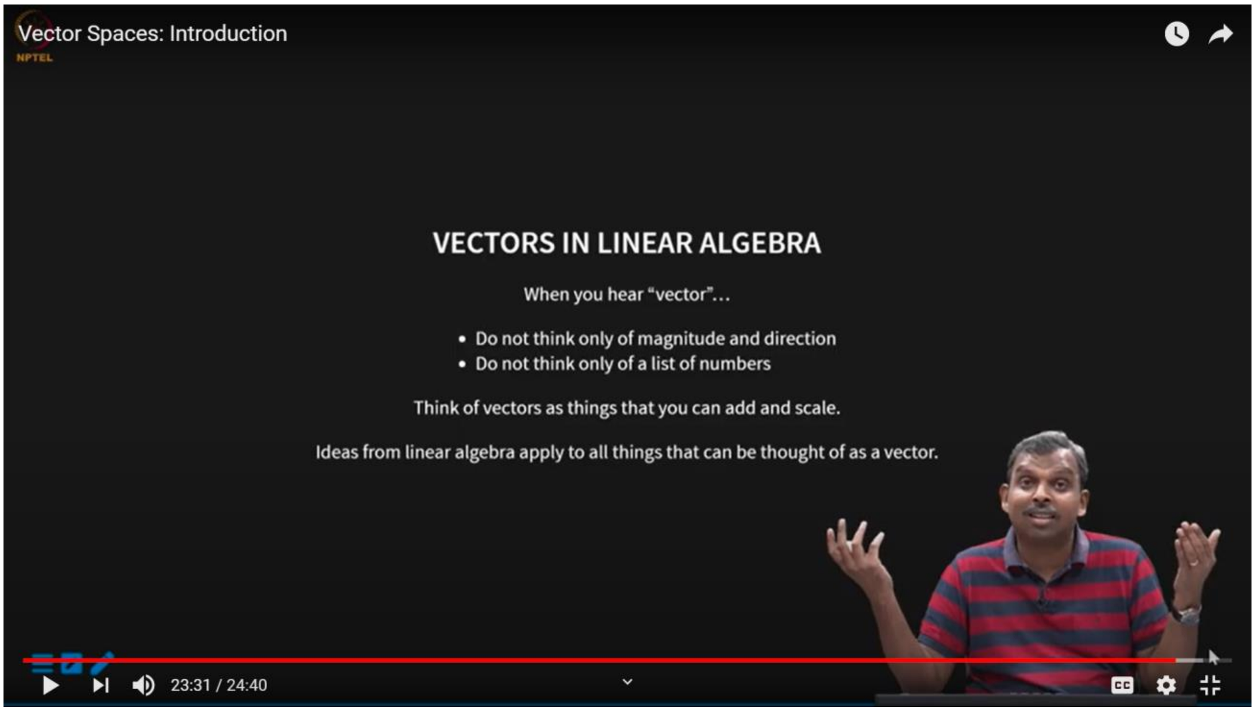

So here is the last slide of my first lecture. What are vectors in linear algebra? Ok so when you hear vector in physics you always think of magnitude, direction, okay? I don't know if you watch cartoons, there is this famous character called Vector in Despicable Me and he says that very clearly - I have magnitude and direction, okay? So is that the only way you need to think of vectors? Another way that a lot of people think of vectors is, it's a list of numbers, right? So is there other things, are there other things that could be thought of as vectors? Is there any benefit to doing things like that?

It turns out yes, okay? So you have to think of vectors in this course as anything that you can scale, and any two things you can add then it becomes a vector for you, okay? As long as those properties are satisfied... Remember the properties. Addition has to be an Abelian group, multiplication with identity has to be, has to just not change the vector and the distributive properties, right? All these properties have to be satisfied. Then you have a vector space and anything that does that is a vector for you okay? So what is the power of this abstraction? That once you abstract it like this and deal with vectors in an abstract way, any non-trivial result you derive for vector spaces applies for all the things that are vectors. So you don’t have to go separately prove it for functions, prove it for matrices, prove it for images or anything like that. You just prove it for vectors then whatever you want to do here will apply, okay? So that's the wonderful thing about doing abstract things in linear algebra, thinking of abstract vectors and deriving results, okay?

So that is the end of this lecture. I will see you again in the next one. Thank you.