Applied Linear Algebra
Prof. Andrew Thangaraj
Department of Electrical Engineering
Indian Institute of Technology, Madras

Week 05
Invariant subspaces, Eigenvalues, Eigenvectors

We have studied about polynomials and roots and we are now ready to jump into linear algebra once again, okay? What is the connection, why do we need all this? This is the crux of what we are going to study in this lecture. Invariant subspaces, eigenvalues and eigenvectors, okay? So these are very important, very crucial, central ideas in the study of linear operators. This is what makes operators more interesting and much more easy to work with and deal with. Eigenvalues, eigenvectors are so crucial in applications. They show up all the time, okay? So you have to have a very good understanding of where they come from. So let us start looking at it. We will study them in multiple ways. We will spend at least two, three lectures on them, look at various applications, how to think of them, where they, what is the correct way to imagine these things, and this notion of invariant subspaces, their connection to eigenvalues and eigenvectors is the central idea in this whole eigenvalue definition, okay? So let us jump into it.

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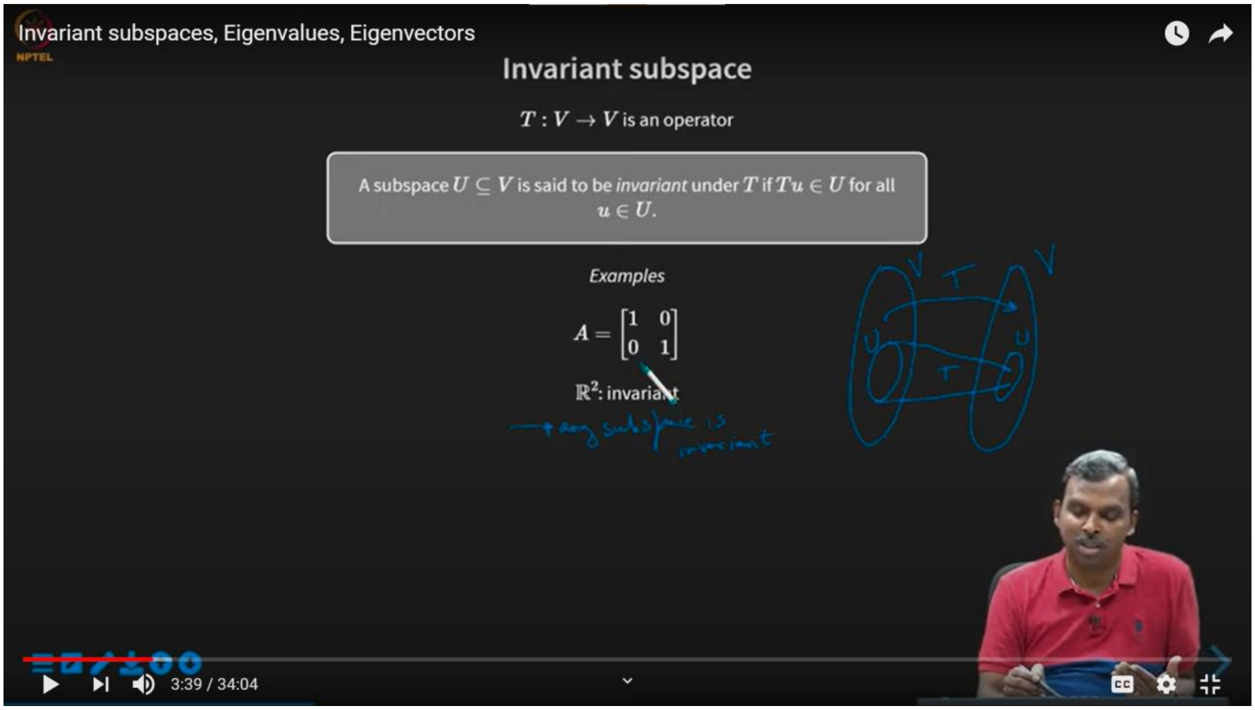

A quick recap. It is the same as before so I am not doing much of a recap in this lecture. So let's jump into invariant subspaces, okay? So what are we studying? We are studying operators from V

to V. There is a vector space V over a field $\mathbb{F}$ as usual and there is an operator, linear map, linear operator which goes from V to V, okay? We will say that a subspace of V is invariant under this operator T... It's a sort of an obvious definition, we say invariant, it doesn't vary, it does not change. If the operator acting on that subspace results only in vectors inside that subspace, okay? So if for every u in the subspace U, Tu belongs to that same subspace, okay? So that is the picture. If you want a picture, you can draw a picture. So I have v, and then I have another copy of v and I have my T going from V to V in general. But guess what? There is a subspace U and I am calling it invariant, the same subspace will be here also, right? So T should not leave U, okay? So while T takes V to V, this subspace, invariant subspace U is special because T after acting on vectors in U does not get outside of U, okay? It keeps you inside U, okay? So this is sort of interesting.

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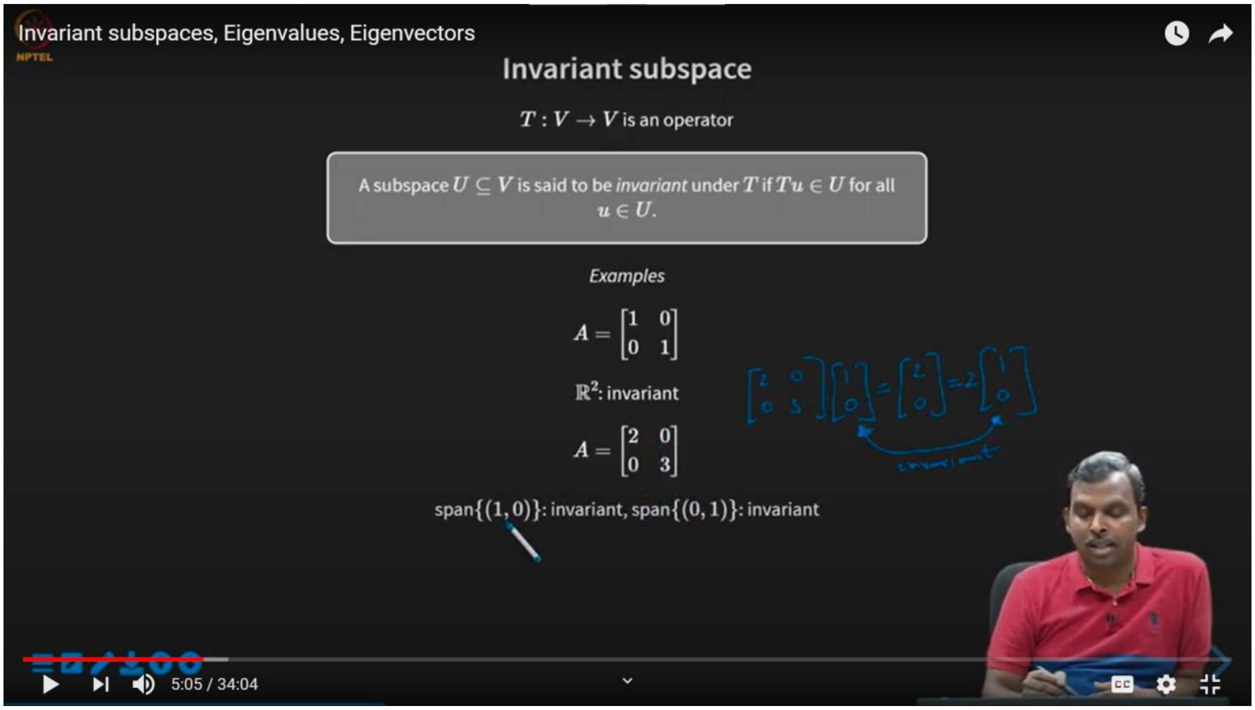

Now the first example I have put here is a very simple example. I am talking about the identity map, right? [1 0; 0 1]. I have shown you that $\mathbb{R}^2$ is invariant but that's a sort of a trivial example, right? For any operator, the entire V will be invariant, okay? So it is not so interesting. But what about this identity map? In fact for the identity map, any subspace is invariant, isn't it? What does the identity map do? It does not do anything. It maps everything to itself. So you take any subspace, what are subspaces in $\mathbb{R}^2$? Lines through the origin, right? So every line through the origin, identity map remains the same, okay? So identity map is sort of like a trivial example. And notice whenever we looked at operators, we like maps with, like this, right? So the operator like this is very easy to understand. Identity, lots of zeros in the off diagonals, only the diagonal is non-zero. We really like that and we already see that that form is very helpful in terms of invariant. So when you have identity map and the identity matrix, lot of subspaces seem to be invariant, so maybe, you know, or we are looking for such things... So maybe this invariant subspace has some connection. We'll see this connection soon enough. But this is important to know, okay? So this is the first example.

The next example is again diagonal. Notice we like diagonal matrices, right? Diagonal matrices describe very simple operators, okay? Diagonal matrices are nice. Once again we see that there are a lot of invariant subspaces, okay? In fact if you look at $span\{(1,0)\}$, okay? The x-axis. That's invariant, okay? Span of $\{(0,1)\}$, the y-axis, that's also invariant, okay? So notice what happens. So how do I sort of confirm that this is invariant? You can, you can check this. It's not very hard. So you see it's diagonal. When I multiply with $(1,0)$, what happens, okay? I get $(2,0)$ which is 2 times $(1,0)$, okay? So this is what is most important. So any, any vector in the x-axis is going to be some x times $(1,0)$. So if I operate $[2\ 0;\ 0\ 3]$ with $x(1,0)$, what am I going to get? $2x(1,0)$. So I do not go outside of the subspace, okay? Invariant. Same thing with $(0,1)$, okay? So if you multiply by $(0,1)$, you are going to get $(0,3)$ which is $3(0,1)$. So it becomes invariant. So we see that this is a diagonal matrix. Very simple with lots of zeros and it has lots of invariant subspaces obviously. One can notice that they are there, okay?

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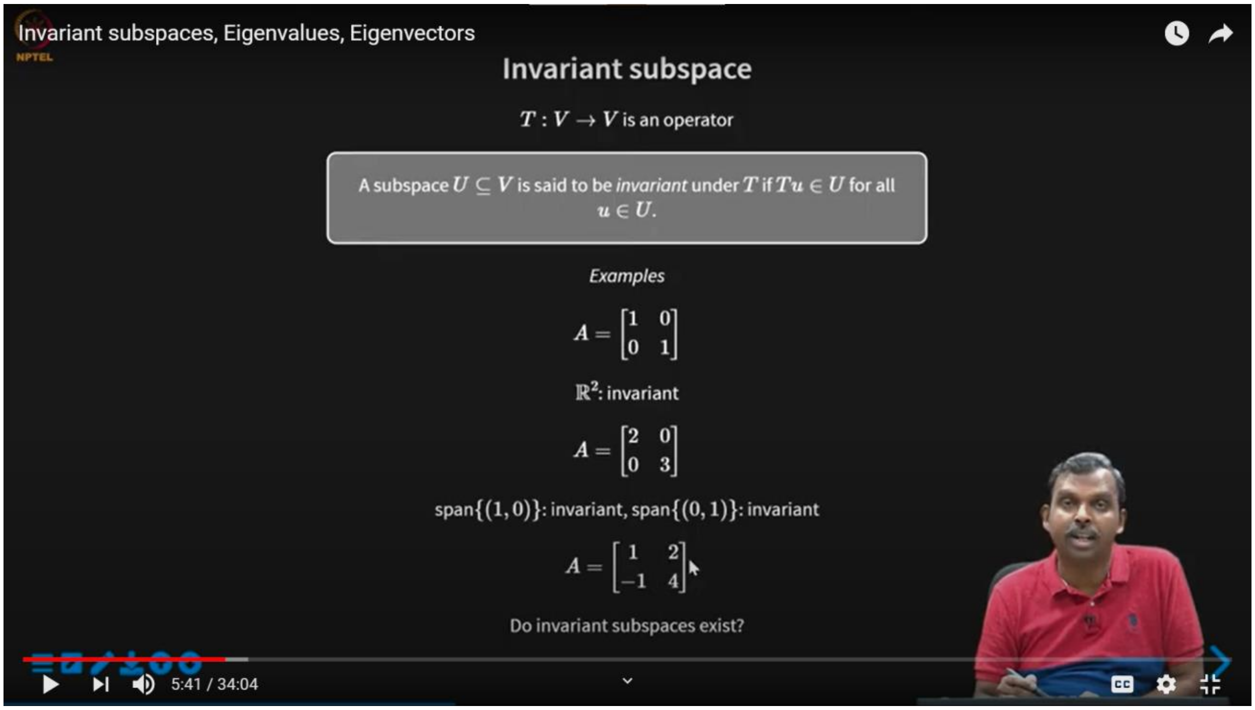

Now what about other types of cases? What about a matrix like this? $[1\ 2;\ -1\ 4]$? Do we have invariant subspaces, okay? How do we find invariant subspaces? Is it easy to find? And do invariant subspaces help? Will invariant subspaces help you in simplifying the matrix associated with this linear map? Will you get something better, okay? So let's see this example, this particular example a little bit more closely. I will give you the answer for this and we will talk about why that helps etc. and then we will look at how to find those invariant subspaces and look at them, define things properly and do it properly. So here is the answer. And why invariant subspaces, why they are interesting. So it turns out for this $[1\ 2;\ -1\ 4]$ there are in fact two different subspaces which are invariant, okay? So what are the invariant subspaces? So let us look at the first one. $(2,1)$ apparently is invariant, okay? So let us check that. Can you check it? It's not very hard. So if you do this $(2,1)$, what am I going to get? I get $(4,2)$ which is $2(2,1)$. So you see that this is invariant, isn't it? You start with any multiple of $(2,1)$, you are still going to get a multiple of $(2,1)$, okay? Same thing with $(1,1)$. You can check that when you multiply with $(1,1)$ you will simply get $(3,3)$ which is $3(1,1)$, okay? So we see that this matrix, while it is not apparent immediately that, maybe you cannot quickly guess $(2,1)$ and $(1,1)$, I am not expecting you to guess. By the end of this lecture, you will be able to find this. But still at this point even though this does not have so many zeros and it is not very clear how I got this, but somehow once I give you, you can check very easily that these are indeed invariant subspaces. There are two of them. Two one dimensional invariant subspaces. Now what is the big deal? What happened because I had these invariant subspaces? Here is what happens when you have lots of one dimensional invariant subspaces.

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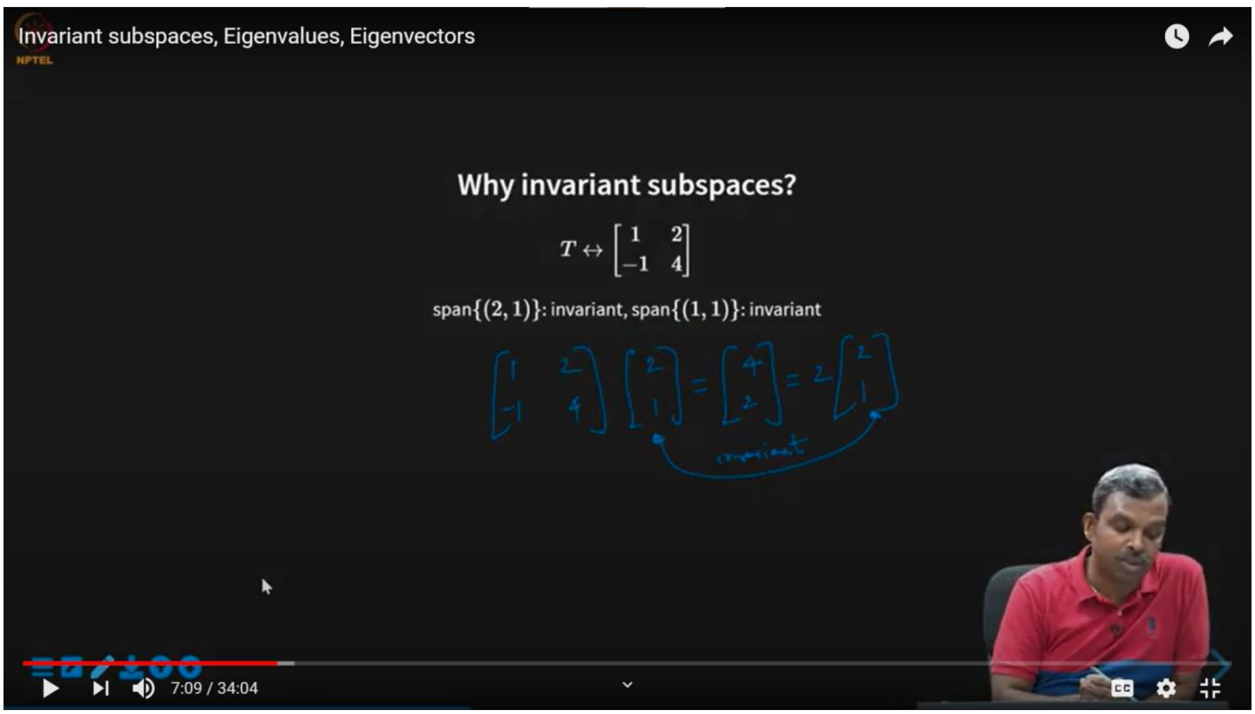

Notice what happens. You can do a change of basis. How do I change bases? We know how to change bases, okay? For this operator $[1\ 2;\ -1\ 4]$ in the standard basis I'm going to change basis to $\{(2,1),(1,1)\}$. Notice where I got $(2,1)$ and $(1,1)$ from. Those were the basis vectors of the one dimensional invariant subspace. Notice what happens when I do $\{(2,1),(1,1)\}$. In these spaces the linear map T, the operator T becomes diagonal, okay? $[2\ 0; 0\ 3]$. It's very easy, it's obvious, right? What is the first column of the matrix? It is, right, the first basis vector transformed by Tv_1 represented in the basis B, okay? If this v_1 is invariant, what is going to happen? It is completely going to get scaled and when something gets scaled, I get a diagonal matrix representation, okay? So you can see it will be just $(2, 0)$ and $(0, 3)$. So everything is only getting scaled and you get an obvious diagonal matrix, okay? So these invariant subspaces in particular, it seems like these invariant subspaces will give you lots of zeros on the associated matrix when you go to the suitable change of basis and that's very promising. And this is something that we wanted to do, right?

When you are given a linear operator, maybe it has a lot of non zeros off-diagonal and you are confused about what it is doing to your various coordinates. It turns out if you find invariant subspaces, there is some hope, okay? And notice this is the same basis, right? Input basis and output basis is $\{(2, 1), (1, 1)\}$. I am not changing the input basis and output basis, I am keeping it the same and I get $\{(2, 1), (1, 1)\}$. When I change to that basis from this basis, I get a diagonal matrix $[2\ 0;\ 0\ 3]$, okay? So that is very nice to see. And that's very promising. So these invariant subspaces are something very important to study and they're very, very, very crucial to understand operators, okay? So let's get going and figure out how to find these things. How did I find $(2, 1)$ and $(1, 1)$? I will show you how I did it. It's quite easy. Let's see that and that will lead us naturally to this definition of eigenvalues and eigenvectors, okay?

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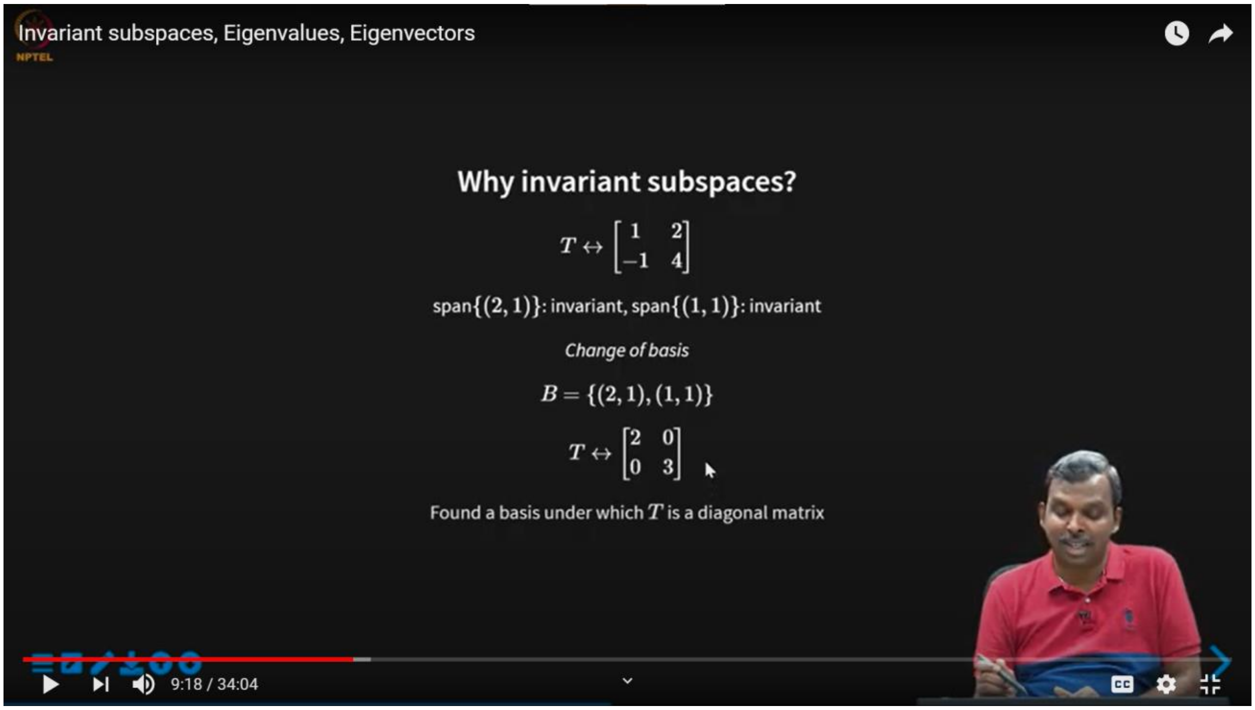

So eigenvalues are closely associated with one dimensional invariant subspaces. So let us do the setting carefully. I have an operator $T: V \rightarrow V$. I will assume it’s finite dimensional and there is a matrix A associated with the linear operator T in some basis. You pick your favorite basis and you have a matrix A, okay? So let us say you have a one dimensional invariant subspace, okay? So $span\{v\}$, right? So there will be one vector v and its span is the one dimensional invariant subspace let us say, under T. Then what will happen if I do Tv? Tv has to be equal to λv, okay? So there should be some λ such that Tv is equal to λv, okay? What about the other case? Supposing v is a non-zero vector, okay? So v is non-zero. So let me also put that down here. v is non-zero, okay? And so I have put if and only if here. So suppose v is non-zero and $Tv = \lambda v$. Then what happens? $span\{v\}$ is the one dimensional invariant subspace. It's easy to check, okay? So you notice that having a one dimensional invariant subspace is similar to looking at this equation $Tv = \lambda v$, okay? So these are the same things, okay? So and that is crucial in our upcoming definition of eigenvalues, okay? A scalar λ, eigenvalue is a scalar, a scalar λ from the field $\mathbb{F}$ is called an eigenvalue of T if there exists a nonzero vector v such that $Tv = \lambda v$, okay? There should be a non-zero vector v which when T operates on it only gets scaled, does not change to something else. You can draw a picture if you like. You can visualize if you like. I want a vector v, a non-zero vector v which when T operates on, you should get a scaled version, you should not get some other combination, no other combination is allowed. Every coordinate should only get multiplied by the same constant, okay? So if that happens, that constant value λ is called an eigenvalue of the operator. So notice I am talking about operator here, not any particular basis representation. We will come to that soon enough. But this is what the property is and you can see eigenvalues are closely connected to one dimensional invariant subspaces. So in short, if I can find an eigenvalue, I have found a one dimensional invariant subspace. And if there is a one dimensional invariant subspace there has to be an eigenvalue, okay? Both of these are connected, okay? So hopefully this definition made sense.

So naturally the next question is, do these exist? Do eigenvalues exist? So let us look at this equation a little bit more closely. If $Tv = \lambda v$ and $v \neq 0$, you can push λv to the other side, you're going to get $(T - \lambda I)$. Now notice this identity map enters the picture, times v equals zero. So this is the same as that you multiply v inside, you get $Tv - \lambda v = 0$, right? Same equation, and $v \neq 0$. Now this is possible if and only if $T - \lambda I$ is non-invertible, right? So see, notice $(T - \lambda I)v = 0$ and v should not be 0, which means the null space of $T - \lambda I$ should be not empty, okay? It should have a non-trivial null space. Only then this is possible, right? And which is also equivalent to $(T - \lambda I)$ being non-invertible, right? We know we are in, you know, square, finite dimensional world. So this is equivalent to this, isn't it? So this is a very simple condition. So λ is an eigenvalue if and only if $(T - \lambda I)$ is non-invertible, okay? So that is a very nice result. $null(T - \lambda I)$ has to be non-trivial. Only then λ is an eigenvalue, okay? So that is nice to see.

Now this non-invertibility we can deal using determinants very easily, right? So $(T - \lambda I)$ is non-invertible if and only if $det(A - \lambda I) = 0$, right? A is a matrix with respect to some basis. Now

I am going to some basis. So far I was only with linear maps. Suddenly I am picking a particular basis. We will see quickly that this is not a problem. But anyway, but this is something I can do. I can pick a basis then look at determinant of, pick a basis, pick a matrix and then look at determinant of that matrix minus λI. That should be equal to 0. Now what is $det(A - \lambda I)$? Looking from the formula, looking at the cofactor expansion etc. you can easily see the $det(A - \lambda I)$ is a polynomial of degree n in λ, okay? So what did we go from? We started from invariant subspaces and defined something called eigenvalue. And what do, what are eigenvalues related to? Roots of a polynomial of degree n, okay? How do you find that polynomial? You go find a matrix for that linear map in some basis and then take $det(A - \lambda I)$, you will get a polynomial in λ of degree n. And that polynomial's roots, you go to the complex field, whatever field that you are in, you find the roots and those are all eigenvalues for the linear map T, okay?

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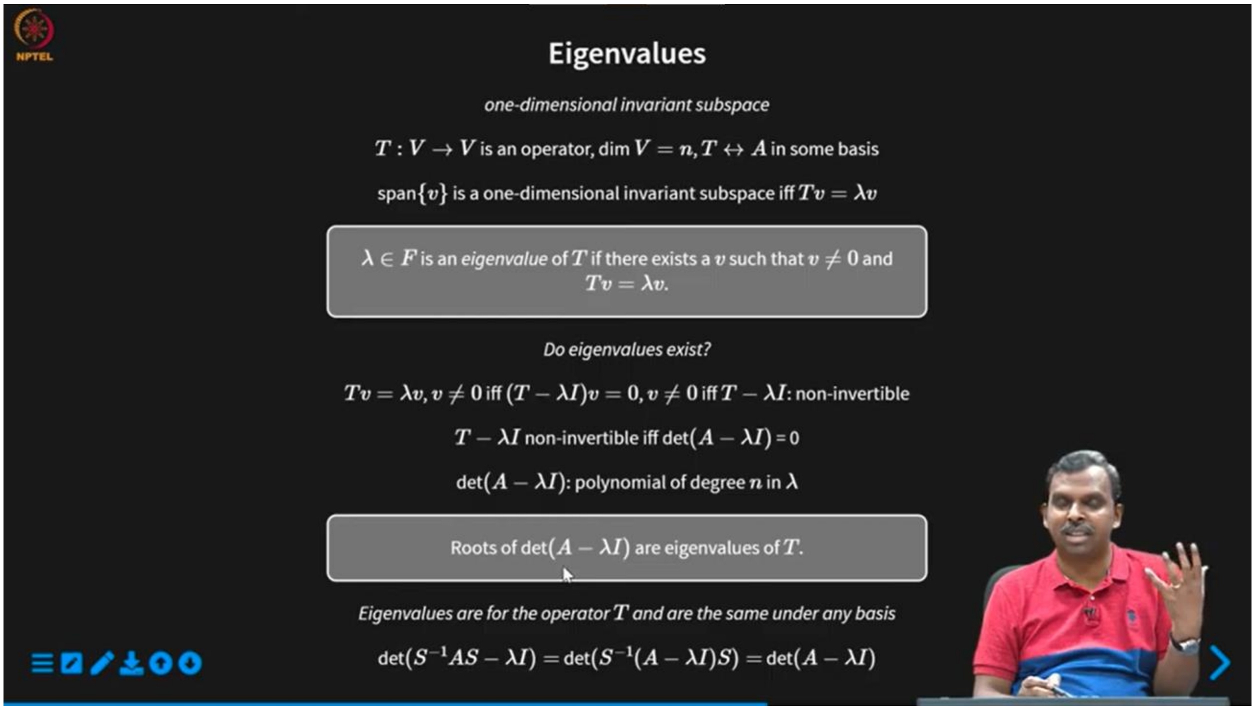

So now why do I keep saying linear map T while I have a matrix A? What if I change bases? What if this matrix changes? Will this polynomial change? It turns out the polynomial will not change. That also is something we can quickly see. The eigenvalues do not depend on the basis. Whatever basis you pick to find the matrix A, you will get the same eigenvalue. Why is that? Because if I, if instead of A, I did some other basis, I know it will be of the form $S^{-1}AS$, right? S is an invertible operator, change of basis operator. But if you look at this determinant, you can pull S^{-1} and S out on both sides. Because this is identity, I can pull it on both sides. And what do I know about determinants? Determinant of a product is product of the determinants. And what is determinant of S^{-1}? $1/det(S)$. That will cancel. I get the same thing, okay? So this $det(A - \lambda I)$ doesn't matter what A is, whatever basis you pick, you get the same polynomial, you get the same eigenvalues, okay?

So this one slide sort of captures a lot of things that are important about an operator. Given an operator, we know that one dimensional invariant subspaces are good. They seem to give us nice matrices. That's very promising. How do I find these one dimensional invariant subspaces? I have to find eigenvalues, okay? Because eigenvalues are directly related to one dimensional invariant subspaces. And how do I find eigenvalues? They are roots of a polynomial of degree n, okay? You go find the roots, you know eigenvalues are there, okay? So soon enough we will see a version of this without determinants also, okay? And in the next lecture, we'll see that. But this is the classical way in which eigenvalues are introduced. To show existence and how they. how one can think of them etc., okay? But we will come back and define eigenvalues in a different way. We will study them more closely than this. But right now this is a good way to introduce eigenvalues, okay?

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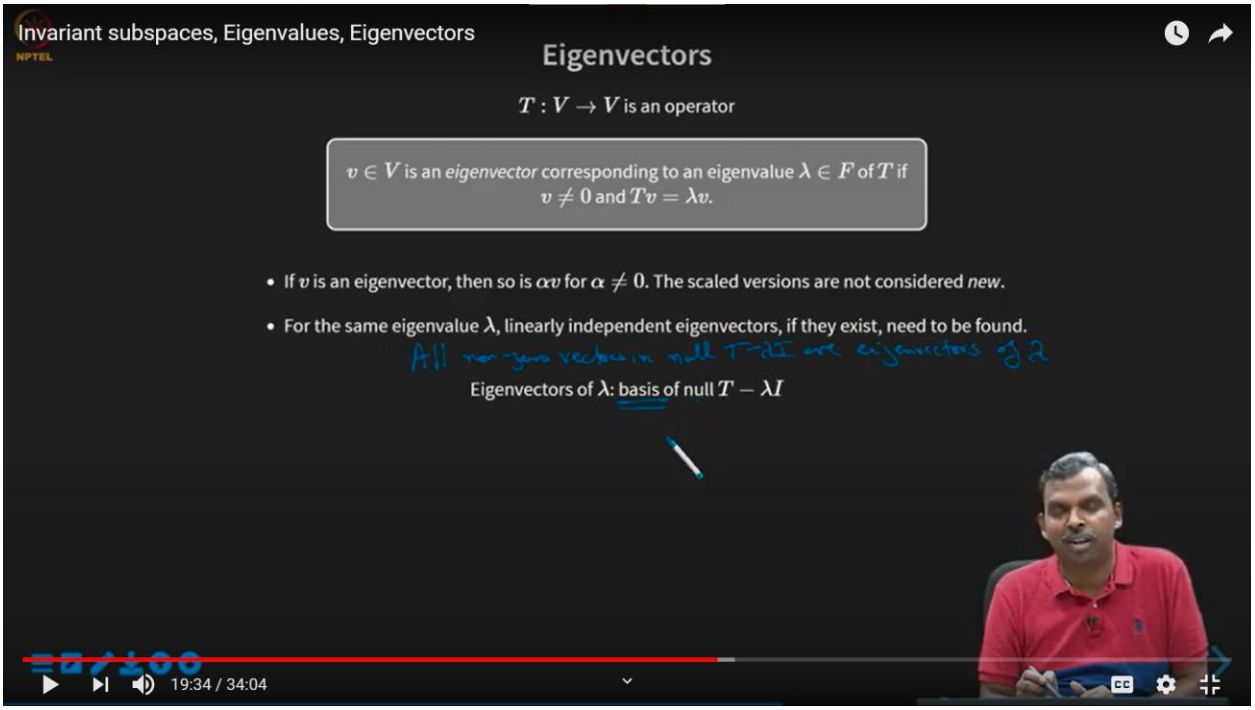

All right. What are eigenvectors, okay? So eigenvectors are very closely related to eigenvalues. It is almost there. I am putting it out as a separate definition here. If you have an operator, a non-zero vector v is called an eigenvector corresponding to an eigenvalue λ if $Tv = \lambda v$, okay? So that other v, right? So the previous equation $Tv = \lambda v$ we defined λ as the eigenvalue. v is the eigenvector, okay? So anyway, every time you think of an eigenvalue, you should also have an eigenvector. Without that there is no question of an eigenvalue, right? So the other part is the eigenvector, okay? So that is the eigenvector. So we see very easily that if v is an eigenvector, then any scaling of it is also an eigenvector, right? Of course you shouldn't put zero. Any non-zero scaled version of an eigenvector is also an eigenvector. So these non-zero scaled versions we will not distinguish. Just because we have multiple non-zero scaled vectors, we will not keep adding them to the eigenvector. All eigenvectors we can think of them as eigenspace or something like that. But they are not really distinct or new in some case. The reason is that they correspond, they are not linearly independent from this other guy, right? But for the same eigenvalue λ, you may find linearly independent eigenvectors. If you find linearly independent eigenvectors, it's still interesting, okay? Because then, you know, just by scaling you won't get them, okay? So you have to find linearly independent eigenvectors. We will, I will comment a little bit more on this soon enough.

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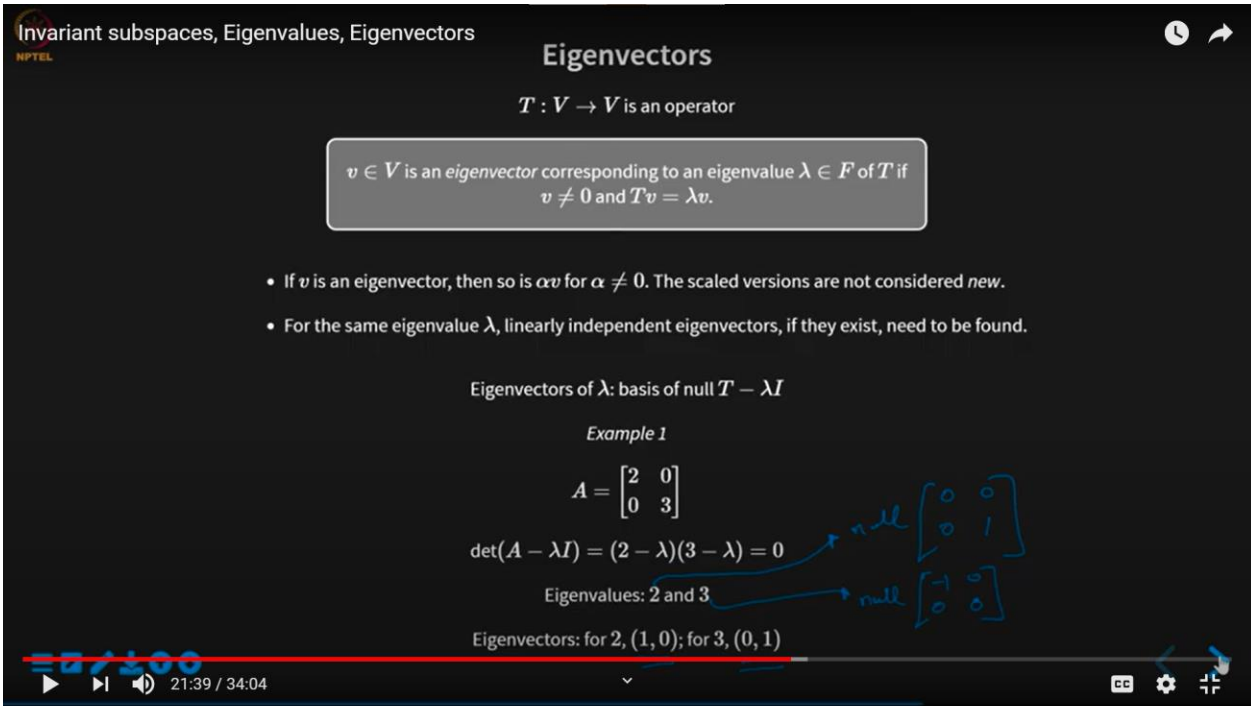

But notice one very easy thing. How do I characterize eigenvectors? Eigenvectors of λ once we understand, we only want linearly independent eigenvectors, is simply a basis for $null(T - \lambda I)$, okay? So why am I saying basis? If I want all eigenvectors, then all possible scalings also if I accept, then the entire null space of $(T - \lambda I)$ is the eigenvectors. Except for the zero vector, all non-zero vectors in... Maybe I should write that down. All non-zero vectors in $null(T - \lambda I)$ are eigenvectors of λ, right? So this is definitely true according to the definition. But, you know, when we write down, usually you do not want to write all possible eigenvectors, right? There might be infinitely many of them. So we simply stick to the basis, okay? Just write the basis and say that, you know, span of all these things is also eigenvectors, okay? Span of eigenvectors is also an eigenvector. So you can simply say basis of $null(T - \lambda I)$, okay? So given an eigenvalue λ, I know $(T - \lambda I)$ has a non trivial null space, right? You go find the basis of that null space, you get the eigenvectors, description of all eigenvectors, alright? So this is the way to find eigenvectors, okay? So eigenvalue, you solve polynomial roots, find roots for a polynomial. For eigenvectors, you do your familiar thing, okay? So you know an operator, you define the null space of an operator. You know how to find null space, right? You just use that same method, you will find basis for the null space, okay? So let's see a whole bunch of examples, okay? So we will see very simple examples. I will only do 2×2 at least in class. You can do maybe more examples with larger matrices. In your assignments, you may have more examples for getting practice. But anyway today you just go to a computer and ask it to give you the eigenvectors of a matrix, it will give you, okay? So you don't need eigenvalues and eigenvectors, you can find. So it's not a big deal to calculate these by hand. But they give you some, you know, the purpose of me doing it is to illustrate to you what are the various things that can happen, okay? So you will see some interesting things can happen.

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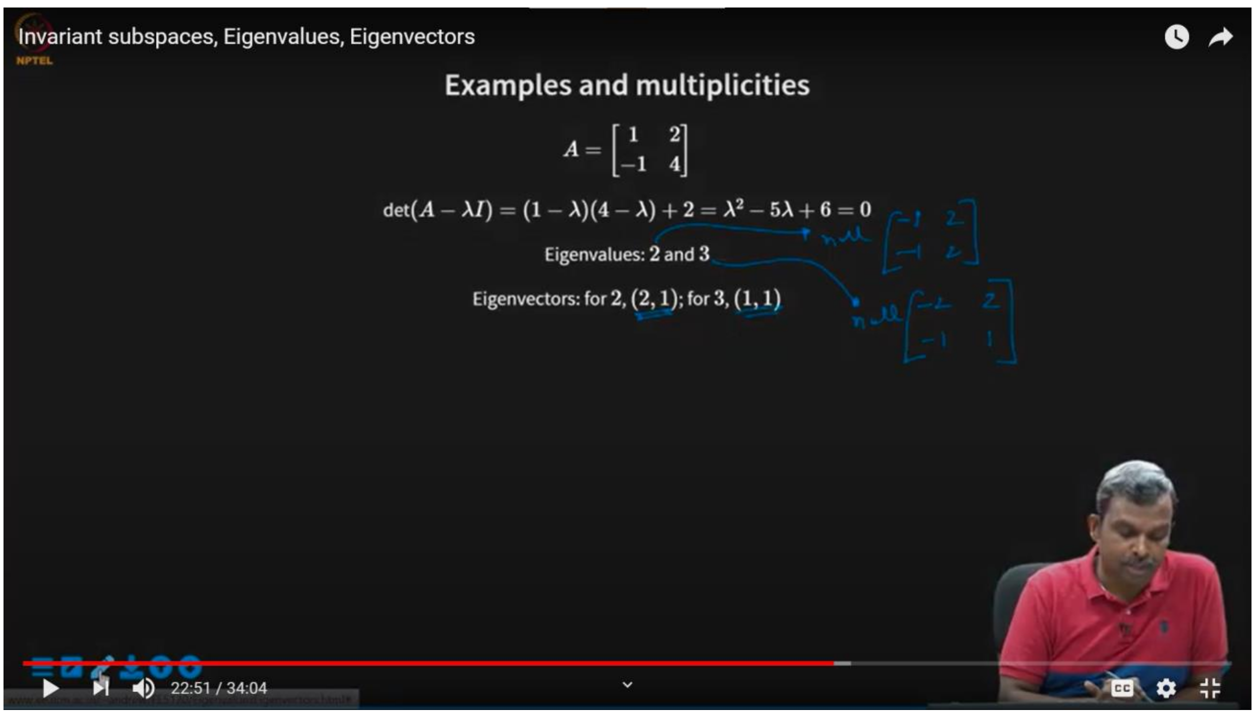

So let us begin with the diagonal matrix [2 0; 0 3]. $det(A - \lambda I)$ is $(2 - \lambda)(3 - \lambda)$. So the roots are simply 2 and 3, okay? One can easily see the roots are 2 and 3. And the eigenvalues are 2 and 3, okay? So easy. How do you find eigenvectors? For eigenvalues corresponding to 2, okay, so for eigenvalue corresponding to 2, it’s the null space $(A - 2I)$. What is $(A - 2I)$? [0 0; 0 3], okay? And for 3 it's the null space of [2 0; 0 0]. Oh no, not, I think I made a mistake here, let me just do this. 1 here, -1 here, sorry? Okay? So $(A - \lambda I)$, right? So it is $(A - 2I)$. So I have to subtract 2 from the diagonals. So I will get 0 1 here. And for 3 I have to subtract 3 from the diagonal, so you get -1 0. So you can see the null space of this guy, the first one is clearly $(1, 0)$, okay? So this is $(1, 0)$ and for 3 it is $(0, 1)$, okay? So they have just dimension one null spaces.

You can see that the matrix has rank 1, no? Rank is one. Nullity is one. So you have dimension 1 null spaces and the basis is very easy to find in this case, okay? Very easy example, you know? I keep doing only easy examples in class, but the more difficult examples you will get in your assignments, okay?

The next immediate example we are going to take up is the more complicated example I did, if 2×2 can be considered complicated, okay? [1 2; -1 4]. You do $det(A - \lambda I)$, you get $(\lambda^2 - 5\lambda + 6) = 0$. And lo and behold you have the eigenvalues 2, 3, okay? So now solving for this may be slightly more complicated, okay? So you have to find, you know, for 2 it is the null of, okay, $A - 2I$. So you have to subtract 2 from the… Okay, I think $[-1\ 2;\ -1\ 2]$, okay? And for 3 it is null of, I have to subtract 3, so $[-2\ 2;\ -1\ 1]$, okay? So you can see these things work out quite okay. You can see that in the null space for 2, you have $(2, 1)$, okay? So if you multiply by $(2, 1)$, you get 0. For 3, the null space is $(1, 1)$, okay? So you can see these are rank 1 matrices and you get (1, 1). All right. So this is how you find eigenvectors, eigenvalues in simple cases.

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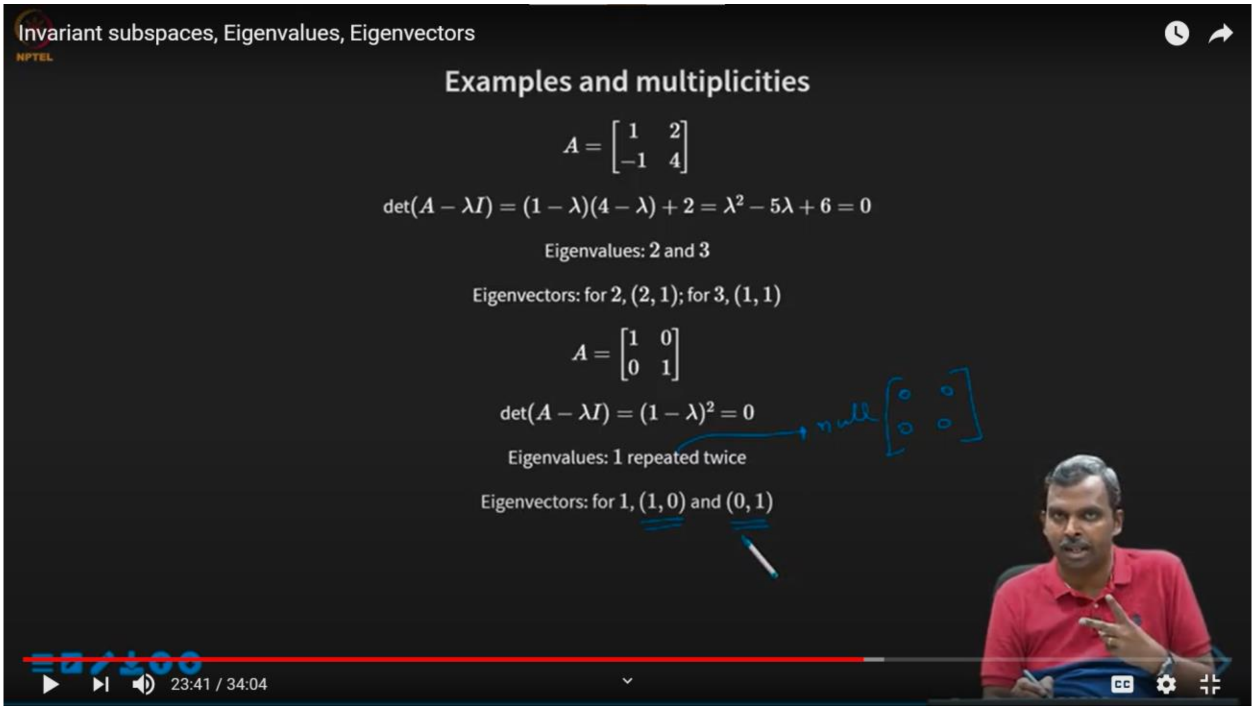

The next case I am going to take up is slightly more interesting. You slowly start getting more interesting cases. The identity matrix. Notice what happens in the identity matrix. My $det(A - \lambda I)$ has repeated roots now. $(1 - \lambda)^2$. And that becomes equal to zero. So eigenvalues are 1 repeated twice, okay? That's one. And what about eigenvectors? Eigenvectors for this is null space of $(A - \lambda I)$. So that's 1, so it is just [0 0; 0 0]. So rank is 0. Nullity is 2, so you have 2 basis vectors in the null of this guy. $(1, 0), (0, 1)$. You can take any other thing also. Might as well take

$(1, 0), (0, 1)$, okay? Interesting case. For one eigenvalue, we got two different linearly independent eigenvectors, okay? And the multiplicity was 2. That's good. So let's move on to a general definition, okay?

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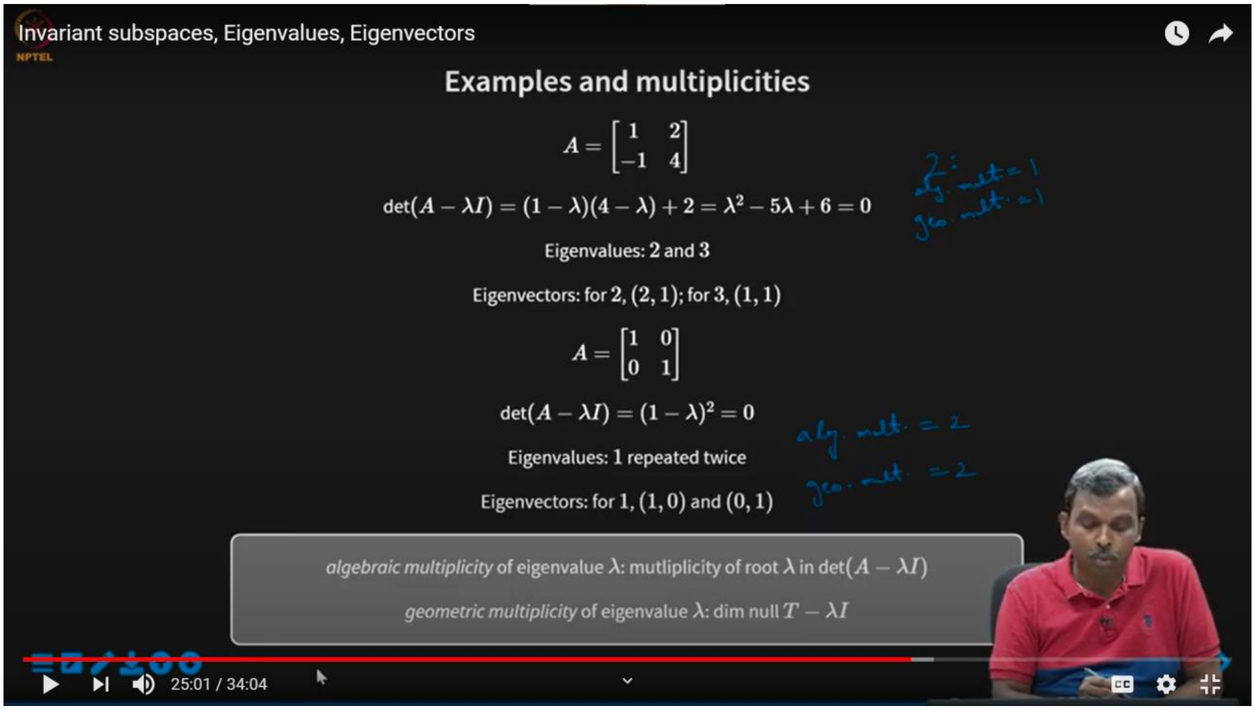

So let me give you a general definition. There are two types of multiplicities associated with a particular eigenvalue λ. One is what's called algebraic multiplicity. This is how many times λ appears as a root in the polynomial $det(A - \lambda I)$, okay? That's called algebraic multiplicity. Another multiplicity associated with the eigenvalue λ is what's called geometric multiplicity. That is basically: you have $null(T - \lambda I)$, right? $null(T - \lambda I)$ is, this is the subspace of all eigenvectors. The dimension of that is called the geometric multiplicity, okay? So in the above example, okay, if you go to this example, algebraic multiplicity equals 2, geometric multiplicity equals 2, okay? What about for this guy? For the eigenvalue 2? Algebraic multiplicity equals 1 and geometric multiplicity equals 1, okay? So these are just examples. You can see more such examples going on. Just definitions, nothing big here. There is a more interesting case and that is the next example I am going to take up.

Let us take up this simple little change from the identity. Instead of putting $[1\ 0; 0\ 1]$, I put $[1\ 2; 0\ 1]$ there, okay? But notice what happens to $det(A - \lambda I)$. You simply get $(1 - \lambda)^2$. Nothing changes because it's upper triangular. You use your determinant method, you know upper triangular matrices also it's just product of the diagonal elements. So the determinant is only $(1 - \lambda)^2$ for $(A - \lambda I)$. So eigenvalue still has algebraic multiplicity 2, right? 1 repeated twice.

Algebraic multiplicity is 2. But notice what happens to geometric multiplicity. I need to find for this guy $null([0\ 2; 0\ 0])$. Now rank is 1 here. So nullity is only one. So you will only get one linearly independent, one basis vector for the eigenspace. So in this null of $[0\ 2; 0\ 0]$, you only get $(1, 0)$, right? So the geometric multiplicity becomes 1. So now this can also happen, okay? Notice in the previous cases we had algebraic multiplicity and geometric multiplicity being equal. In this case geometric multiplicity became less than algebraic multiplicity. That turns out that can also happen, okay? So we will see these, more, more interesting results of this type we will see going along. But at least we see that these are all the calculations. So in general what do you do if you have an $n \times n$ matrix? You find out $det(A - \lambda I)$, find out the roots of λ, roots of that polynomial and then go find the basis of the null space, okay? So that you might think is the method.

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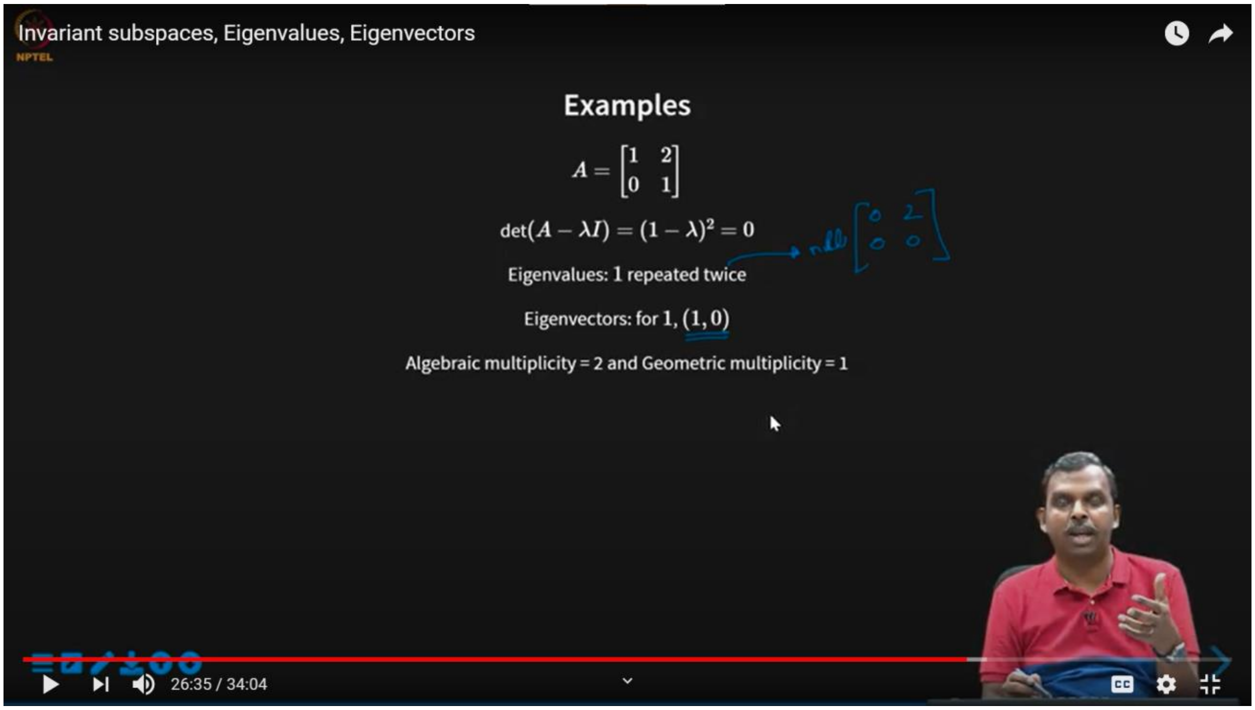

So it turns out the general case $n \times n$, you better use modern numerical methods. There are so many tools out there, so many, you know, software which can easily find it. And most of them don't do this route, okay? This finding the determinant, finding the polynomial which is, you know, $det(A - \lambda I)$, finding the roots of the polynomial and then solving, that's not the method that is preferred. There are many more powerful methods today and this is not a course on Numerical Methods in Linear Algebra so we're not going to see all that in great detail. But there are many more methods today which are powerful and in fact it's so powerful that you know, how do people find roots of polynomials? In many cases you find the matrix for which that polynomial is the $det(A - \lambda I)$ and then find the eigenvalues of that matrix, okay? So there are things like that. So in some sense, eigenvalues are more fundamental than the roots of the polynomial in some vague way because they can be, at least from a numerical point of view, right, let me not say fundamental of course. So from a numerical point of view, finding eigenvalues is much easier using direct other methods, okay? But the determinant is good for small cases and of course when you have to do hand calculations in quizzes, determinants are extremely useful. So we have to study the determinant approach first. So later on we'll come back and see how to think of eigenvalues, eigenvectors in a fundamental way without using determinants. We will see that in the next few lectures and you will see that is more illuminating, it gives you more ideas about where they really come from. But this notion of one-dimensional invariant subspace related to, you know, making matrix of the operator simpler, all of that and eigenvalues and eigenvectors and associating it with determinant is not a bad way to study eigenvalues to begin with, okay?

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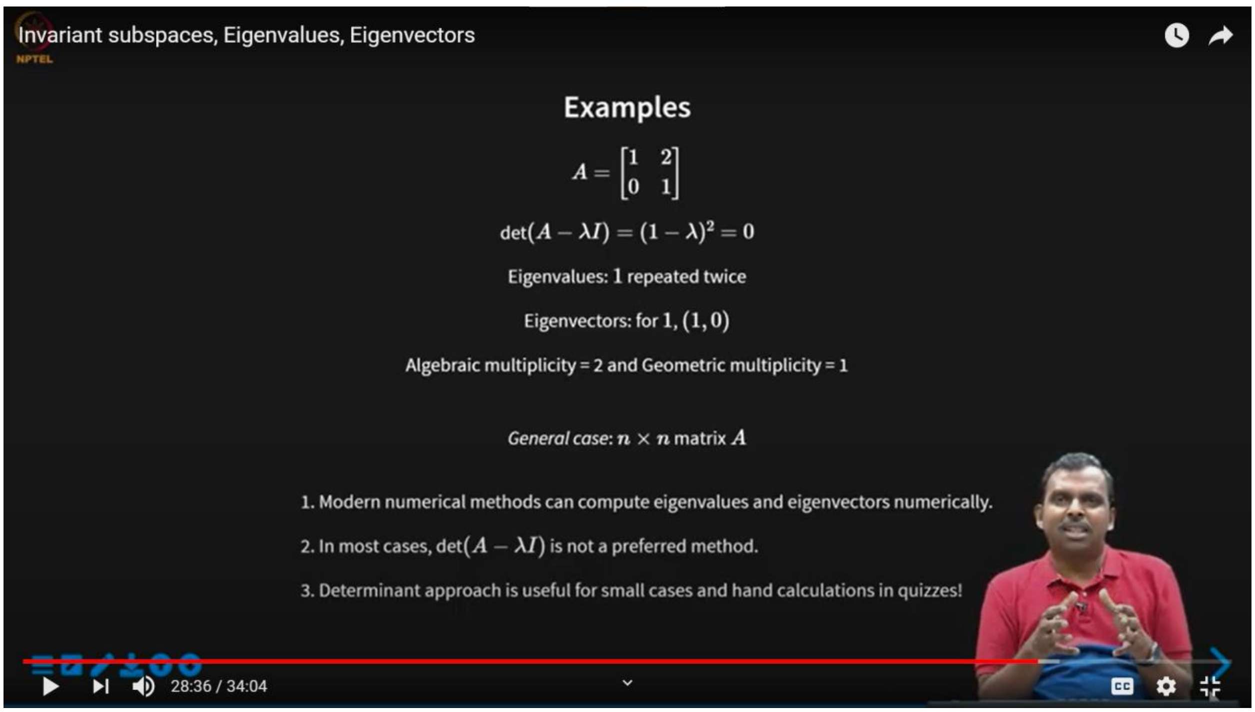

So, and like I said, I mean we are not just interested in computing eigenvalues, eigenvectors, we are also interested in understanding what is going on a little bit more, okay? For that we also want to see some theoretical results around eigenvalues, eigenvectors. The next result we'll see is a very interesting theoretical result. The first one we are seeing about eigenvalues, eigenvectors. I will leave you with this for this lecture. I will pick up from here in the next lecture. So here is a very interesting theoretical result on eigenvectors. This is about linear independence, right? If you have distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ and there are eigenvectors associated with each of these eigenvalues... Remember once they have an eigenvalue, there will be one eigenvector, that is assured, right? Why? Why is one eigenvector assured? That's how I define the eigenvalue. $(T - \lambda I)$ has to be non-invertible which means there is some non-zero vector in the null space. I know there is at least one eigenvector out there, right? So I have m distinct eigenvalues. I have m eigenvectors associated with them. It turns out if the m distinct, if the m eigenvalues are distinct, then the m eigenvectors are linearly independent, okay? It's a very nice result. The proof is not very complicated, I will walk you through the proof. But notice what this tells you. So if you have more and more eigenvectors, more and more linear invariant subspaces, they all cannot be linearly dependent. The invariant subspaces of an operator have to be linearly independent, okay? When you have these linear independence, again beginning to look nice, no? I mean those things then start to form a basis, you know, I mean all of this is becoming very, very interesting. So this is what's very critical. So let's see a quick proof.

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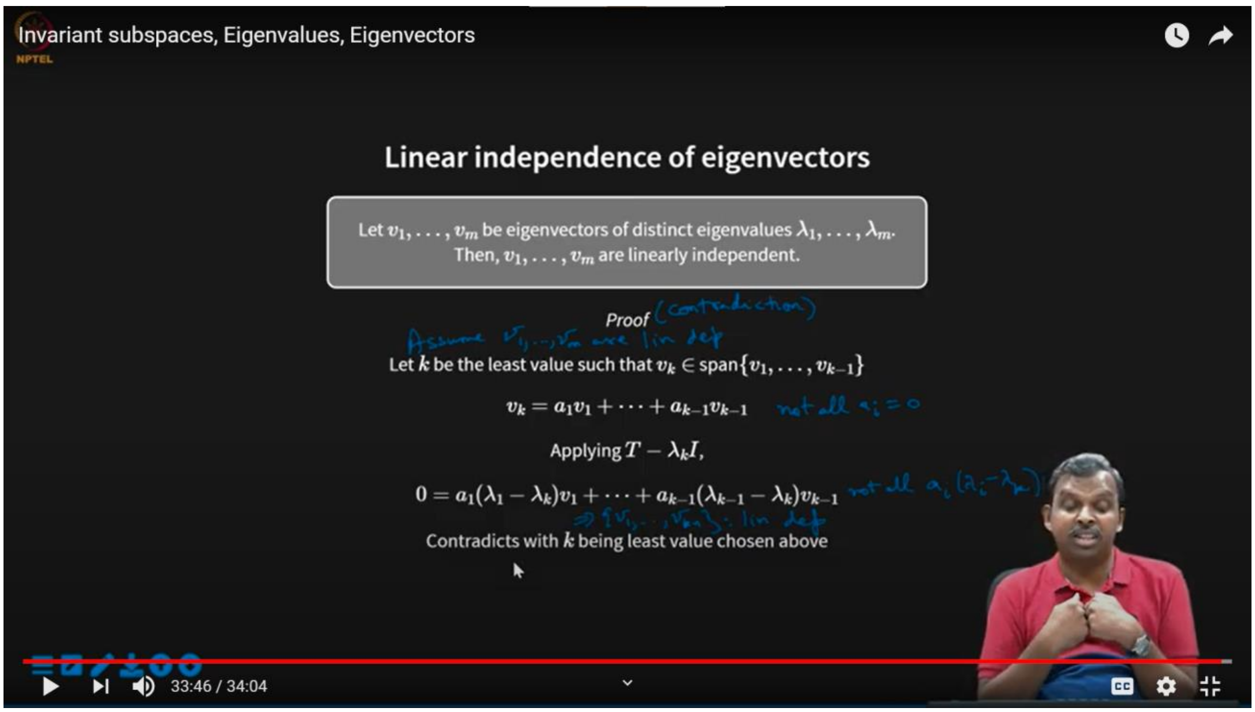

So the way the proof works is by contradiction, okay? Sorry. The proof is by contradiction. What is contradiction? If you want to show something is linearly independent, what do you assume? Assume v_1 to v_m are linearly dependent, okay? So proof is by contradiction. So you assume they are linearly dependent. Once you assume they are linearly dependent, what happens? You can use your linear dependence lemma, okay? So you can see how many times we use this linear dependence lemma, right? It is very fundamental and powerful. So I know that there will be a v_j such that $v_1, \ldots v_{j-1}, v_j$ lies in the $span\{v_1, \ldots ,\ v_{j-1}\}$. At least one j exists. Now out of all the j, I will pick the least such j and I will call it k, okay? So k is the least value such that v_k belongs to the $span\{v_1, \ldots, v_{k-1}\}$, okay? So sort of a critical choice here this k being least. And I will get my contradiction on that, okay? I will say that k cannot be the least, okay? I am starting with the least and it cannot be the least. So you will get a contradiction, okay? So why am I assured that such a value exists? Linear dependence lemma, you know this at least one. Out of all that I will pick the least. So now what does it mean? v_k is a linear combination of $v_1, \ldots, v_{k-1}$. Now I know all these guys are eigenvectors, they are not regular vectors, right? If they are regular vectors, I cannot do more. But they're eigenvectors. So notice what I am applying. I am going to apply $(T - \lambda_k I)$, okay? That operator. Notice that operator, that operator is very powerful. It's an interesting operator by the way. So $(T - \lambda_k I)$ I'm operating on both sides. What will happen to the left hand side? It will go to zero. What will happen to the right hand side? You will get this form that I have here, okay? $a_1 * (\lambda_1 - \lambda_k)v_1 + \ldots + a_{k-1} * (\lambda_{k-1} - \lambda_k)$. So notice not all the a_i's will be 0, right? That's the linear dependence lemma and this $\lambda_1 - \lambda_k$, all of these are non-zero. So out of this also there will be something non-zero, right? This $a_i * (\lambda_i - \lambda_k)$, at least one will be non-zero. Which means what? $v_1, \ldots, v_{k-1}$ is linearly dependent. If $v_1, \ldots, v_{k-1}$ is linearly dependent, you use the linear dependence lemma, there should be something else which is in the span and my k being the least is violated, okay? So $v_1\ to\ v_{k-1}$... Maybe I should write this down. So not all a_i equal to zero. So here also not all $a_i * (\lambda_i - \lambda_k) = 0$ because λ_i is not equal to λ_k, right? So the distinct value is very important. So that implies $v_1, \ldots, v_{k-1}$ is linearly dependent. If it is linearly dependent, by the linear dependence lemma, there should be a j between 1 and $k-1$ such that v_j is a linear combination of $v_1, \ldots, v_{j-1}$ and that violates, this contradicts this choice of k, k is the least such value, something like that cannot exist. And I have a contradiction. So any time v_1 through v_m is linearly dependent, I am going to have this contradiction which means v_1 through v_m have to be linearly independent, okay? So that's a nice result to have. So you see that these invariant subspaces, eigenvectors are becoming very, very interesting. Particularly when you have distinct eigenvalues, they are becoming linearly independent. And we will start using these things in the next few lectures to simplify the matrix corresponding to linear map, try to understand it better, try to get a better feel for what it represents, okay? So thank you very much.