Applied Linear Algebra
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Week 05
More on Eigenvalues, Eigenvectors, Diagonalization

Hello and welcome to this lecture. We are going to continue looking at eigenvalues, eigenvectors. Idea of diagonalization in particular will be introduced and talked about in detail. So this is sort of a continuation of what we did in the previous lecture where we defined eigenvalues, eigenvectors. We saw the notion of, I mean how to use this determinant to find this polynomial, find the roots and then solve for the eigenvalues and then solve for the null space to find the eigenvectors, okay? We saw what the definition was. We're going to continue that, we'll study it from a different point of view. You know, always studying things from multiple points of view helps when you're looking at the basics, okay? So let's get started. Particularly this diagonalization idea is extremely important. It's one of the pillars of the applications of linear algebra in most areas of engineering, okay? So let's carry on.

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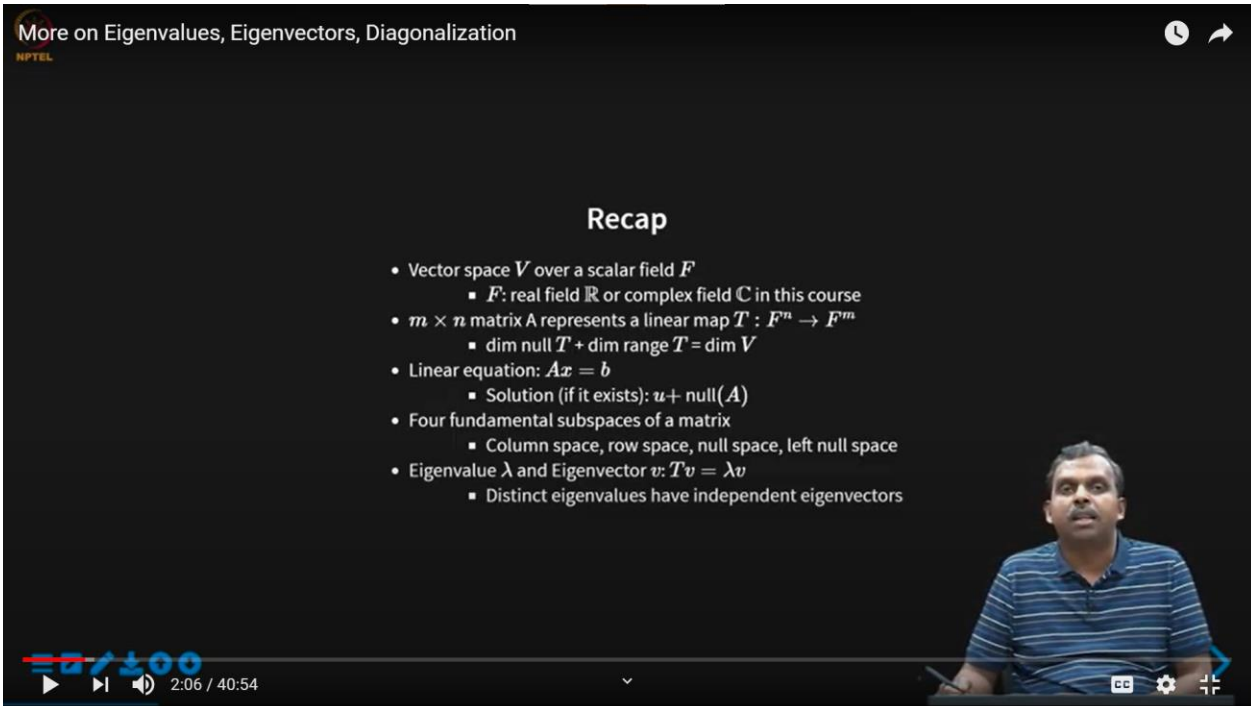

A quick recap. Once again let us go back to the very beginning of this course where we introduced vector space V over either the real field or the complex field primarily in this class. And then we saw a linear map, once you pick a basis, is represented nicely by a matrix. And then there are all sorts of interesting relationships between the null space, range space and all these other spaces associated with the matrix. And then this helps us solve linear equations in a very clean way. And the solution always has a certain nice form which you can quite easily find using elementary row operations, okay? And of course we looked at the four fundamental subspaces and their connections and that gave us a better understanding of the linear map. And then we saw that, you know, you can do elementary row operations, pick your bases. If you're allowed to pick input and output basis you get this row reduced echelon form which is really a very simple form for any linear map. Now when you go to operators which are linear maps from the vector space to itself, then you do not want to have that freedom of different input and output bases, you want to have the same basis. When you want to have the same basis, there are some restrictions. It is not possible to use elementary row operations the way we did. So we have to think of what is possible and that is how we ended up with this notion of eigenvalues, eigenvectors and they seem to be really really useful in simplifying the matrix corresponding to the linear map, okay? And we saw towards the end of the previous lecture this very interesting result that distinct eigenvalues, the eigenvectors, corresponding to that, corresponding to distinct eigenvalues are linearly independent. And that is a very promising and very interesting result and we will put such results to use in this lecture when we talk about diagonalization. But before we do that…

So the way we introduced eigenvalues and eigenvectors, it looked like determinants played a very crucial role. But in reality you can develop all of that without determinants, they do not really, they are not very fundamental to the theory of eigenvalues, okay? So what we will see in the beginning of this lecture is how to think of eigenvalues in sort of a linear map, linear operator fashion without bringing in the determinant. So that will be the first part of the lecture. And then we will move on towards diagonalization. Okay. So what is it that the operator can do which distinguishes it from a general linear map, right? So when you have an operator from a vector space to itself, right? The operator can be applied repeatedly over and over again to a vector, okay? So you have a non-zero vector v. You apply T to it once, you get Tv, if you had already ended up from V to W which is another vector space, maybe of a different dimension, it is not natural to use T again, right? So on the other hand, if you are in the same vector space V, maybe using T again will tell you something, okay, will tell you something about T itself, right? So when you repeatedly use the matrix, what happens to a particular vector is something that is interesting. And this ends up happening a lot in practice in real life. So you will see. It may not be very immediately apparent to you. So maybe in the next week when we look at applications, I will give you so many applications where this notion of operator occurring and operating again and again on a vector and how we want to understand, how that works plays a crucial role. And in fact eigenvectors show up naturally. Eigenvalues, eigenvectors show up naturally when the operator is applied repeatedly to a non-zero vector, okay? So let me look at that and show you how this happens.

So T can be applied repeatedly to v and for this purpose we will restrict the field to be complex because, you know, the roots and all exist in complex field so it's easy for me to restrict to complex field. So let's look at the T being repeatedly applied to v. So what would happen when you apply it repeatedly? I mean you can keep on doing it. So I'm missing some dot dot dot here, okay? So you can imagine there needs to be a dot dot dot, okay? So repeatedly doing it, v, Tv, you know. T^2v, T^3v, you know, T applied repeatedly to v, you can go on up to n, okay? You can in fact maybe imagine you go $n+1, n+2,$ etc. but, you know, something interesting happens when you go to T^nv, okay? The reason is you now have $n+1$ vectors, okay? When you have $n+1$ vectors in a vector space of dimension n, what do we know? The $n+1$ vectors have to be linearly dependent, okay? So this is very interesting. You started with a non-zero vector v and then you kept repeatedly applying the same operator to it, eventually you see you should have a linearly dependent set of vectors. Because you can keep on doing it, but you know, there is only finite dimension, so you should have a linearly dependent set of vectors. So there will be constants $a_0, a_1, \dots, a_n$ and a_i in complex such that this is true and not all a_i are 0, okay? So in fact this little thing here about this equation, because I know v is not 0, so in this equation a_0, you know… So we usually say this, right? So we say not all a_i are equal to zero, okay? This is a condition that is needed for linearly dependent.

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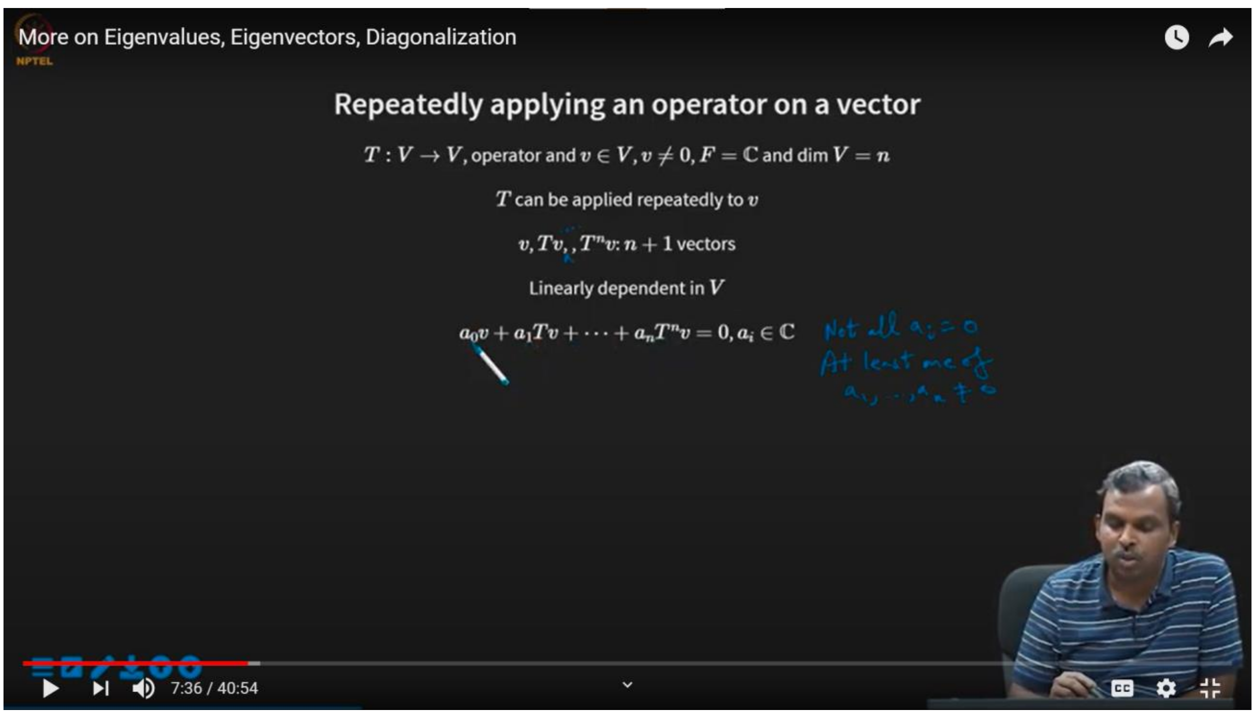

In fact you can say a little bit more, you know? In fact in this case you can say at least one of a_1 through a_n is not equal to zero, okay? Why is that? Because if all of the a_1, through an were 0, then what will happen? a_0 also should be 0, right? Because v is not 0. Because I know v is not 0, it has to turn out that a_1 through a_n something should be there to cancel out the a_0v, you know? I mean if a_0 is 0, that's different, but this v itself is non-zero, so in case if all of a_1 through a_n were 0, then there is a problem, you can never get this to be equal to 0, right? So that creates an issue.

So you, at least one of these a_1 through a_n will not be 0 also okay? So this particular type of linear combination will definitely involve a T, okay? Without involving a T you are not going to be able to cancel out the v. So anyway think about why that is true. Of course if a_0 is 0 or clearly then one of these guys has to be non-zero. That's okay. a_0 is non-zero then one of these guys has to be also, has to be also non-zero, okay? So that's the way to think of this result.

Okay. So what we will do is we can let m be the maximum i such that a_i is not 0, okay? Since we know that this a_1 through a_n something here is non-zero, this m will be at least 1, okay? It cannot be just zero, it has to be 1 or 2 or 3 or something. So some T power is definitely involved and that a_m will not be zero, okay? So this equation becomes this equation, okay? So what is m? m is the maximum a_i such that that is non-zero. So everything to the right of $m, m+1, m+2$ and all is zero, okay? m is the maximum index for which ai is not zero. Beyond that it becomes zero. So you get this equation, okay? Where a_m is non-zero, okay? So basically I want the equation where this last term, the leading term is non-zero, okay? So I am finding out the maximum value of the index where it is non-zero and I am making an equation like that, okay? So here I do not know if a_n, a_n could be 0, right? So I go find the m, maximum index for which it is non-zero and get my equation here. And here I can say a_m is non-zero, okay? So I have sort of like a polynomial equation here, right? So this should remind you of polynomials immediately, right? $a_0 + a_1 T + \ldots + a_m T^m$. a_m not zero. So degree m polynomial. Except that this variable is not some unknown x anymore, it is your operator T, okay? So you have to worry about if that makes sense, I mean can you think of polynomials where, you know, the unknown is an operator and does that make sense? You'll see quickly enough that it's perfectly fine, it makes sense. But before that, let's remind ourselves, if instead of this T, we were to think of our familiar unknown variable x and think of the polynomial $a_0 + a_1 x + \ldots + a_m x^m$, remember the field is a complex field, so I know this is going to factor into linear factors, right? So that was the Fundamental Theorem of Algebra repeatedly used on this polynomial. I know that there are m roots, possibly with multiplicity, but there are m roots and I can write it as $a_m(x - \lambda_1)\ldots(x - \lambda_m)$. So you will have linear factors like this, okay? So now it turns out instead of this x, you can put T here, okay? And since these operators have an algebra, right? So you can add them, you can multiply them, okay? So all you have to do for polynomials is to be able to add and multiply, okay? So then everything works out quite okay and I don't want to go into great detail here for why the, you know, you can have the unknown variable... Think of it as an operator in the polynomial domain and everything works out quite okay. But still you can sort of imagine why that is true, okay? So that's always true. So this factorization of polynomials in terms of this unknown variable x translates into a factorization of this polynomial in T, into a product of sort of linear factors, okay? So $a_m(T - \lambda_1 I)\ldots(T - \lambda_m I)$ okay, so how do you make sense of this? You can sort of multiply this out, right? So because I know there is operator algebra, right? I can, I can add operators, I can multiply operators. T into T will become T^2, T into I will become T, you know? So I can multiply this out. When I multiply this out, the same rules of whatever I use for multiplying with this x will also hold here. So if this polynomial factors in this fashion, this operator polynomial will also factor in this fashion, okay?

So what do I have here? This polynomial equation, right? Where something into v, some polynomial into v, is equal to 0, now factors into a_m which is non-zero times a bunch of linear factors $(T - \lambda_1 I)\dots(T - \lambda_m I)v = 0, v \neq 0$.

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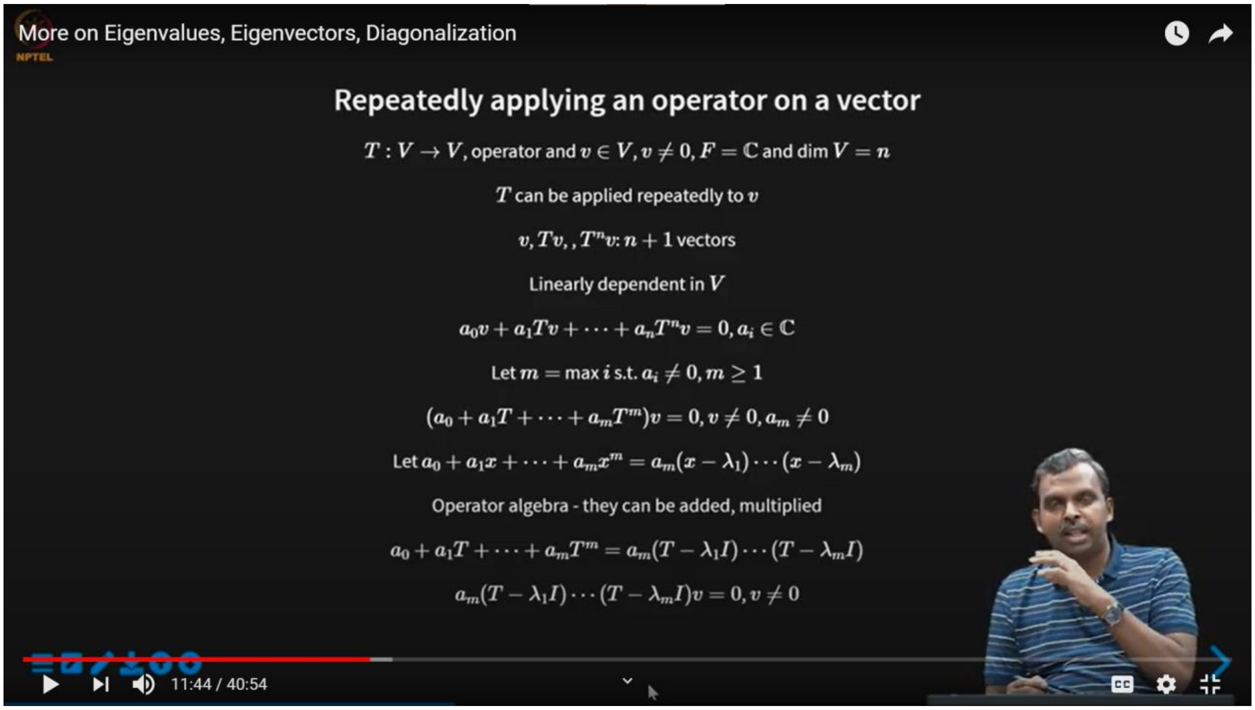

So think about what all we have accomplished. We have done a lot in this slide, lot of steps here. You started with an operator, picked up a non-zero vector, looked at that non-zero vector and all the repeated applications of this operator on that vector. You notice that eventually there has to be a linear dependence, so you find the linear dependence equation and that is nothing but a polynomial times a vector being equal to zero. Polynomial in T. And then you take that polynomial and then factor it into linear factors. So eventually you get a bunch of linear factors multiplying a v equal to zero and v is non-zero and a_m is non-zero, okay? So now notice what should happen. You can give a proof for existence of eigenvalues without using determinants or anything like that. So far I've only used linear dependence, linear independence nothing else, right? So notice why that is true, okay? Since this product $(T - \lambda_1 I)\dots(T - \lambda_m I)v = 0, v \neq 0$, there has to be at least one i for which $(T - \lambda_i I)$ is non-invertible, okay? So notice this product has m such terms $(T - \lambda_i I)$. Out of these m terms, at least one needs to be non-invertible, okay? Why? I mean it is easy to see why, right? So if all of them were to be invertible, you can simply invert them one after the other, right? So $(T - \lambda_1 I)$ is invertible. That times some vector is equal to 0. So you just simply, I know that what is inside itself should be 0, right? So if you have invertible operator, say some S. Sv equals zero means v is zero, right? So the first thing is invertible means the remaining has to be zero. The next thing is invertible, the remaining has to be zero. So eventually you will get to

$v = 0$. But v is non-zero, so that is a contradiction, okay? So all of these $(T - \lambda_i I)$ cannot be invertible, okay? There has to be some i for which this $(T - \lambda_i I)$ becomes non-invertible. If this equation has to be true, okay... So do not be too alarmed by this $(T - \lambda_i I)$, it's just an operator, right? How does it operate on v? $(T - \lambda_i I)v$ is nothing but $Tv - \lambda_i v$, okay? So it's a very well defined operator, okay? So don't be, in case you are wondering what is this $(T - \lambda_i I)$. okay? It acts on v and you get $Tv - \lambda_i v$, okay? So it's a very, very well defined proper operator. That's what I meant when I said, you know, operators can be put together as polynomials and they work very, very decently, okay? So no problem.

There so this, there has to be an i for which $(T - \lambda_i I)$ is non-invertible and I have found an eigenvalue, right? I start with a non-zero vector, okay? Remember once again, I did not start with anything else, I just started with the non-zero vector. I had my operator. I repeatedly applied, I got a polynomial in terms of T times that non-zero vector being zero which, and then I factored that polynomial and then out of that linear factors, one of those terms has to give me an eigenvalue, has to give me a λ_i for which $(T - \lambda_i I)$ is non-invertible, okay? So that gives you λ_i being an eigenvalue. So you can prove quite easily the existence of an eigenvalue without using any determinant or any argument. Only thing you need is the Fundamental Theorem of Algebra applied with the complex field, okay? So this is a very nice and clean, simple proof for why eigenvalues should exist.

So now there are many other results and one of the results that we will see and use in this class is this interesting little result here. There can be at most dimension of V distinct eigenvalues for an operator, okay? So what is the proof? Proof is actually quite simple. The eigenvectors corresponding to distinct eigenvalues have to be linearly independent, okay? How many max linearly independent vectors can I have? Dimension V. So there can be at most $\dim V$ distinct eigenvalues for an operator. So all these results do not really need determinant, right? See, one way of seeing this result is you see $det(A - \lambda I)$, that gives you a polynomial of degree n, so it can have at most n roots, so there can be at most, you know, or it can have exactly n complex roots so it will have exactly n complex eigenvalues is something you can say. But, you know, using these kind of arguments and the linear independence of eigenvectors which, you know, does not need determinants or anything, you can do these kind of results. So I am just trying to point out that in your book for instance, the development of eigenvalues is without using any determinant idea, okay? You do not have to use determinant idea to develop the eigenvalue. But, you know, it's not wrong. Usually traditionally it is done. So I've just mentioned determinants also. But it's possible to build up this whole theory without using determinants at all, okay? But it's a nice way to think about it. Once again I want to remind you the way eigenvalues show up fundamentally is because you take any vector and when you repeatedly use an operator on it, eventually you have to get a linear dependence, okay? And that linear dependence forces something to give in the operator. So some repetition has to happen somewhere and that repetition can always be narrowed down to a one dimensional subspace if your polynomial factors fully into linear factors, right? So that is the basic idea. If your polynomial is not going to factor into linear factors, one of those terms has to be non-invertible so you may not get an eigenvalue type thing, you will get maybe a higher dimensional invariant subspace, okay? So that is the basic way in which this theory of eigenvalue develops. And in a way, if you just use the determinant, you are missing all this important connection. The fact that, you know, this repeatedly using an operator on a non-zero vector creates this invariant subspace, necessitates the notion of the invariant subspace. Otherwise, you know, you won't have that linear dependence, okay? So that is the idea.

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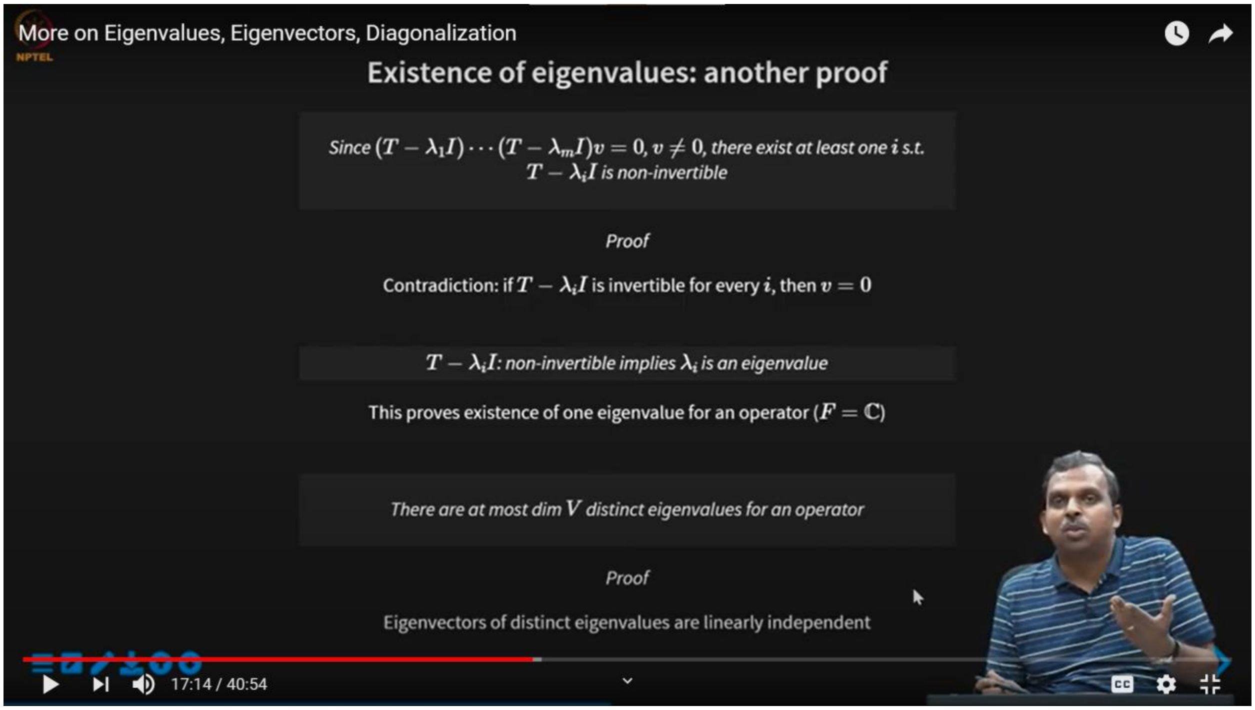

Okay. So now that we know so much about eigenvalues, let's start trying to find eigenvalues for some simple matrices, okay? So we will start with diagonal and triangular matrices. You know what diagonal matrices are, right? Non-zero values only on the diagonal, everything else is zero. What about triangular? You have upper triangular and lower triangular. Upper triangular means diagonal and above the diagonal, you can have non-zero elements. Lower triangular means diagonal and below the diagonal you can have non-zero elements, everything else has to be zero, okay? So the result is actually quite easy, okay? If you have diagonal elements, if you have a diagonal operator, okay, diagonal matrix, sorry diagonal matrix, the diagonal elements become the eigenvalues, okay? The reason is you can look at $(T - \lambda_i I)$ and it will be non invertible if λ is on the diagonal, okay? So it is easy enough to see it. I mean if you want I can write it down. So if you have diagonal elements d_1, d_2, like that, and if you do $(T - \lambda_i I)$, what happens? If you do T minus... This is A and if I do $A - d_1 I$, I will get 0 here, right? And then $d_2 - d_1$, so on, okay? I get a zero here which means this entire row became zeros, okay? So that means $(A - d_1 I)$ is non invertible and d_1 becomes an eigenvalue, right? So that is one important rule to remember, okay?

So this rule is very, very important, okay? So λ is an eigenvalue if and only if $(T - \lambda_i I)$ is non invertible, okay? So it is a very easy rule to see from the definition, but you can see how useful it is for in many simple cases to find out the eigenvalues, okay? For, at least for the diagonal case, one can easily find out the eigenvalues. Same thing holds true for triangular matrices also, okay? All the diagonal values will be the eigenvalues for an upper triangular or lower triangular matrix.

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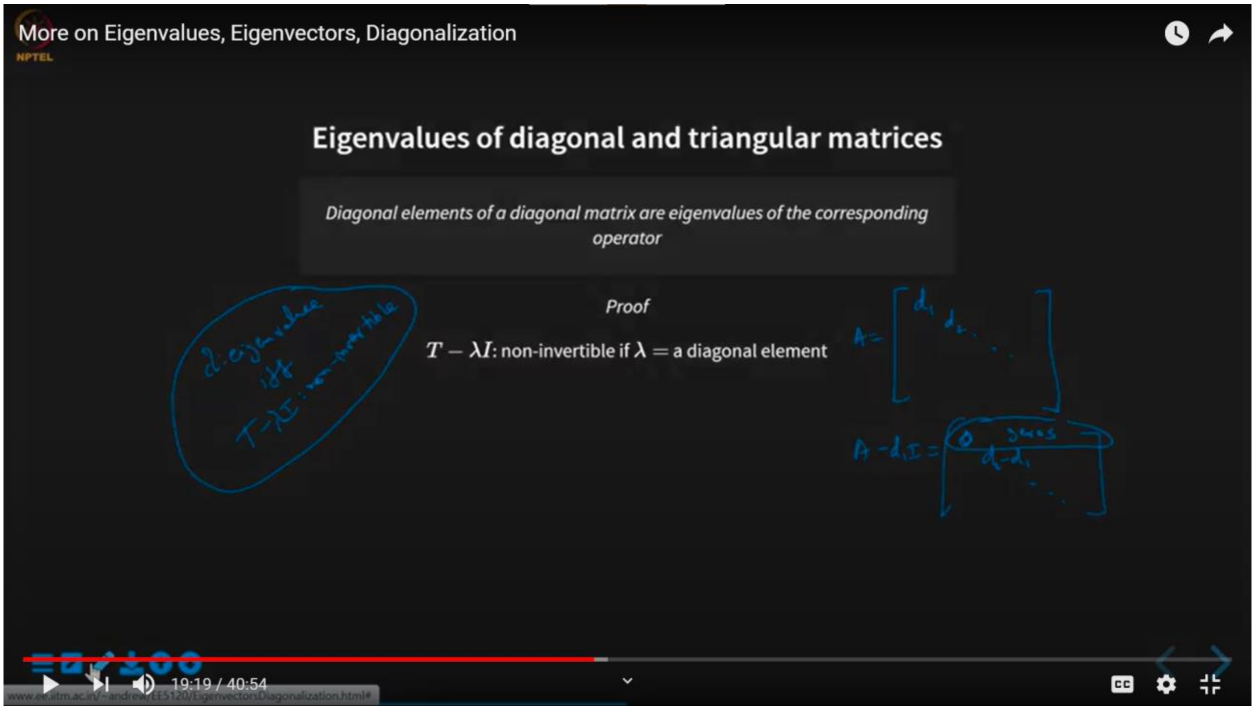

So here maybe the invertibility is not too immediate to see. So let me give you a simple example. Let us say we look at [1 2 3; 0 1 4; 0 0 1] and, oh maybe I should do a different matrix, let me put 1, 2, 3 here. So suppose this is A and if you want to look at $A - I$. That is probably very easy, okay? [0 2 3; 0 1 4; 0 0 2], okay? So you can see here this one. So it becomes non-invertible, okay? What about $A - 2I$? You will have [−1 2 3; 0 0 4; 0 0 1], okay? So you notice here, anytime I have a zero here, okay, you will end up having a linear dependency to the left here, okay? So think about why that is true. Once I have a zero here, because I would have had, this is triangular, right, so it would be triangular non-zero. Maybe here. Once you have a zero here, these parts will be linearly dependent. You can even draw a line like this here and then these parts will be linearly dependent, okay? So think about why that is true. So that zero in the middle really kills the linear independence coming out, right? You can also see that is how the zeros are structured. So this also is non-invertible. So for diagonal and triangular matrices, eigenvalues are very, very easy. You can just pick up the diagonal values, they will become the eigenvalues for you. With multiplicity. If you have the same value repeating, that many times that eigenvalue will appear as a root in the determinant. You can also use the determinant definition. You know the determinant is a product of the diagonal elements for these kinds of things and the result ends up being true.

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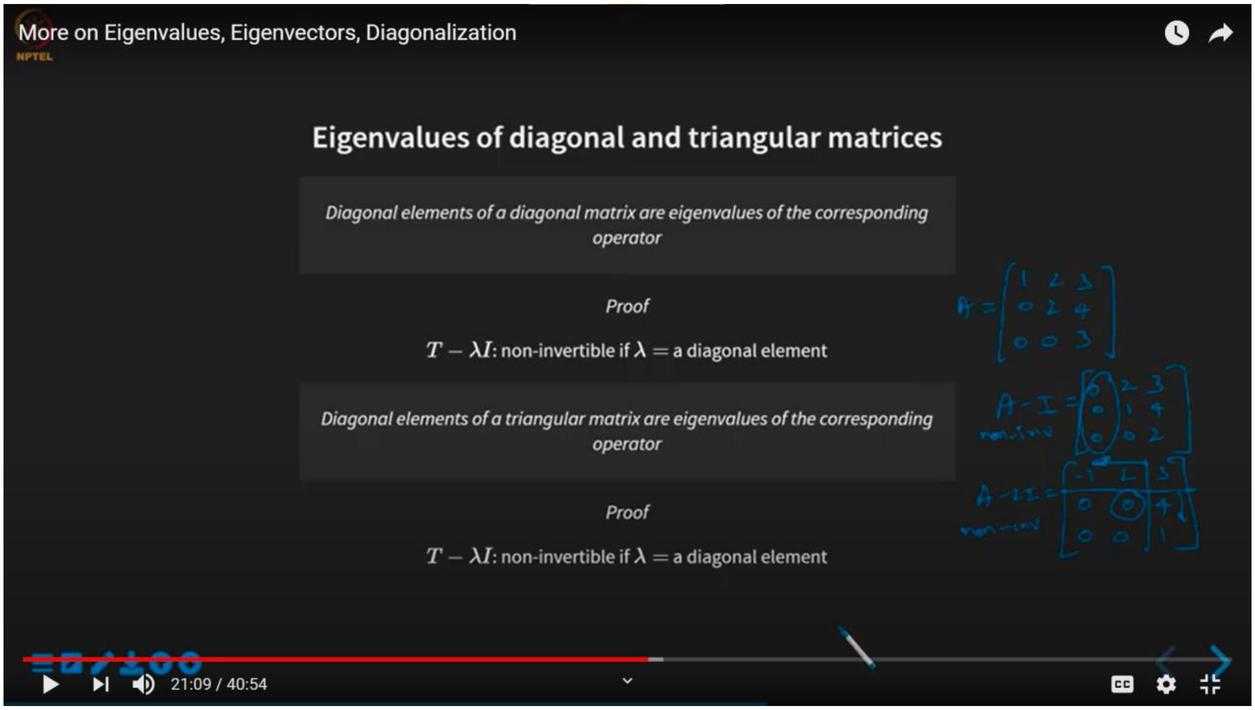

Okay. So now we move on to diagonalization, okay? So we saw that... Like I mentioned, it's, diagonalization is extremely important. I will try to give many examples and repeat this in so many different ways, but still it's one of the crucial ideas in linear algebra. So let's look at why that works. It's actually very easy if you think about why this should work, okay? So if... I'm going to give a sufficient condition for diagonalization, later on we will expand this condition, we'll look at larger linear operators for which diagonalization is possible. For now I will look at a sub case or a sufficient case for diagonalization, okay? So if your linear operator has n distinct eigenvalues, okay, right? What is n? n is the dimension of the vector space V, and T is an operator from V to V, okay? If your operator has n distinct eigenvalues, the eigenvectors form a basis for the vector space V. Why is that? Very simple. Eigenvectors have to be independent for distinct eigenvalues. There are n linearly independent eigenvectors. So you pick up all of them, put them together, that should be a basis, right? So we know that that is true. So that is going to work out, okay? So this leads to the diagonalization idea, okay? If you pick the basis of n linearly independent eigenvectors, then in that basis T will become diagonal, okay? Why is that? It is quite easy to see again. So what is the matrix in the basis of eigenvectors, right? So the first column is Tv_1, second column is Tv_2, so on, right? The last column is Tv_n, okay? So here I have taken $v_1, \dots, v_n$ to be the basis of eigenvectors, okay? So the matrix will be Tv_1 in the first column, Tv_2 in the second column, …, Tv_n. Remember this has to be. When you find the coordinates, this needs to be, you know, coordinates with respect to what? With respect to the basis, the same basis, right? $v_1, v_2, \dots, v_n$, okay? It's the same, same thing here. You find out the coordinates of Tv_n in the basis v_1 through v_n, right? Is that okay? That gives you the matrix representation for T in that basis v_1 to v_n, this is a standard way to find the matrix for an operator, okay? Now notice $T(v_i)$ is an eigenvector, right? v_i is an eigenvector, sorry? So Tv_i becomes $\lambda_i v_i$, okay? So what is Tv_1? Tv_1 is $\lambda_1 v_1$, okay? So Tv_1 is $\lambda_1 v_1$, that's all, there is no other v_2 or v_3 or anything, nothing happens. So Tv_1s coordinates will simply be $(\lambda_1\ 0\ 0\ \dots)$ okay? So what will be the first column here? It will simply be λ_1 followed by a bunch of zeros. What about Tv_2 now? Tv_2 is $\lambda_2 v_2$. Now $\lambda_2 v_2$ if you look at the coordinates, coordinates will be $(0\ \lambda_2\ 0\ 0\ 0\ \dots)$. So what will be the second column? $(0\ \lambda_2\ 0\ 0\ 0\ \dots)$. So likewise you will finish everything and you will get a diagonal matrix here, okay? So it is a very simple and powerful idea. It is very important to write it down. So notice that once again Tv_2 is $\lambda_2 v_2$, okay? So what is $\lambda_2 v_2$? What are the coordinates in the bases v_1 to v_n? This will simply give you the coordinates $(0\ \lambda_2\ 0\ \dots)$, okay? So now when you write these coordinates down as columns, you get the matrix corresponding to T in the basis v_1 through v_n and that is simply a diagonal matrix, okay? So it is very, very easy to see why if you have distinct eigenvalues, you will have a basis of eigenvectors.

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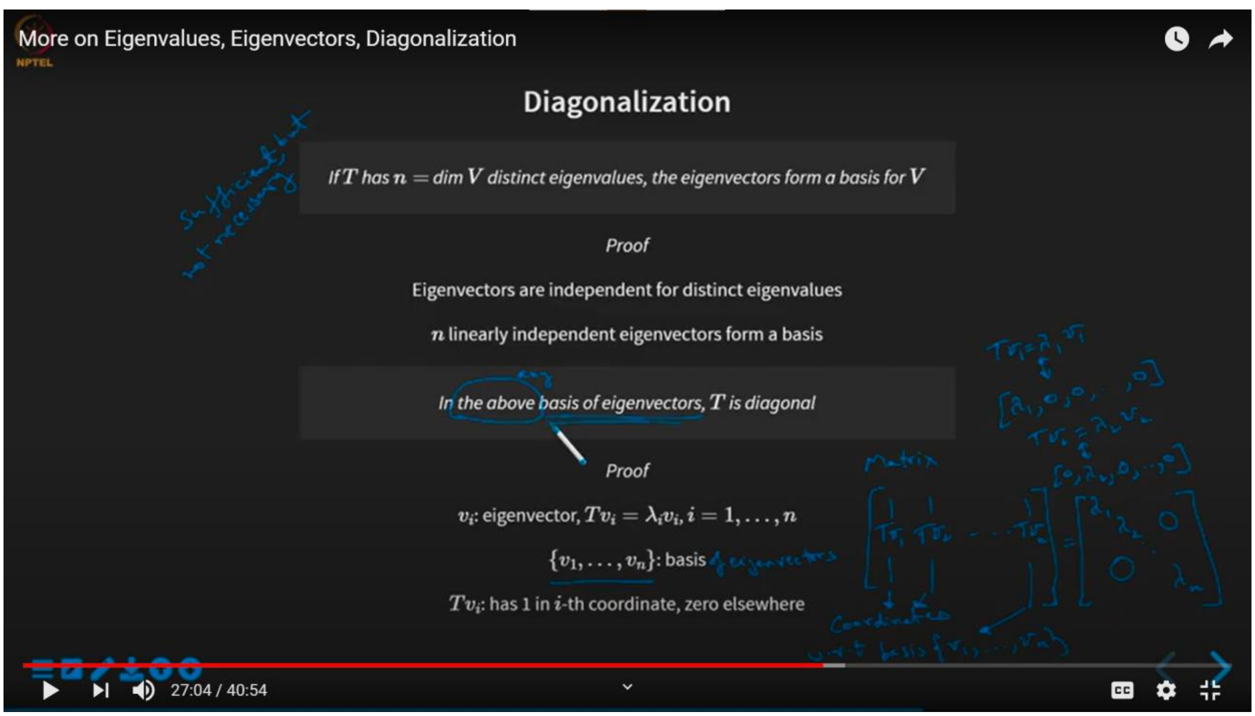

So see, I started by saying distinct eigenvalues which was sufficient to give me a basis of eigenvectors, okay? That's the first result. If you have a basis of eigenvectors, then you have a diagonal T, okay? Something has to give you a basis of eigenvectors. Distinct eigenvalues give me a basis of eigenvectors, okay? But that is only a sufficient condition, okay? So this is sufficient and not necessary. You will see later on even if you have repeated eigenvalues you can get a basis of eigenvectors for an operator. Simple example is the identity operator, right? You take the identity operator, it has only one eigenvalue 1 occurring as many times as you want. But if you look at the, you know, $(A - I)$, it is all zero which means any basis is a basis of eigenvectors for the identity operator. Even if the eigenvalues repeat, you can get a basis of eigenvectors, okay? But a sufficient condition is distinct eigenvalues.

So the crucial result is this one. If you can find a basis of eigenvectors for the linear operator, then in that basis T becomes diagonal. This is a very, very, very crucial result. In any basis of eigenvectors, so instead of, in fact instead of saying the above you might want to think of in any basis of eigenvectors. This basis of eigenvectors is crucial, okay? You have an operator T. If you can find a basis of eigenvectors, then only matrix you have to worry about is diagonal matrix. And imagine how easy it is to think of the diagonal matrix, right? So I want to spend a little bit more time here and give you a picture of what happens when you diagonalize, right? So remember A is a matrix, okay? Let us say it is diagonalizable. What is diagonalizable? Meaning it has a basis of, let us say it has a basis of eigenvectors, okay? Diagonalizable means there is some basis in which it is diagonal, okay? So that also means it has a basis of eigenvectors, right? So these are all related, right? So if you think about it, if the, if a linear map, linear operator becomes diagonal in some basis, that basis is a basis of eigenvectors, okay? So it's quite easy to see it all goes both ways, okay? So this is important. Suppose A is diagonalizable. What does that mean? There is a basis of eigenvectors and so let us say A is specified in matrix, given in standard basis, okay? Okay? And you found the basis of eigenvectors. And let us say you created a matrix S which has the first eigenvector in the first column, second eigenvector in the second column so on and the n^{th} eigenvector in the n^{th} column. What are these? These are coordinates of eigenvectors in standard basis, okay? Is it alright? So you have a basis of eigenvectors. I make a matrix out of that, okay? So why do I make a matrix out of that? Because I know I can do a change of basis quite easily, right? So now I want to do a change of basis to this one: $v_1, v_2, \dots, v_n$. These are the eigenvectors I want to change to this basis. So what will happen to the matrix when I do that? I will get, so let's say, so the eigenvalues corresponding to this is $\lambda_1, \lambda_2, \dots, \lambda_n$, these are the eigenvalues, okay? So I have A, how do I change basis to this? What is the, there is the similarity transformation that we saw, right? You have to multiply by S on the right and S^{-1} on the left, right? So this is the same linear map, same linear operator in the basis of eigenvectors, right? v_1 to v_n, okay, right? So this v_1 to v_n is basically, it will convert from standard basis to the new coordinates in terms of this basis… No no no, I need the other way around. So it will convert from, I am sorry, S will convert from this basis v_1 to v_n, coordinates in the basis v_1 to v_n to the standard basis, right? Because these are coordinates in the standard basis. So this will convert from, this will convert from... Let me call this B. This will convert from B to standard basis, okay?

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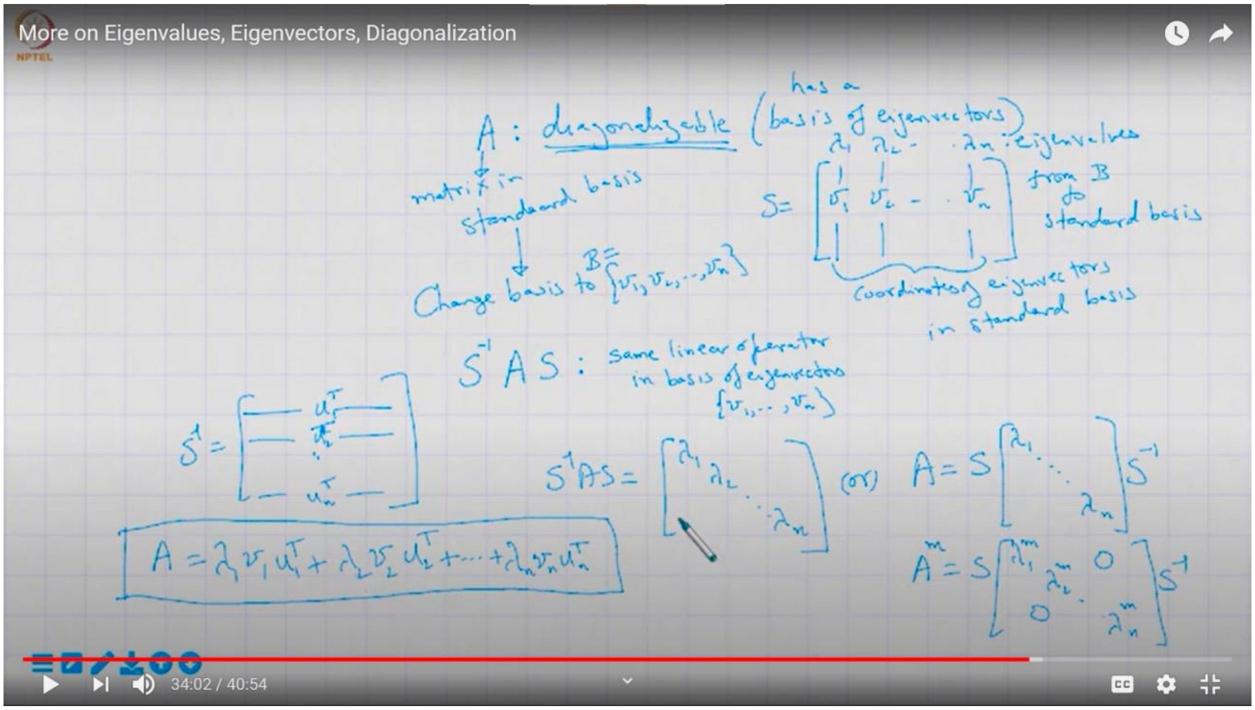

So think this through carefully. So you have a vector in V, it will convert to standard basis. A is in standard basis. So it will give you the output and standard basis. S^{-1} will convert from standard basis to the coordinate system in the basis B. So this is the matrix corresponding to the same linear map in the basis B, okay? But what do I know about it? This is eigenvector basis. So what should be $S^{-1}AS$? This should just be diagonal, okay? $\lambda_1, \lambda_2, \ldots, \lambda_n$, okay? Or another way to write this is A becomes S times this diagonal matrix times S^{-1}. Is that okay? Okay? Think about how I got this. This is sort of important and sort of central, okay? So S, λ_1, S^{-1}, okay? So notice what happens if I repeatedly power A. What will be A^2? You can see any power of A is simply the same S, and λ_1 alone getting that power, S^{-1}, okay? So this is what makes these eigenvalue things so powerful. Repeated application of A is simply just diagonal scaling. Diagonal powering, there is nothing else that you need to do. There is another way in which I like to write this. We have the columns of S, we can also think of the rows of S^{-1}, okay? So maybe S^{-1} I can write as u_1^T. I mean since we always think of column vectors. So let me write the rows as u_2^T, sorry, transpose here, like this and u_n^T, okay? Let us think of the rows like that, okay? S^{-1}, the rows of S^{-1} I am writing as $[u_1^T;\ u_2^T;\ \ldots;\ u_n^T]$, okay? Now what will happen to this product A equals S times a diagonal matrix times S^{-1}? This is just matrix multiplication. And what did λ_1 do? λ_1 to λ_n when it multiplies S^{-1} it will simply scale the rows, right? So λ_1 will come on u_1^T, λ_2 will come on u_2^T like that. And then you will multiply by S on the left and the columns of S are known to you now. This, that product you can write in a very clean way. So essentially you will get A as $\lambda_1 v_1 u_1^T + \lambda_2 v_2 u_2^T + \ldots + \lambda_n v_n u_n^T$, okay? So this is possible when you have a diagonalizable matrix? Remember this crucial, crucial thing is diagonalizable. When a matrix is diagonalizable, all this you can do, okay? Powering of the matrix becomes easy. You can write the matrix as a nice little, you know, outer product sum. See each of these terms is rank 1, you know, your $v_1 u_1^T$, it is just a rank 1 matrix. It is scaled by λ_1 and it adds up. So diagonalizable matrices are so simple to describe, okay? So this view of a diagonalizable matrix is very important, okay? So why it's so crucial and simple and easy to represent, okay? Let's continue.

So what I have done, so I also want to take the opportunity this week to get you, you know, get you used to using computational tools, numerical tools. A lot of tools out there. I'm going to show you three different tools, okay? So for you to, whatever you like you can pick up and use. The first one that I've used and I like, quite easy, it's easy to use wherever you are, is Colab. It's an offering from Google. Since many of you have Google GSuite accounts, your smail account for instance is a GSuite account, you can open Google Colab from your GSuite account and, you know, you can use it. It's quite useful. There are three different libraries. In fact this is shared with you from the sheet. If you click on, open in Colab, you'll be able to download this Python notebook that I have on Colab and use it if you like, okay? So there are three libraries which are quite nice. We will not be using too much of the other ones? But I am just pointing it out to you. Numpy, Scipy and Matplotlib, these are three Python libraries, you just need to import them and once you import them you can start working with them, okay? So in the first step I am showing here is: I am just generating a 10×10 matrix, okay? With some random numbers, okay? So $n \times n$ random integers and I'm making it symmetric. The symmetry you can ignore if you do not like, but I am making it symmetric because the answers will be nicer if that is so. And you can print it, you will see that the values are here. And then there are Numpy, the library Numpy has, you know, functions which give you eigenvalues. So numpy.linalg.eig() will give you the eigenvalues and the eigenvectors, okay? So you can print the eigenvalues. So you can see the matrix that I had before here, this nxn matrix of these kind of numbers all over the place, right? The matrix has so many eigenvalues and they are all distinct. Can you see that? How many eigenvalues? 10 of them. And they are all real, you know, there is a 10th degree polynomial, has all real roots and that is because I picked it as symmetric. We will see later on why this, all these things are true and you can see that, you know, the values are all here and they are all distinct. Notice they are all distinct. And this v is actually the collection of eigenvectors, okay? This is actually a 10×10 matrix. The printing and the formatting didn't come out quite okay. But you can see the first row is given here, you know, the second row is given here. It is a 10×10 matrix, every column is an eigenvector. The first column is an eigenvector corresponding to eigenvalue, this 60.62. Second column is an eigenvector corresponding to 8.9, so on, okay? So you have eigenvectors, your eigenvalues, this is a diagonalizable matrix, okay? And then you can write down all the, you know, forms that you like, okay? So I have written something here to show you how this works. You know the lambdas are distinct, you get a diagonalizable matrix, okay? Pick up this Colab sheet, Colab notebook sheet and then go and, you know, change these things, fool around with this, get used to these ideas of, you know, just see the eigenvalues. I mean just don't think of these as abstract things that I described with lambda and you know, null space, and all that. You can actually work with them and get your hands dirty, okay? So this is very important and this I've shared with you, okay?

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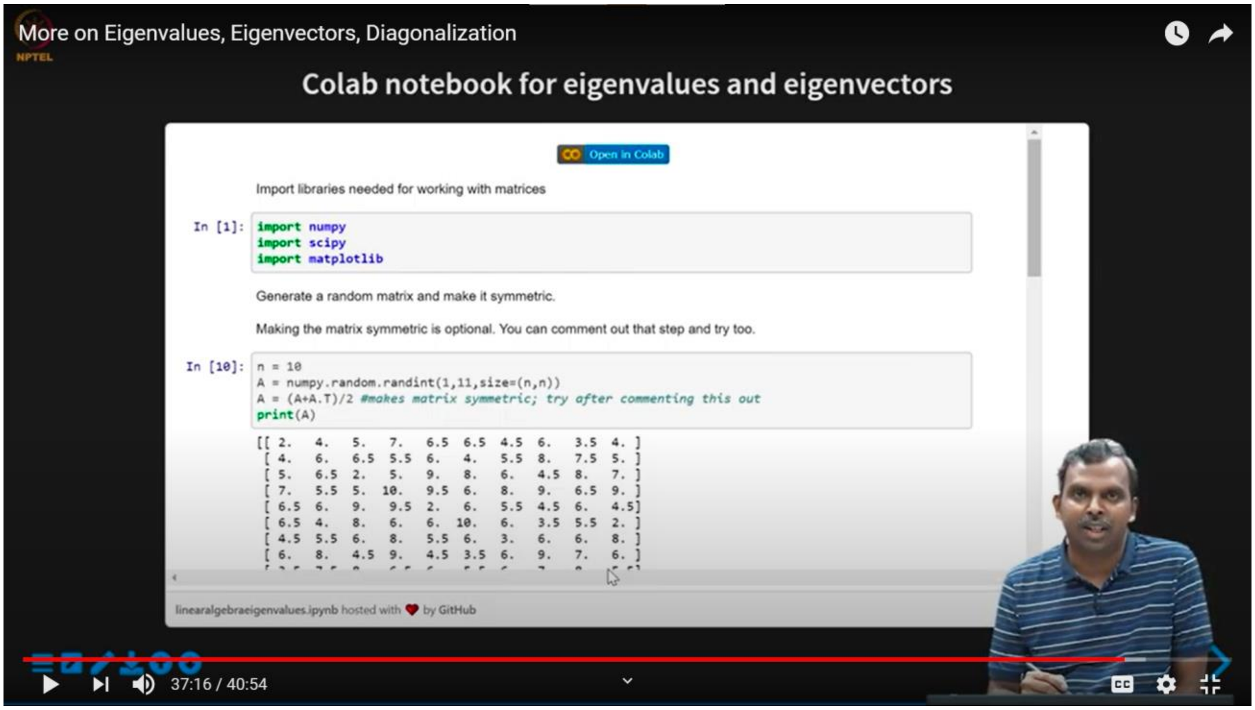

All right. So what happens when you have repeated eigenvalues, okay? So it's all great, diagonalizable when you have non-repeating eigenvalues, distinct eigenvalues. When you have repeated eigenvalues it doesn't look too bad when you have simple cases. The first case is the identity, right? 1, 1, 1, there you see you have three eigenvectors. I have a basis of eigenvectors. So this is diagonalizable, already it’s diagonal, there is nothing much to do here. Here is another case, okay? Once again, you know, you can see I have just added the 2 on top. When I added the 2 on top, notice what happens. I have 3 eigenvalues again. 1, 1, 1. But when I do $A - I$, what will happen, okay? So you can see there is only one eigenvalue. If I do A-I, supposing this is A by the way, you will get [0 0; 0 2] ends up coming there. And notice what this 2 has done. This 2 has messed up one of your eigenvectors, okay? (0 1 0) is not an eigenvector anymore. Because this, the null of this has only, this nullity is only 2, okay? So there are only 2 linearly independent eigenvectors here. The third one is not an eigenvector. So this presence of this 2 has messed it up a little bit for you, okay? So you get only two eigenvectors. So when you have only two eigenvectors, there is no other eigenvector here. So in this, when you have only two eigenvectors, this, you cannot diagonalize as it turns out, okay? So repeated eigenvalues can cause problems and you can make it, you know, the non-diagonal entry enters. So once the non-diagonal entry enters, you know, repeatedly squaring it, you have to take care of that entry, okay? So you cannot ignore that entry, it won't just directly become power of the diagonal, okay? So that power you lose, okay?

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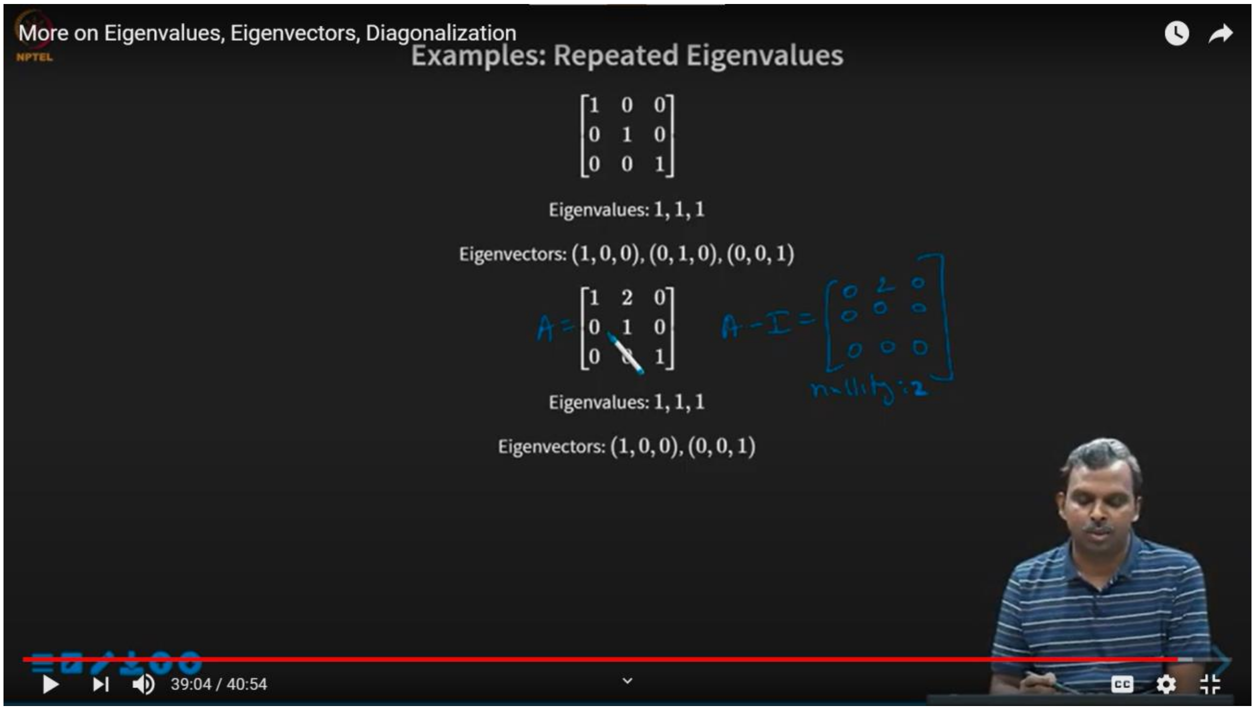

So you can go on and, you know, look at the next step also, you know? You can put one more element here 2, 3. I have added the 3 strategically here, okay? Notice what happens now. You have three eigenvalues, great. But, and all of them are the same, but there is only one eigenvector, okay? So if you look at $A - I$, you will have 0 2 and then there will be the 3, so the nullity is just 1. So you have only one eigenvector, you've lost two of your eigenvectors and again this matrix is not diagonalizable. And you end up with this form, okay? So you can't do better than this form. So repeated eigenvalues can kill your diagonalizability, okay? Something to be watchful about, okay?

So generally to summarize this little lecture on eigenvalues and diagonalization, eigenvalues are extremely useful in obtaining simple matrices. We saw that in particular if the eigenvalues are not repeating, they are distinct eigenvalues, that guarantees a basis of eigenvectors for me, that's very nice. But basis of eigenvectors results in a diagonal matrix, that’s very important. And when the matrix is diagonalizable, it has very simple forms, okay? So you can have a very simple nice form in which you can express it, okay? When eigenvalues are repeated on the other hand, you may not have enough eigenvectors, okay? So what do you, what can we say in general? There are these notions of generalized eigenvectors and eigenvalues and all that, but we may not go in that direction in this class, maybe later on if we have time we’ll come back to it. But for now this is important for us to know, okay? So how do we hunt for bases of eigenvectors and what is the best we can do in the general case? Supposing, what is guaranteed in the general case? Supposing it is non-diagonalizable, then what, okay? What is the maximum number of zeros I have to live with in the off-diagonal, okay? So those kinds of questions are interesting. We’ll look at it a little bit maybe, not in too much detail, okay? So I will stop here for this lecture. We will pick up again in the next one. Thank you.

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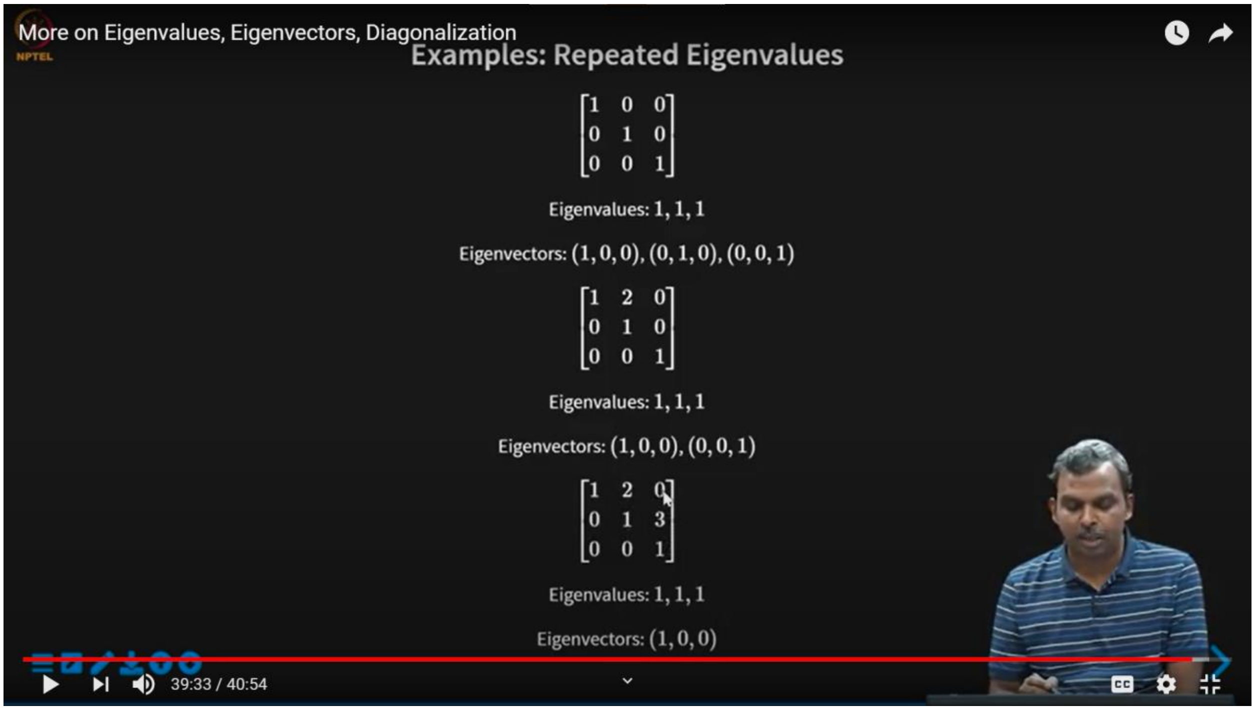

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