Applied Linear Algebra
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Week 06
Linear state space equations and system stability

Hello and welcome to week 6 of the lectures on Applied Linear Algebra. We studied eigenvalues, eigenvectors and how they provide, you know, a good understanding of linear operators. Particularly with respect to repeated application of the operators, when you do A, A^2 and all that, they simplify the, you know, final expressions considerably. And we will see in this week three or four applications of eigenvalues and eigenvectors particularly from an engineering point of view and also from various other interesting modern ideas as well. So let's see, let's get started with the first one which is about, you know, linear state space equations and system stability. This is a very commonly used model for many engineering systems. This linear system, state space system type model. We will see some very simple examples, we will see how linear algebra comes in and how something called stability of a system can be measured using eigenvalues. So let us get started, okay?

So this is a recap. The recap will remain the same for the whole of this week. The applications are varied. Eigenvalues, eigenvectors have a huge number of applications. We will see a few prominent ones this week, okay? So what is this system state and evolution? So typically when you have some, when you're building or when you're engineering something, it's usually a dynamic system in the sense that there are a few variables for, that capture the state of the system at any point in time and as time evolves these variables change. This could be a mechanical system. There's something rotating, moving, etc. It could be an electrical system where a signal comes in and it gets processed etc. So all engineering systems are like this. There are a few key state variables and these state variables evolve with time because of your actions, because of some environmental thing and you want to keep tracking that as time progresses, okay? So usually there will be a set of state variables and this will... So I will keep time discrete. It's like a discrete time system. Quite often people use continuous time models. But in this class, that requires tools which are slightly beyond us. So we will use discrete time. So time goes as $K = 0,1,2,3$, etc. So this is not a bad model. Many systems today work like this. You have a discrete time clock and every clock you take some action, you take some measurement. So you can have time quantized in this fashion, okay? So this is my state variable, system state so to speak, at time k. I will call it as a vector x_k. I will assume that there are n state variables $(x_{k1}, x_{k2}, \dots, x_{kn})$, okay? So once again, I have a system, some engineering system, if you will. It has n state variables which describe it at time k. These variables are denoted as x_k, a vector x_k and these evolve with time as k increases from $0, 1, 2$ etc. Maybe the initial state is known to us and they evolve with time according to some equation that we control, okay? So that evolution is controlled in a linear fashion, okay? So we will assume that there is an $n \times n$ matrix A which will multiply x_k to give x_{k+1}, okay? So this is not a bad model. Many systems evolve like this. Maybe not in this exact simple way, maybe A has some more complications. Maybe there are some mild non-linearities. Maybe there is some noise. Maybe A itself varies with k. So you can do a lot of modifications to this model. But the essential simple model is $x_{k+1} = Ax_k$. It's very popular. Quite often, you know, if you look at, say, control systems, they would say state evolves based on the input also. Every k there will be an input which will also be included in the model etc. But I mean all of those are just artificial additions. As long as you have linear property, you can always redefine everything as one big state and write one equation like this. Think about why that is true.

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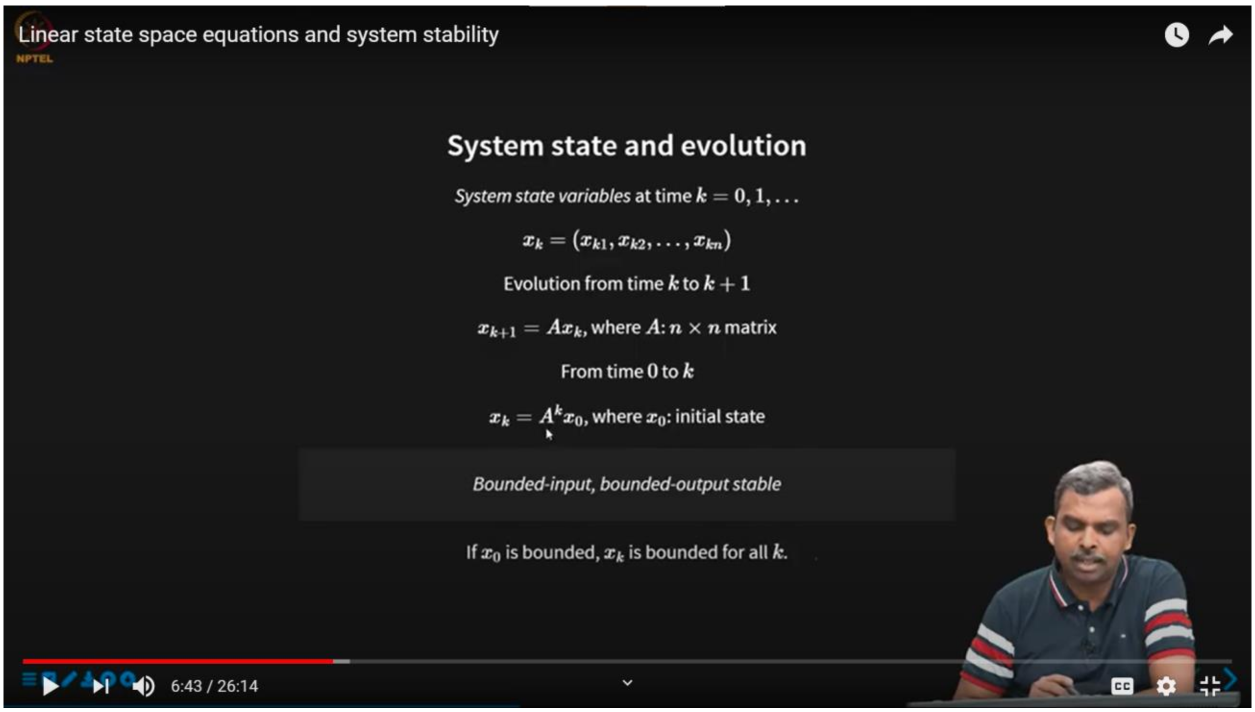

So you can write like this. So this $x_{k+1} = Ax_k$. It's some sort of a linear model for evolution. So this is the model that we will look at in this lecture at least to study what happens when a system with so many state variables evolves with time in a linear way like this. So what happens particularly as k becomes larger and larger and larger. So that's what you want to see, right? So that's the part which we will look at. Okay. So like I said, so from k to $k+1$, x, the state variables get multiplied by the matrix A. And once again remember I'm denoting the vector in this form but it's actually a column vector, okay? So this just for convenience I am writing like that. It's a column vector. Remember that. That's some notation we've been using in this class. So what will happen if you look at the time from 0 to k, the evolution from 0 to k? You will start getting multiplication, right? So you can see why, you know, Ax_0 will give you x_1. And then you multiply with A again, you get Ax_1 which is A^2x_0. Likewise you will keep getting it. So x_k will become A^kx_0. And this

x_0 is the initial state, okay? So you see how this A^k enters the picture because of this kind of evolution, right? So repeatedly this A gets, you know, applied to this, the state of system variables and then you get A^k, all right? So now we are interested in what happens to A^k for different types of A. Can we say something about A? And you will see naturally these eigenvalues will enter in a very nice way and determine what happens in this case, okay?

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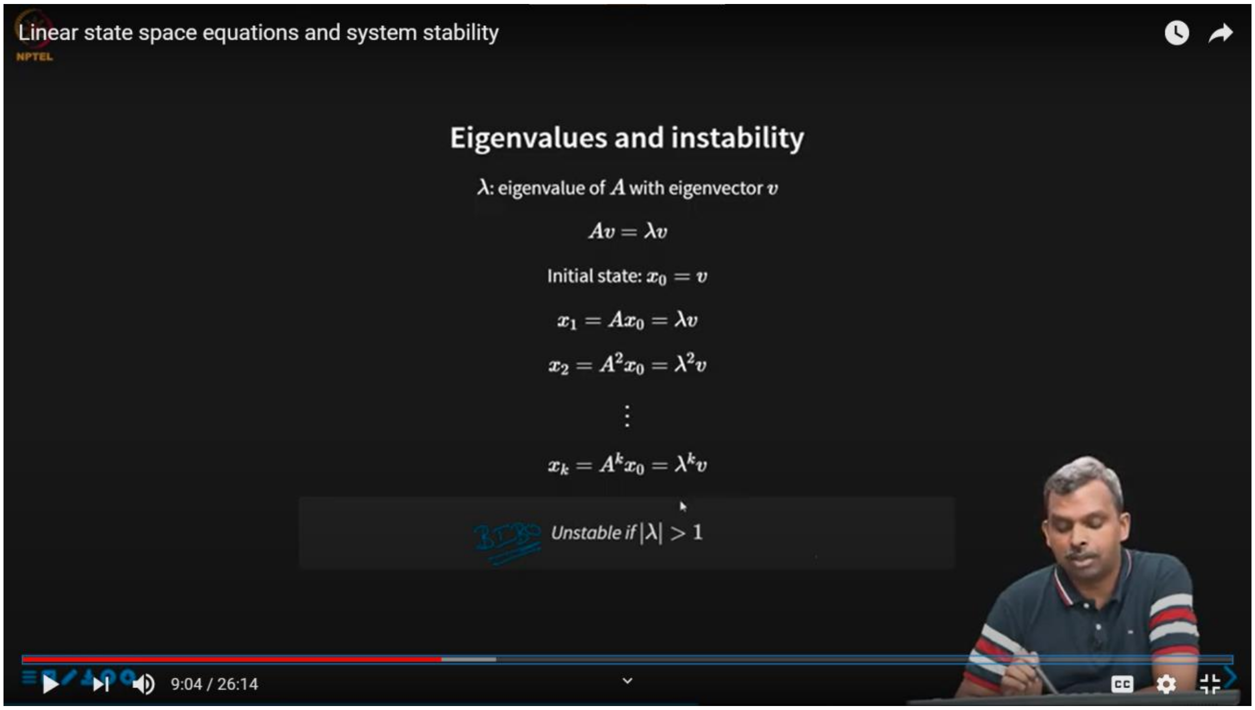

Okay. So in particular there is this notion of stability which is quite popular in engineering systems. Bounded input bounded output stable in some sense, okay? So I say input here but what I mean is bounded initial state and then bounded final state stable, okay? So I'm just sort of abusing notation here to say input, output, okay? So basically what it means is: if x_0 is bounded, if the initial state is bounded, nothing is infinity etc. right, which is mostly what's going to happen, x_k should be bounded for all k, okay? So I want, I want to have this condition. If this is true, then this evolution, this A is supposed to be bounded input bounded output stable, okay? So BIBO stable is a very common abbreviation in many engineering systems. So this is a common requirement. So now we will try to look at A and see when A will result in a BIBO stable system. So what is the kind of consideration that we need to come up with, okay? So that is going to be the main objective of this lecture and we will see the eigenvalues of A will nicely enter the picture and give you a good answer. Okay. So this slide sort of captures the instability and the connection to eigenvalues, okay? So for what type of eigenvalues will you have BIBO instability, okay? So let us say you have an eigenvalue λ for A with eigenvector v. So we know that this Av is going to become λv, right? So if your initial state x_0 is v, notice what happens. x_1 becomes λv, right? Ax_0. x_0 itself is v, so Av =

λv. Then what will be x_2? $A^2 x_0$. And that will become $\lambda^2 v$, right? Because it's, $A(Ax_0)$, so that is $\lambda^2\ v$, right? One more Av will come. Av is again λv. So we will get $\lambda^2 v$. So on, you see that x_k becomes $\lambda^k v$, okay? So it's very easy, isn't it? So you see how these invariant subspaces, one dimensional invariant subspaces help you when you want to look at stability, BIBO stability, right? So you know that the thing is invariant. Only this λ matters. It keeps on getting multiplied, right?

So one thing we can conclude from this is: evolution by A will be BIBO unstable if $|\lambda| > 1$, okay? So this is easy enough to see. Just let me, just make sure I get this point right. It is BIBO unstable, right? So that is the criteria we are looking at. So clearly if $|\lambda|$... See, I am imagining that λ could be complex as well. Even if A is real, λ could be complex. So that's why I put $|\lambda|$. If $|\lambda| > 1$, and then you are allowed to pick the initial state to be this eigenvector v which could also be complex... So we in general will allow our numbers to be complex just to be safe. Maybe they are all real, maybe all that works out, but we will allow things to be complex as well, okay? So x_0 is, if you pick it to be the eigenvector v of an eigenvalue, corresponding to an eigenvalue whose absolute value is greater than one, so you see clearly when λ, $|\lambda| > 1$, λ^k is going to blow up, okay? So when this blows up for that input v, clearly you don't have BIBO stability. So its stability is violated, okay? So if the modulus of any eigenvalue becomes greater than 1, then you have instability. Absolutely no problem here.

So what if this is not true? What if all absolute values of λ are less than 1, okay? Will you have stability? So that is something we are yet to prove. We have only shown: if there is an eigenvalue with absolute value greater than 1, then the system is going to be BIBO unstable. What if this is not there? What if all the eigenvalues are less than 1, can we guarantee stability, okay? So here we need to look a little bit closer, okay? So we will start with the simplest case where A is diagonalizable, okay? So if your A is diagonalizable, remember, just because, you know, you have an operator it does not mean it has n independent eigenvectors and it always ends up being diagonal, right? So there are cases where you do not have as many eigenvectors. So it may not be diagonal. But the diagonal case is the easiest to consider when you want to particularly raise it to higher powers. We know that for sure, okay? So if A is diagonalizable, we know that there is a basis of eigenvectors for A, okay? v_1 to v_n. And there are corresponding eigenvalues λ_1 through λ_n. Remember there may be repetitions here, okay? This λ_1 to λ_n, I am writing it like that but there could be repetitions, okay? So remember that. Keep that in mind. I am writing it for simplicity as λ_1 through λ_n. So any initial state x_0 since this eigenvector basis is there, I can write it as a linear combination of the eigenvectors, right? x_0, any initial state, in fact any vector v, right, I can write as a linear combination of the eigenvectors v_1 through v_n, okay? So those coefficients I am simply giving them some names. Some $(\tilde{x}_{01}, \dots,\ \tilde{x}_{0n})$ etc. okay? So the coordinates of x_0 with respect to the eigen basis I am calling as $(\tilde{x}_{01}, \tilde{x}_{02}, \dots,\ \tilde{x}_{0n})$, okay? So x_0 may have another, other coordinates in the, I mean if you multiply this out maybe you will get its coordinates in the standard basis but I am interested in the coordinates in the eigenbasis. Once I write it like this, I can do repeated applications of A quite nicely because I know each eigenvector is well behaved with respect to the operation A, right?

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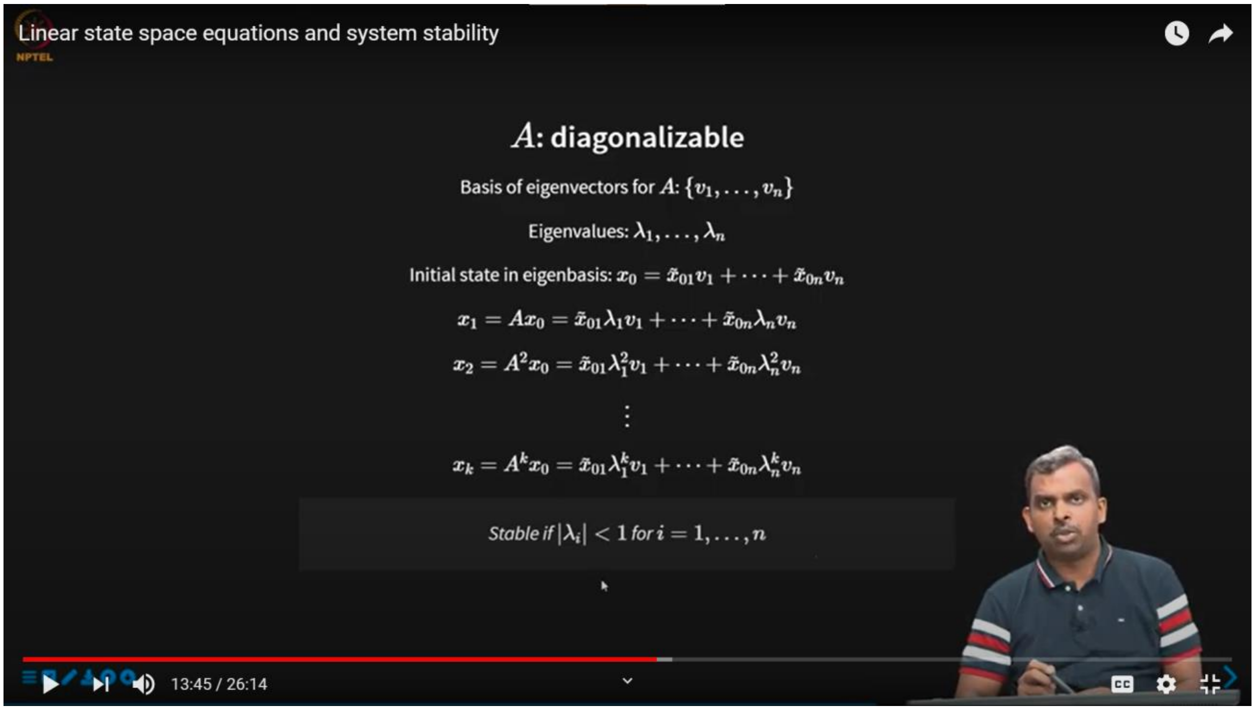

Notice what will happen if I do... x_1 is just Ax_0. It just gets multiplied by the corresponding λs. Next is x_2 which is A^2x_0 and then it gets multiplied by λ_1^2 to λ_n^2. So that's it. So you can keep on doing this. You see the k^th state from the start time x_k is simply the same coordinates, but instead of v_1 I would have $\lambda_1^k v_1$ and instead of v_n I will have $\lambda^n v_n$. Quite easy to see, okay? So we see interestingly if $|\lambda_i| < 1$, right, for all i, then I will have BIBO stability. In fact it will be you know x_k will always tend to 0, right? Eventually as k becomes very large. It depends on, you know, how close λ_i is to 1 etc. But if it were to be less than 1, the absolute value, if it is less than 1 then each of these terms will tend to 0 and you will have stability. If any of them is greater than 1 then we also know it will be unstable. But if it's less than 1 and A is diagonalizable, clearly, easily we are able to see that this is, this will have this kind of behaviour, right? $\lambda_1^k, \ldots, \lambda_n^k$. And you can imagine what else can happen here, you know? What if $|\lambda| = 1$, right? In that case you can have more interesting behavior, right? Some of the eigenvectors will not vanish. They will continue to stay there whatever happens, how many other times A appears, right? So all those kinds of things will happen. Interesting behavior like that can be observed if $|\lambda| = 1$. So you will have some oscillatory or, you know, such behavior which is possible. But if $|\lambda_i| < 1$, then everything will go to zero eventually. If $|\lambda_i| > 1$, there will be some inputs for which you will get blow up, okay? So that's what happens. Most inputs will also end up being like that, okay? So good. We've seen two different cases. One where, you know, you have instability because eigenvalue is greater than one.

Then you are blowing up. And you have stability when A is diagonalizable and absolute value of λ is, all the eigenvalues are less than one, okay? So you see this absolute value of eigenvalue, absolute value of the eigenvalue, okay, control stability in these kinds of linear state space systems, okay? So that's a good result to remember, okay? So it's a wonderful application of eigenvalues. There's so many systems out there and all of them should, for stability, have eigenvalues controlled. That's a good thing to know.

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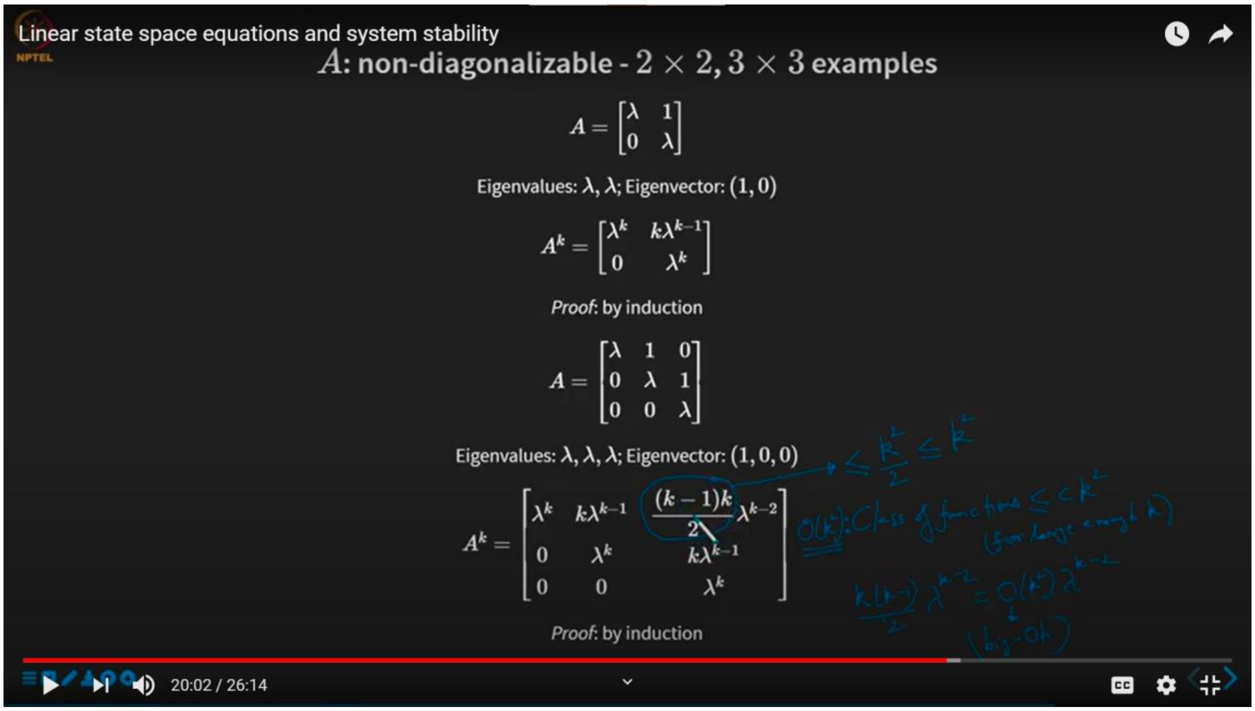

What if it's non-diagonalizable? I want to give you a few examples for what happens in the non-diagonalizable case and point you to a powerful result which helps you capture all the non-diagonalizable cases also. We won't quite see a proof of this result, but you know, it's interesting to know this result at least, okay? So let's take non-diagonalizable cases. We'll start with very simple examples, okay? So the example I will start will be the 2x2 case. This is one of the simplest cases of a non-diagonalizable 2×2, right? So you have λ on the diagonal, the lower value is zero. Then you have 1 on the upper value, okay? So there are two eigenvalues here. And there is only one eigenvector, okay? So we have seen this before, okay? So we have seen such examples before. This is not a diagonalizable case. You do not have enough eigenvectors, as many as the multiplicity of eigenvalues, okay? So now what happens to this when you keep raising it to higher and higher powers? You can do this proof, it's not very hard, you can show A^k is $(\lambda^k \; \lambda^k)$ on the diagonals, but it will have a non diagonal value also. It will have $k\lambda^{k-1}$, okay? You can prove this for example by induction. It is not very hard. You just do A^k, you can go to A^{k+1}. You just multiply by A on the left for instance on the right here. You will see you will have λ^{k+1} on the diagonal, and this one will become $(k+1)\lambda^k$. So you can see why, how that is set up. It will come out quite easily, okay? So you might argue: what about other 2×2 cases which are non diagonalizable? So it turns out this is a very typical case for 2×2 when it is non diagonalizable. There is no other case really, right? So see the two eigenvalues have to be the same. So you know that I can make A into an upper triangular matrix with λ here. And then this value, you know, can it be something else? Not really, right? If you think about it, you can adjust so that it becomes 1. So think about how you do another basis transformation to get 1, okay? So it's not as simple as row elimination. You need a careful invertible matrix multiplication on both sides and change the bases to get this, okay? So it's possible. So this is, you don't lose any generality in the 2×2 case, okay? So that is good to know. So this is nice, okay? I mean I wasn't really going into detail there, but I'm just saying that this is enough. You don't need anything more.

What about 3×3? So once again I will consider a very simple case, okay? So I will consider, in the 3×3 case I will consider a case where there is $(\lambda\ \lambda\ \lambda)$ on the main diagonal. Below that is all zero. I know I can take it like this upper triangular. Is okay. But on the upper part, I will only take 1 on the next immediate diagonal and then 0 after that, okay? So I will only consider a case like this. So we will see later why these kinds of things are enough, okay? But for now just believe me. Let's see, let's handle this case first, okay? So this seems to be a simple case. And if we can't handle this, it’s very unlikely we will be able to handle other things. So we will handle this case first. So once again there are three eigenvalues, as in eigenvalue λ repeated thrice. But there is only one eigenvector, right? So it's only $(1\ 0\ 0)$. There are no other eigenvectors. So once again if you do A^k here, this is again a proof you can do by induction. You can even sort of see how this generalizes from the previous one. You will have λ^k on the diagonal. And then you will have $k\lambda^{k-1}$ on the slight, one off diagonal. And then the next one will be, you know, summation up to $(k-1)$ then, you know, $\frac{(k-1)k}{2}$. So you can see why this is true. You can sort of multiply and check by induction that this will give you the correct formula going forward for A^k, okay? So this is a proof that you can do. Notice the important thing here is, okay... So quite often you want exact expressions. But the point here is this one. This function is less than or equal to, let's say, $\frac{k^2}{2}$, okay, isn't it? Right? It's some, you know, $(k-1)$... So in fact if you don't like even $\frac{k^2}{2}$, you can say it's $\leq k^2$. So some something into k^2, okay? So this detail, this $\frac{(k-1)k}{2}$ is sort of, is confusing to us. What really matters is it grows like k^2, okay? Like k^2. There could be some constant there. Half or something. But it grows like k^2, okay? So there is this convenient notation for functions like this. There are these classes of functions, okay? Which are all upper bounded by some constant into k^2, okay? So we will define this class of functions. And this notation is very common. $\mathrm{O}(k^2)$, okay? What is $O(k^2)$? These, this represents the class of function. It is not one function, it is actually a class of functions, there are many functions which satisfy this upper bound $\leq ck^2$. Usually for large enough k. But even in, you know, we don't even need that for this. Large enough k, okay? So this is how people define this $O(k^2)$. Even though for small k there may be some, you know, skirmishes here there, once k becomes large, this function is upper bounded by ck^2, okay? So that is $O(k^2)$. So it is very common to write this instead of writing $(k-1)k/2$. I might as well write like this, right? $\frac{k(k-1)}{2}\lambda^{k-2}$ seems very confusing. I can write it as $O(k^2)\lambda^{k-2}$. So you get a sense of what it is, right? So this is how, this is used you get a sense that, you know, k^2 is sort of the dominant term in this term. I don't really bother about the details of how that term is, I just need to know that it is order of $O(k^2)$. This is called big O by the way. Big-Oh. So there is a similar small-oh notation and all that. We do not need that as of now. But, right now this big O is a useful thing to quantify things like this. Otherwise you know, you will be worrying so much about the exact expression here. But while it is not so important to you in the study, okay?

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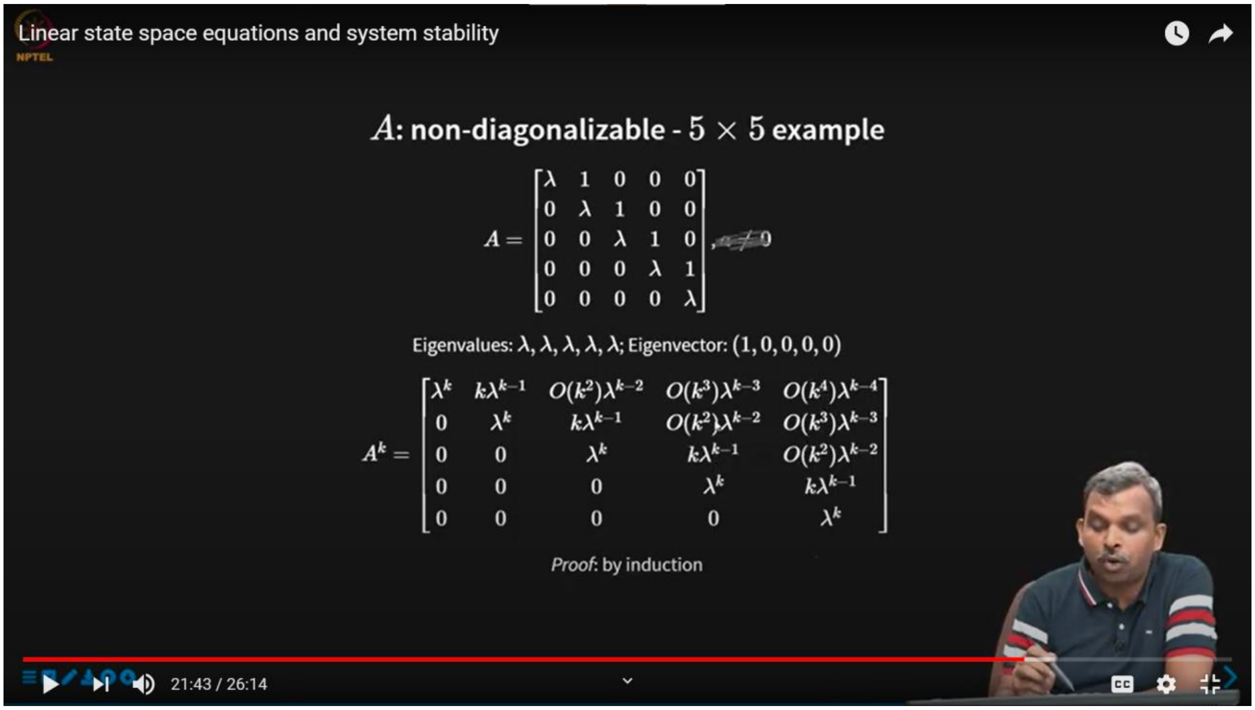

So this is A^k for this case, okay? Can I generalize this? It turns out yes. Let's take a slightly bigger example. 5×5. And once again I will take a very simple form where lambda is on the diagonal 5 times and then there is just 1 showing up on the one off diagonal, okay? So this a, this is not needed, just remove it. So there is only, there are five repetitions of lambda. There is only one eigenvector. And what can we say about A^k? So once again A^k you can do a very careful count and, you know, find these exact functions if we want. But it's really not needed, you can use this wonderful big O notation that we have found to simply write, you know, the first... On the diagonal you will have λ^k, one off diagonal you have $k\lambda^{k-1}$, that's okay. In the second diagonal one, you know, the next diagonal, you will have some function of k^2. You know exactly what that function is, $k(k-1)/2$ or something like that, but I am just simply writing it as $O(k^2)$. In the next one you will have k^3, okay? So that will, you can also find out what that is exactly if you want to. If you put in some effort, you will get it. But that's not so important to us. All that matters is it's $O(k^3)$. And the next final thing will be $O(k^4)\lambda^{k-4}$, okay? So you can prove this, you can prove it by induction or various other methods are there. You will get this answer, okay? So the point is A^k has these kind of terms. $k^4\lambda^{k-4}$ and then λ^k, λ^k. λ^k shows up and then k power something shows up, okay?

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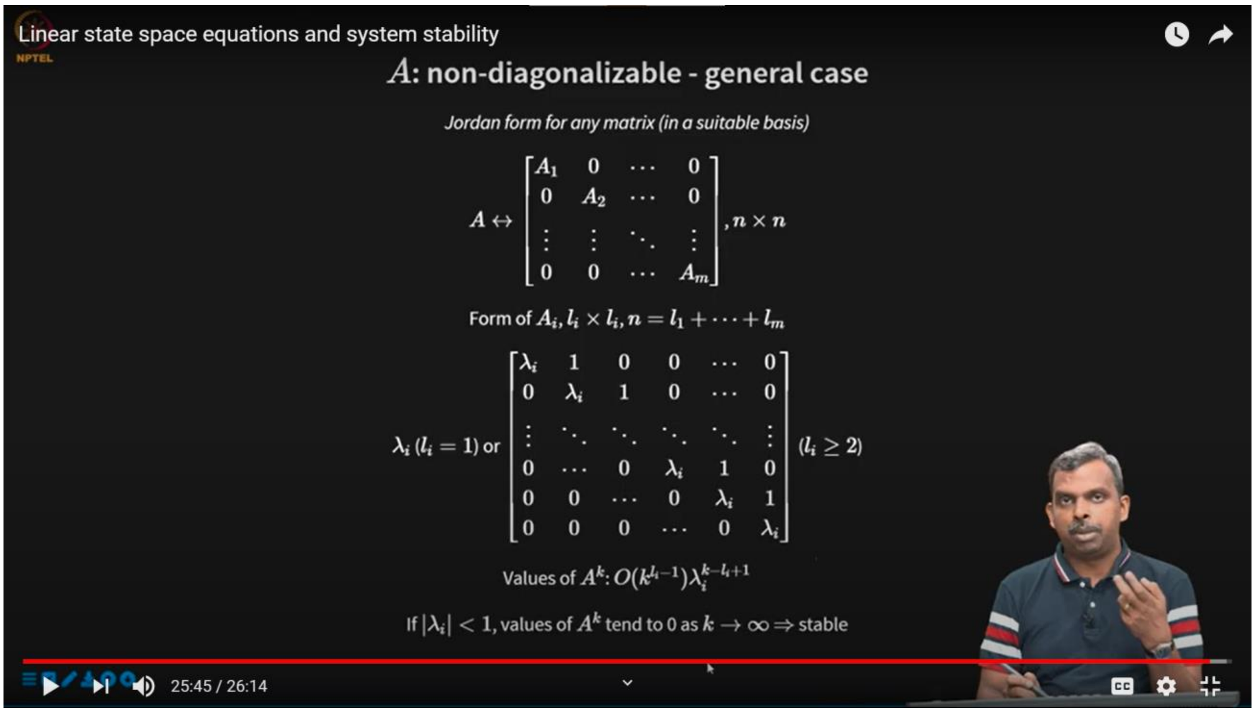

So this is what happens for matrices of this form, okay? So λ on the diagonal and ones on the off diagonal, okay? So you might rightfully ask what about other forms of this matrix, you know? Right now this thing looks so restrictive. What about other forms of this matrix? So it turns out there is a very powerful result in linear algebra which says that these kinds of things are enough, okay? So what is that result? That result is what is called this Jordan form for any matrix, okay? So it turns out whatever the matrix may be, okay, so whatever non-diagonalizable, diagonalizable, general case. Any matrix A has something called the Jordan form, okay? What is this Jordan form? You can write it in a block diagonal structure like this. $A_1, A_2, \ldots, A_m$, okay? Each A_i is an $l_i \times l_i$ matrix, okay? So square $l_i \times l_i$ matrix. And this n which is the overall number of columns or rows is simply $l_1 + l_2 + \ldots l_n$, okay? And what is each A_i? The form of each A_i is either λ_i if l_i is 1, okay? Say the λ_i if l_i is 1. Or if l_i is greater than or equal to two, it has a form that we have been assuming so far which is λ_i appearing on the diagonal and just ones appearing on the one off-diagonal. Everything else is zero, okay? So once again let me repeat myself. Every matrix or any matrix, diagonalizable, non-diagonalizable, whatever has something called a Jordan form. What is the Jordan form? There is a basis in which it becomes block diagonal and every block is either a

1×1 λ or $l_i \times l_i$, this simple form we have assumed, okay? λ showing up on the diagonal and one showing up off diagonal, okay? Now these λ_is may repeat, okay? They may repeat across these A_is but the form will always be like this. You can always write it like this, okay? So this is a fantastic result and this uses ideas called generalized eigenvectors. If we have time later on in the class we come back to it. But for now let's assume this and see what it means for our result.

So once we know this, what happens now? A^k will simply become $A_1^k, A_2^k, \dots, A_m^k$, right? I do not need to bother about anything else, okay? Because this is block diagonal, right? When you multiply, you will simply get $A_1^k, A_2^k, \dots, A_m^k$. So the only thing I have to worry about for a general matrix A when I raise it to the power k is matrices of this form and of course constants. Constants just, you know, just 1×1 matrix. They just go up. Not constant, I mean scalars. Just λ_i, that raised to the power k is simply that raised to the power k. And matrices of this form raised to the power k which I already know how to do, right? I did it for the 5×5 case. You can easily extend that to the general case, right? So that is possible. All right. So now values of A^k will always be of this form $O(k^{l_i - 1})\lambda_i^{k - (l_i - 1)}$, okay? So this is the form for values of A^k. So non-zero values, okay? So we see that if λ_i, $|\lambda_i| < 1$, these values will tend to zero ultimately as $k \to \infty$. So you will have stability, okay? So this is the result you might quote. But I mean you might want to be careful because, you know, there is this $k^{l_i - 1}$ and if this l_i is very large, right, even if λ_i is reasonably close to 1 this may not die down that fast, okay? So you really might need a very large k before it dies down. Something to worry about. But still asymptotically at least as k becomes really, really, really really large, this will ultimately become stable, okay? So this Jordan form is quite useful for these kinds of results. But we will not see it immediately. We will come back and see it later, okay? So to summarize we have seen that, you know, if an evolution is linear with an operator A, then the eigenvalues of A, the absolute value of the eigenvalues of A control its stability, okay? So if the absolute value is less than 1, you end up BIBO stable. If it's greater than 1, you would not be BIBO stable. It can expand. Equal to one is the case we did not consider. It results in some oscillatory type behaviour, okay? So that is the end of this lecture. Hopefully you are convinced now why eigenvalues are so widely used in engineering, in systems. They play a very central role. There is an equivalent expression for continuous time as well but we are not going into that in this lecture, okay? Thank you very much.