Applied Linear Algebra
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Week 06
Discrete-time Linear Systems and Discrete Fourier Transforms

Hello and welcome to this lecture on applications of eigenvalues. In the last lecture we saw how eigenvalues helped us determine stability of linear systems when, with respect to the, you know, when you look at the operator A. How to use its eigenvalues to determine stability. Absolute value had to be less than one for it to be stable and all that, okay? So that is a good result we saw. So we will look at another similar application which is very, very powerful. I mean you will see how, why this is extremely powerful in this lecture. It leads to this notion of Fourier domain and Discrete Fourier Transforms. It's very powerful in, particularly in electrical engineering but even in other areas. This notion of Fourier domain is extremely powerful and it naturally comes because of the linear algebraic and eigenvalues involved in systems and their input and output. So in particular we look at discrete time linear systems, linear time invariant systems and how this Fourier domain sort of naturally enters through that, okay? So let's get started.

The recap is similar to before so I'll skip it. So let's get into this linear system. So we've seen this before even in the previous lecture. But let me just repeat. So what is a system? In engineering you think of something that takes in an input, generates an output, has some state, that becomes a system. Many electrical systems work in discrete time and when they work in discrete time you can always, you know, figure out that my input signal is $(x_0, x_1, \dots, x_{K-1})$ over a period of time, okay? Over a certain period of time. Usually you are always interested in a certain period of time. You look at an input signal, you want to study it and put out an output signal. Your system might do something like that, okay? So your output signal y for that period of time is $(y_0, y_1, \dots, y_{K-1})$. Once again time is discrete 0, 1, 2 etc. You can think of a clock. Discrete time system. It's very common. So this xn and yn could be real or complex also. So complex numbers are just pairs of real numbers with some connections. So we can think of the inputs as being complex as well, okay? So that is good, that's the setting. And LTI systems. So linear time invariant systems are very, very popular. So most systems that you know in electrical engineering uses or even in other areas it is used, the output y_n will be related to the input x in this fashion, okay? The fashion is described there. y_n the output at time n is given by h_0 times x_n. What is this h_0, h_1, etc.? This is called the impulse response of the system, okay? So you can think of it as some scalars that describe the system. So this $h_0, h_1, \dots, h_L$ are used to specify the LTI system. So once you give me that, okay, the output at time n is simply $h_0 x_n + h_1 x_{n-1} + \dots + h_L x_{n-L}$, okay? So this operation is called linear convolution in electrical engineering. So it's a very simple definition, right? So all you are doing is: to generate the output at time n, you take a linear combination of the previous, you know, L or $L + 1$ inputs, input values. How do you do it? You scale each of the values by some constants that you have, add them up and put them out as the output, okay? Seems like a simple way to design systems. But this is very powerful. You can do so many different things with this, okay? So you can imagine how almost anything that you want to accomplish with system, you know, processing of inputs to generate desired output, this you can do. You can do, you know, filtering out of some noise, this, that etc. Averaging you can do. You can imagine. So many things are possible with this simple and powerful description. And it's not an exaggeration to say, you know, almost all electrical systems at least use this model, exploit this model for design, analysis etc. okay? So let us look at this model. And this model naturally leads to the Fourier domain, okay? So that is what we will see. You will see how eigenvalues sort of enter this picture and then give you the Fourier domain that you need, okay?

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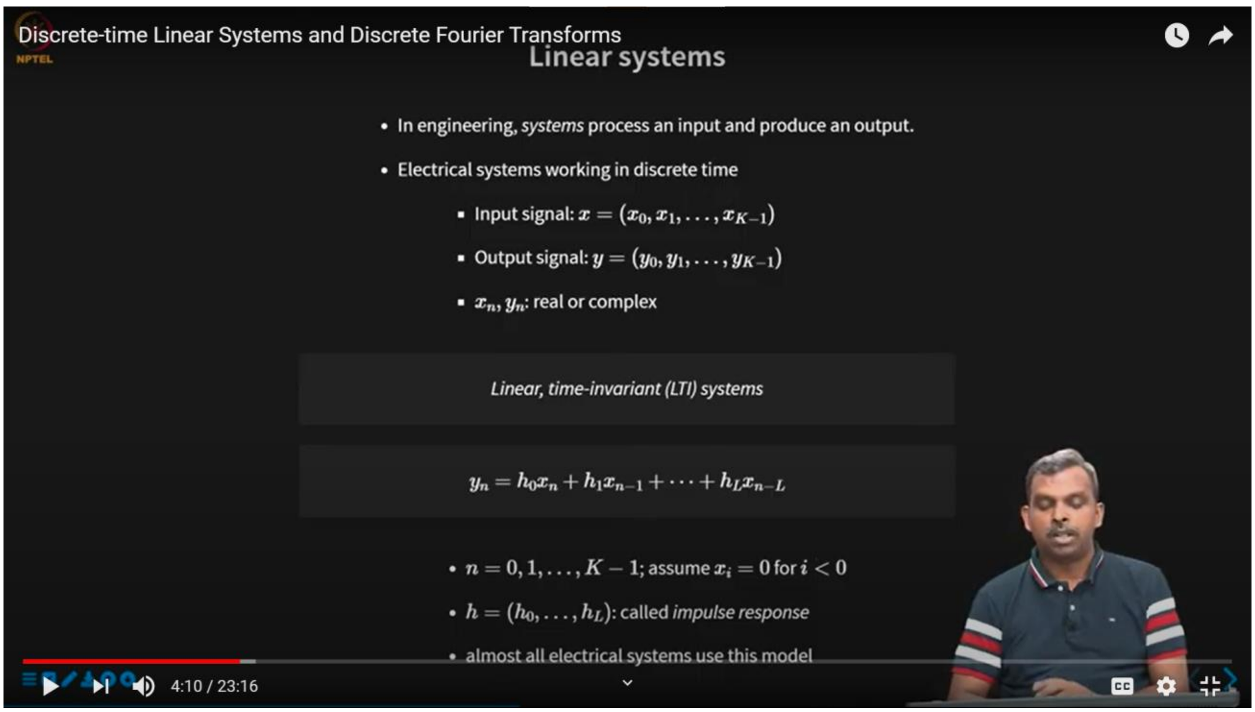

So the previous linear convolution can be written as one big matrix transform $y = Hx$, okay? So y is the entire vector $(y_0, \ldots, y_{K-1})$. x is the entire vector $x_0, \ldots, x_{K-1}$. I can write a matrix which I have called as H linear, H_{lin} which captures this product, right? See, remember y_0 is going to be simply $h_0 x_0$. All the previous ones, the x_{-1}, x_{-2} and all we are assuming is 0, right? So system's, there is no input in the time before that. That's something we can do. So we can write like this and you can see how this, you know, h_1 enters the picture. Next h_2 enters also all the way up to h_L and then, you know, h_L itself will shift. There will be zeros in the middle and then you sort of have the circular shift. I mean not circular, right shift kind of behavior that you have in this matrix. And it's easy to see why this is true. It's the same equation as before. $y_n = h_0 x_n + h_1 x_{n-1} + \ldots$ written in full glory, you know? In some sense every value is written out and you have this big matrix which represents linear convolution and linear time invariant systems, okay? Given an input signal x, you produce an output signal y, okay? I am going to make a change here to this equation, okay? So I will convert this right shift into a circular type of shift, okay? Why is that important? That gives me a lot of simplifications in the study. It brings in this Fourier and all that. So it's just very crucial.

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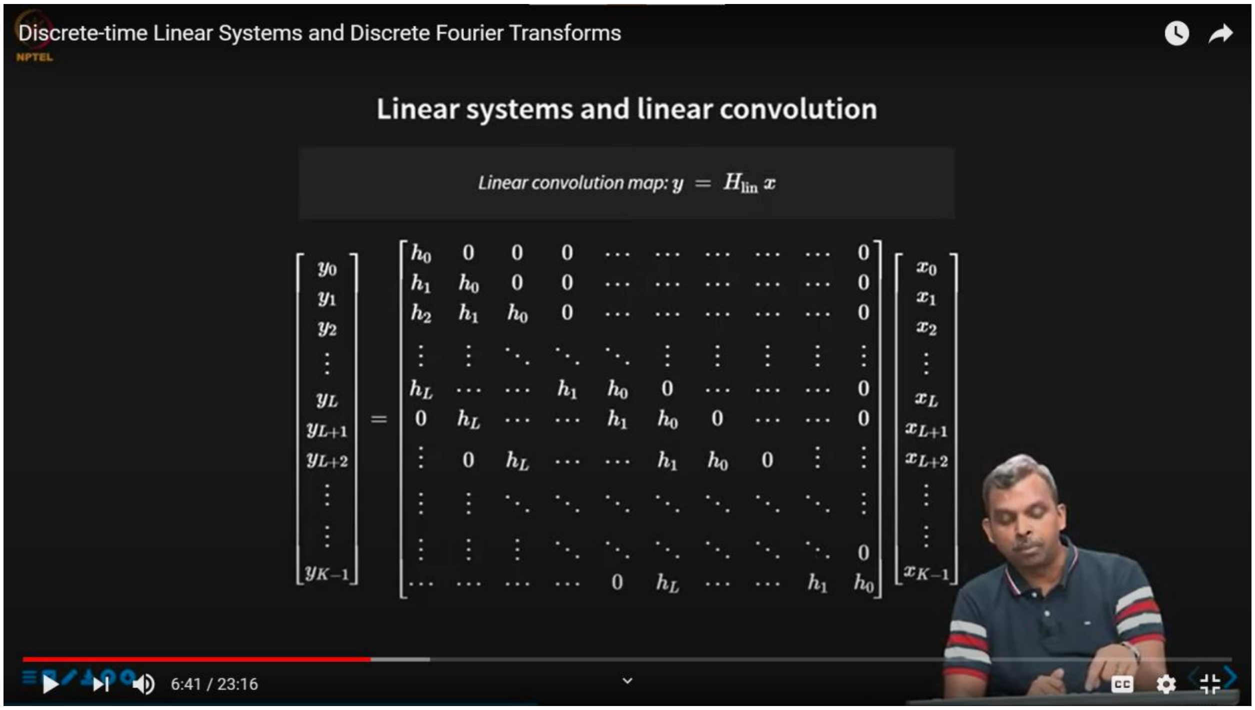

And for that what we will do is: we will imagine that this k is under our control, okay? Supposing, you know, the number of inputs is not k, something lesser than k. I can always do some zero padding and control this k, you know? Increase the input and output length and that's okay. Okay? So I will imagine something like that. So from x_0 to x_k I am not interested in all the k inputs. Only up to some point I am interested. After that I have zero padded or something like that, okay? So we will assume that. Once you assume that you can see that I can introduce some additional numbers in this matrix and make it into a circular shift instead of just a right shift with h_L to h_0, okay? So in the, particularly in the first part you see that, you know, this is not quite a circular shift, you know? The first $L + 1$ rows are different from how the remaining rows are working here. So I want to have the same process for the whole thing. So for that I need to introduce something here on the right top, right? On the right top I need to introduce something here and for that I will conveniently assume that I can do some zero padding to do that, okay? So that's how we move towards this circular convolution, okay? So if you can assume that there is some zero padding possible, you can convert this linear into circular convolution and I can introduce these guys here, okay? So notice this is the part that got introduced. This part got introduced here and that was not there in the previous linear convolution, okay? So I have to assume that correspondingly somewhere here, okay, I have zeros so that this part does not quite enter the picture. So the last few things are not crucial for me and that is how it works, okay? So once I do this, it's very simple to describe. So you, so think of it column wise. Column wise is also very easy. You think of the column. Column is just the impulse response $h_0, h_1, \ldots, h_L$ and the next column is a shift, okay? Then you keep shifting. If you go below the bottom, you come back to the top. Another way to describe it is: you look at the first row, you write it as h_0 and then $h_1, h_2, \ldots, h_L$ like this. And the next row is a circular right shift, okay? So from row to row you simply do one circular right shift to the right, okay? Right shift to the right of course, right? So you know row $i+1$ equals circular right shift of row i, okay? So this is very important, okay? So such matrices are called circulant matrices and we see that just because we had an LTI system, okay, and just because we had that convolution operation describing the output for a given input, and we were able to look at a finite number of bits, finite number of symbols, finite number of signals points right, up to capital K, and that was under our control, we could zero pad or do something like that, we are able to get a circulant matrix to represent the input-output operation, okay? From input to output the operator that's working on it can be made a circulant matrix. So the circulant matrix is very, very important because the eigenvalues and eigenvectors of a circulant matrix can be described conveniently in a very nice way and leads very naturally to the Fourier domain, okay? So let us see that next.

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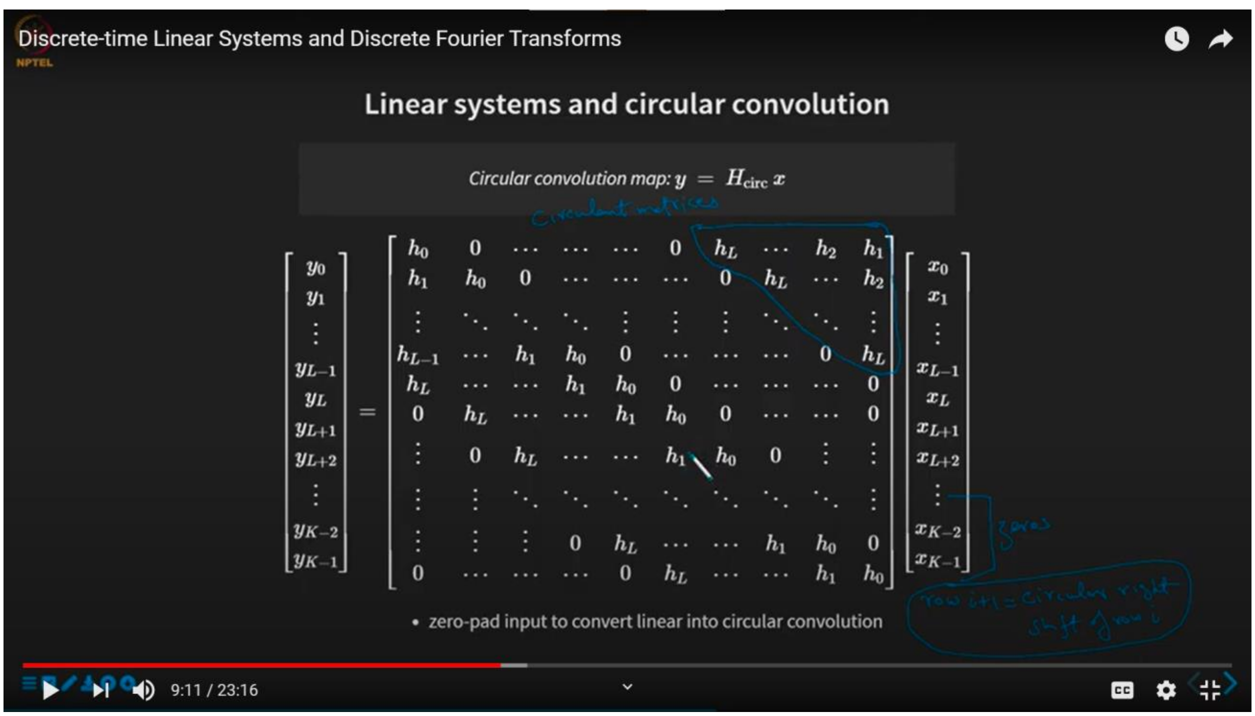

So linear maps which are represented by circulant matrices are particularly easy to describe. Why is that? We see that, you know, the first column is the impulse response and everything else is circulant. What are the eigenvectors and eigenvalues of H_{circ}? Any circulant matrix, it doesn't matter what the entries of the circulant matrix are, its eigenvectors and eigenvalues can be very, very conveniently described, okay? So this slide tells you that, okay? So remember normally you expect the eigenvectors and eigenvalues to depend on the entries of the matrix, okay? Yes the eigenvalues in almost all cases definitely depend on the entries of the matrix. But it turns out for circulant matrices the eigenvectors do not depend on what you put in the matrix. As long as the matrix is circulant, the eigenvectors are fixed, okay? So it's a very nice result and it comes because of the way the operation works, right? So think about what the operation does. The operation simply takes, you know, every row is a circulant shift of the other, okay? So if you want to have an eigenspace for it, if you want to have a one dimensional invariant subspace for such an operation, your eigenvector, you know, something very simple should happen when it, you know, shifts right by one place, right? It should just get scaled by something when it, for every linear shift, every circular shift it should be scaled by the same value. As long as you have a vector like that it will become a one dimensional invariant vector, okay? So think about why that is true? I mean, I would give you the answer here and then I will expect that you go back and work it out and convince yourself for how the circular right shift operation, okay, can create, you know, one dimensional invariant subspaces. What is the connection between those two and all you need is for your eigenvector when it does a circulant right shift, when a circulant shift, it should be scaled by some constant. As long as that is true, it will become an invariant subspace of the, you know, the circulant matrix operation, okay? And that you can guarantee using these, you know, roots of unity, complex roots of unity.

So complex roots of unity will enter the picture. What are those? I'm assuming some familiarity with complex numbers here. We know that this ω that I am writing here, $e^{i\frac{2\pi}{K}}$ is a complex K^{th} root of unity, right? And ω^K will be equal to 1. And there are some nice results like this. Like $\omega^{(K-l)k}$ is ω^{-lk} , right? So because its ω^{Kk}, ω^K is 1, okay? So only ω^{-lk} can be left. So these kinds of properties are very important. Particularly ω^K being 1 is very important, okay? So notice what happens now. If I define a vector v_k, notice how this is defined. You have ω^0 as the first coordinate. ω^k, k can be anything, k is from 0 to $K-1$ some integer. ω^k , ω^{2k} so on till $\omega^{(K-1)k}$, okay? Okay. So I have a length K vector here. What will happen if I do ω^{Kk}, okay? You will get 1 again, right? So because ω^{Kk} is 1, okay? So remember that, okay? So now notice what happens if I right shift, right circular shift v_k, okay? As in if you do a circular shift on v_k... Let me write that down here. Circular shift of v_k will simply be, let me do it in some direction. $\omega^{(K-1)k}, \omega^0, \omega^k, \omega^{2k}, \ldots, \omega^{(K-2)k}$, isn't it? This is the circular right shift of v_k. If you want, you can do circular left shift also. I mean both are both are the same.

Now what is this? If you look at it very carefully, you will see that this is nothing but ω^{-k}, okay? Times v_k, okay? Do you agree? Okay? The first one is $(K-1)k$, but that is the same as ω^{-k} , right? So if you multiply ω^{-k} by v_k, you get this circular shift, okay? So you can also do a left circular shift. You will get $\omega^k v_k$, okay? So that is the crucial connection here, okay? This v_k, because this ω^k is 1, I can define a vector v_k like this which when circular shifted simply gets scaled, okay? A circular shift is a scale, okay? So I put ω^{-k} because I did the right shift. If you do a left shift, you will get ω^k, okay? So circular shifts are scalings for v_k and you can quite easily prove that for any circulant matrix, this v_k will be an eigenvector, okay? So this becomes an eigenvector. So I will leave the proof as an exercise. It's quite easy and the central idea is this thing, the fact that when this circular shifts you will simply get a scaling for v_k, okay? And this ω^{-k} sort of enters the picture here. So in fact the eigenvalue λ_k is $h_0\omega^0 + h_1\omega^{-k} + h_2\omega^{-2k} + \ldots + h_L\omega^{-Lk}$, all right? So think about why this is true. I am leaving the full proof as an exercise.

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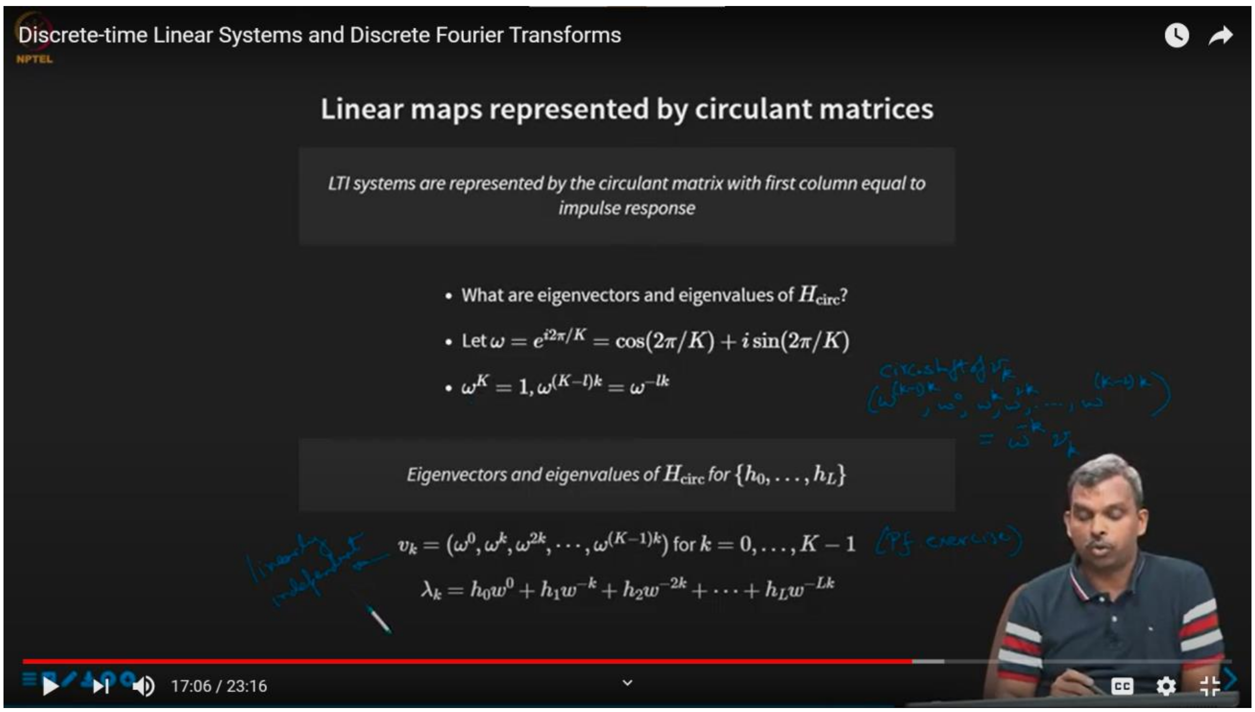

And I know there's lots of notation here. This complex K^{th} root of unity and all is entering the picture. Think about it calmly and try and work it out on your own. You will see this happening. And the crucial idea is this. The fact that shifts becomes scaling for these type of vectors and that's clearly because this ω^K is 1, okay? So think about why that worked out. So these complex roots of unity are playing a crucial role here. And you have these eigenvectors. So those of you who are familiar with Fourier transforms will immediately say that λ_k becomes the Fourier transform of the impulse response, right? It's not very surprising. People in electrical engineering must have seen this before. But if you are not from there maybe you do not quite see the connection. But this is, this is very crucial, okay? So we know, so the crucial point is LTI systems are, you know, are characterized by their impulse response and the input-output operation is characterized by multiplication by a circulant matrix, okay? Of course with some zero padding, whatever, all that is okay. So you have a circulant matrix and circulant matrices are, particularly they have a very,

very interesting eigenstructure. What is the eigenstructure? It is connected to complex K^{th} roots of unity and you know, you can easily write down an eigenvector. The same eigenvectors for any circulant matrix. The eigenvectors are the same, okay? And you have different eigenvalues. So notice these v_k in fact are linearly independent, okay? You can show that these are linearly independent, okay? So this is also very important. This is, these are linearly independent, okay? So this proof may not be too easy but you can do it. This is linearly independent, okay? Again that is also an exercise, you can do that. So you have K, you have capital K linearly independent eigenvectors for any circulant matrix irrespective of what the, you know, entries in the circulant matrix are. The eigenvectors remain the same and that's because of this circulant property that this nice vector satisfies because of this complex K^{th} roots of unity, okay? So this is what you get.

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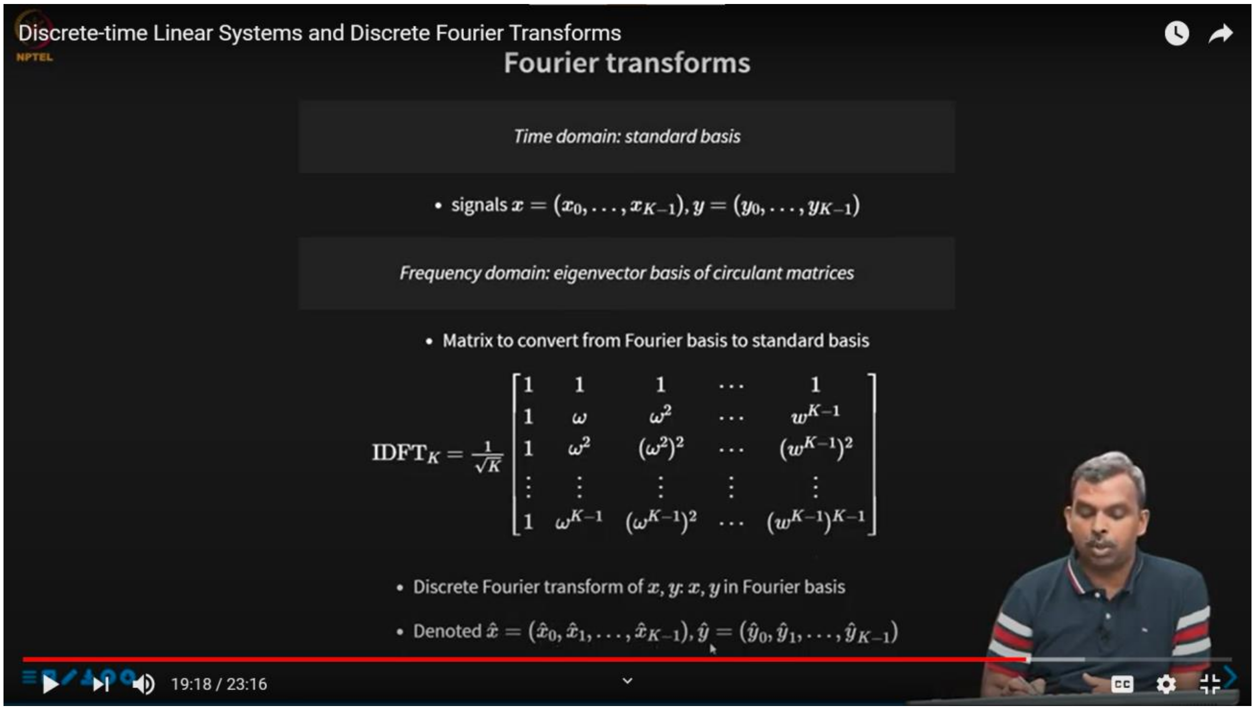

All right. So let's see how we can put this to use, okay? So once you have this notion of, you know, the eigen basis which is the same for any LTI system, whatever the LTI system may be, the eigenbasis that I'm going to use to describe, you know, the operation is the same, okay? And it's an eigenvector basis. You have a basis full of eigenvectors for the circulant matrix, okay? So that's linearly independent set of eigenvectors. That's particularly powerful, okay? So what people do in... So to go to this so-called frequency domain of Fourier domain is to take your signals and then simply try and express them in the eigenbasis. The eigenbasis is also called the Fourier basis, okay? The eigen basis for circulant matrices, you can also call it the Fourier basis, okay? And that leads to this frequency domain representation for signals, okay? Instead of thinking of the signals as $(x_0, x_1, \ldots, x_{K-1})$ etc. you change coordinates, you change basis. Change basis to what? The eigenbasis of the circulant matrix, okay? This has tremendous advantages in describing what the LTI system is going to do, okay? So we'll see that soon enough. But how do you do that? It's easy to see, you know, the matrix to go from Fourier basis to standard basis is simply this, okay? Standard basis is what people call time domain. Fourier basis is this frequency domain. How do you go from Fourier to standard basis? You simply multiply by this matrix and this is called IDFT_K, okay? So this is a very popular matrix. It's called Inverse Discrete Fourier Transform matrix of size $K \times K$. I've put this $\frac{1}{\sqrt{K}}$ here. I mean it's just for convenience, it's not so crucial. But this is the operator which will take you from the Fourier basis to the standard basis, okay? So basically this leads to the definition of this Discrete Fourier Transform of signals x, y. It is nothing but the x, y which was given in standard basis or time domain expressed in the Fourier basis or frequency domain, okay?

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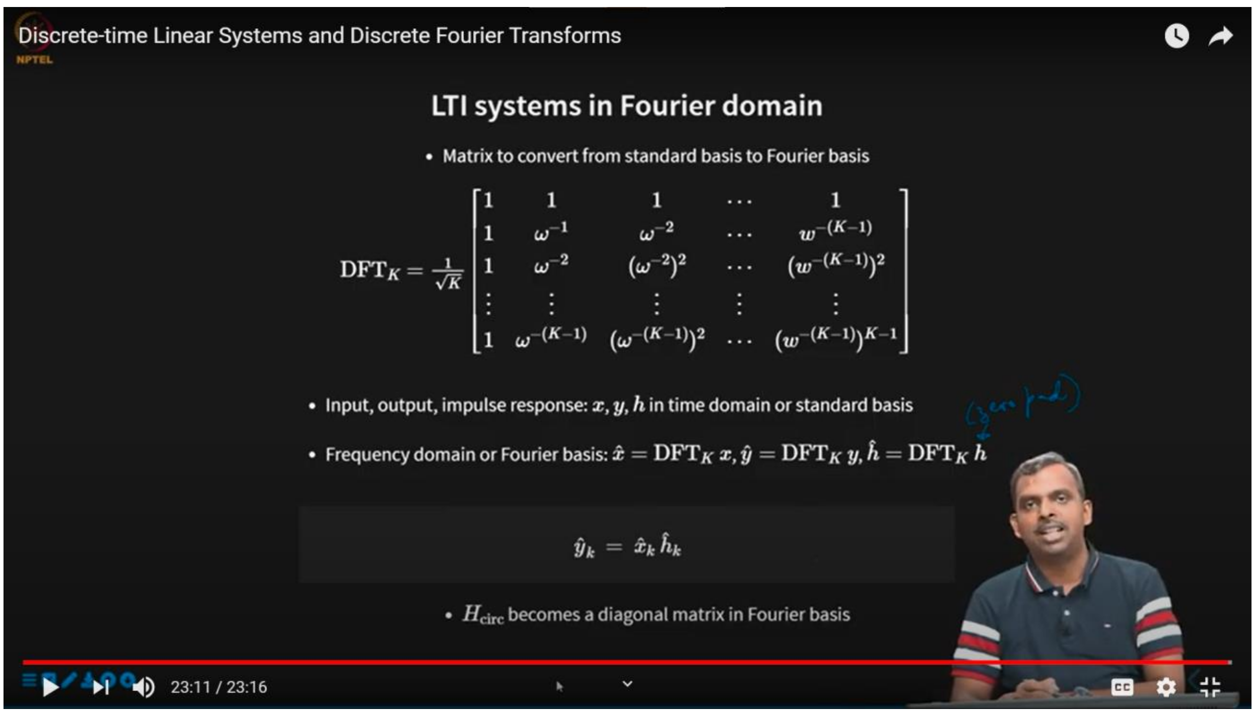

So the notation one can use is $\hat{x}$, okay? The Fourier basis represented coordinates of x, I will call it $\hat{x}$. $(\hat{x}_0, \ldots, \hat{x}_{K-1})$. $\hat{y}$ is $(\hat{y}_0, \ldots, \hat{y}_{K-1})$. How do you go from $\hat{x}$ to x? You simply multiply by IDFT, okay? Going from $\hat{x}$ to x is multiplying by IDFT. How will you go from x to $\hat{x}$? You have to multiply by the inverse of this matrix. It turns out the inverse of this matrix can be very easily found and that is called the DFT matrix, okay? Matrix to convert from standard bases to Fourier basis, it is the DFT matrix. Instead of ω, you will have ω^{-1} . You can imagine, okay? So why this has to be true, the properties of this, you know, complex K roots of unity are such that this is the inverse of IDFT, okay? And the $\frac{1}{\sqrt{K}}$ also works conveniently for us to give the proper inverse, okay?

So what people do when they want to study LTI systems is not only use the time domain or the just the signal domain representation. For the input output and impulse response, people convert this x, y and H from time domain or standard basis to frequency domain or Fourier basis. How do you do that? $\hat{x} = (DFT_K)x$. $\hat{y} = (DFT_K)y$. $\widehat{H} = (DFT_K)H$. You may argue H has only length up to n. You have to zero pad, okay? Okay? So you do zero padding to convert the, you know length to K, okay? So that you can use DFT_K here, okay? So once you do that, what will happen in the Fourier domain? In the Fourier domain the circulant matrix becomes a diagonal matrix. What occurs on the diagonal? The λ_k. What is λ_k? It is nothing but $\widehat{H}$, okay? So $\widehat{H}_1$ to $\widehat{H}_K$ will occur on the diagonal for this H_{circ}, okay? And you see $\hat{y}_k = \hat{x}_k \hat{h}_k$, okay?

So H_{circ} is diagonalizable. It has a full set of eigen eigenvectors, no? Full linearly independent set of eigenvectors and that gives rise to this whole Fourier domain, frequency domain business of studying LTI systems, okay? So instead of worrying about a complex convolution with, you know, multiplying, scaling, addition and all that, once, you go to the Fourier domain it's only multiplication. Your H_{circ} became just a diagonal matrix, okay? And that's the powerful thing. And not only that, it doesn't matter what the impulse response is. As long as it's circulant, the Fourier basis is fixed. So it's very powerful, okay? So you take any signal, you write it in Fourier basis, you will get an idea of what's going to happen to it when it goes into the LTI system. So instead of looking at the impulse response, you want to look at the frequency domain representation or the Fourier basis representation for the impulse response. You get a sense of what it does to the different eigenvectors, you know, in the eigenvector basis. So the eigenvector basis gives you a fantastic grasp of how the whole, you know, LTI system works, okay? So this is put to heavy use in electrical engineering. Many subjects in electrical engineering talk about frequency domain, its various properties, you know? How do the coordinates behave in frequency domain with respect to the coordinates in time domain. So many nice problems can be phrased and ,you know, this behavior, if you do something to the time domain, something will happen to frequency domain, all of that can be quite easily derived and understood and they are used quite extensively, okay? So this notion of eigenbasis representation to simplify the operator is very, very, very crucial. Look at how we used it, okay? The circulant otherwise was having a very complicated looking matrix. We went into the Fourier domain or the eigenbasis domain, we got a very simple diagonal matrix and that's been used very heavily in applications in electrical engineering, okay? So hopefully this again convinced you that this eigenbasis representation is not some idle theory, it is very, very powerful and it is used quite a bit in engineering, okay? Thank you.