Applied Linear Algebra

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Week 07

Dot product and length in Cn, Inner product and norm in V over F

Hello and welcome. In this lecture we are going to start looking at inner products. Inner products are extremely important. They also lead to something called norm in vector space. That's also extremely important. And both inner product and norms are sort of generalizations of what you might have learned before as dot product and length in, particularly in real vector spaces. In complex vector spaces we learn dot product and we learn about length of a vector, the magnitude which is defined in a standard way. Inner product and norm generalize those notions. That's one way to think about them. Another way to think about them is that, you know, they capture one property of the vector. So vector usually sometimes, in today's applications vectors tend to be very long. The vector itself could be length thousand, you know? There may be a thousand coordinates. And how do you get a sense of how big it is or how small it is? Or suppose somebody gives you two vectors. How do you get a sense of: are they close to each other in some sense, right? In a loose sense. As just a single number as opposed to the entire vector, you know. I mean if you have two length 5 vectors, maybe you can stare at them for a while. You would know whether they are close, whether they are big or small. The coordinates are easy to study. But what if it's thousand long, you know? I mean then it's good to have some generic method by which you can look at the, you know, some metrics of those vectors, some simple functions of those vectors which give you a good idea of, say, how big they are. If there are two vectors, how close they are together, how related they are together. So this inner product and norm can be thought of as giving you good metrics and good functions of the entire vector and to give you an idea of these kinds of things, okay? So keep that intuition in mind as we describe the theory of how inner products are defined, how norms end up coming up, showing up and what are the various properties that they are all related to, okay? So let us get started.

A quick recap. We've been talking about vector spaces over a scalar field F. And the F typically is real or complex. We saw matrix representations and how, you know, various subspaces and, you know, are defined around a matrix and how they contribute to solving linear equations etc. And finally we also saw eigenvalues and eigenvectors and how these one dimensional invariant subspaces always exist when the field is complex. And how they give rise to very good properties for characterizing how a linear operator actually behaves, okay? So this... Finally we also saw this nice result that upper triangular matrix representation is possible for every operator. Particularly for operators eigenvalues and eigenvectors are very, very useful, okay? So this is a quick recap. So let us jump into inner products, okay?

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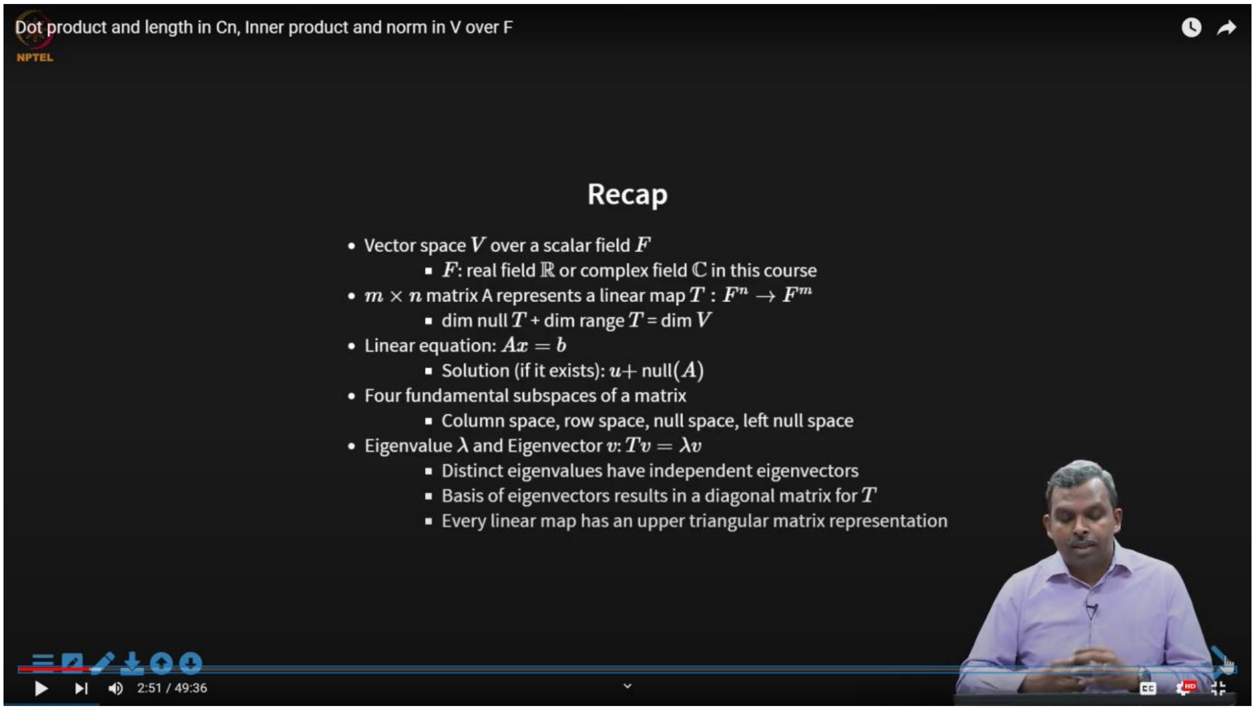

So I will begin by talking about the familiar dot product and notion of length that you have in real, you know, euclidean vector spaces $\mathbb{R}^2$ and $\mathbb{C}^2$, $\mathbb{R}$ and $\mathbb{C}$ and things like that, okay? So this is something that you're familiar with. We must have studied this from school. If you have two vectors, let us say in $\mathbb{R}^2$. One is (x_1, x_2). The other is (y_1, y_2). Then the dot product of these two is defined as $x_1y_1 + x_2y_2$. So you immediately see the dot product takes two inputs, two vectors x and y and... I've written it in this order but this is the definition, right? The output of the dot product, after you evaluate the dot product you just get one real number, okay? In this case, right? So that's, so you can think of the dot product as mapping two vectors to one scalar, okay? So that's a good way to think about it. That's one thing that's nice. So why is dot product interesting in linear algebra? So the first property that I put there is at the heart of why dot product is very interesting in linear algebra. If you fix y, okay? If you make one of the two vectors fixed and if you vary x, okay, dot product gives you a function from one vector to real numbers, right? So dot product is generally a function of two vectors to a real number. If you fix one of those vectors, it simply becomes a function of one vector to a real number, right? So that's what the dot product is. And that function will be linear in x, okay? So linear in the sense that, you know, your additivity will be satisfied, homogeneity will be satisfied. So that gives you the connection to linear algebra, okay? So this linearity is sort of fundamental and nicely built into the inner product, okay?

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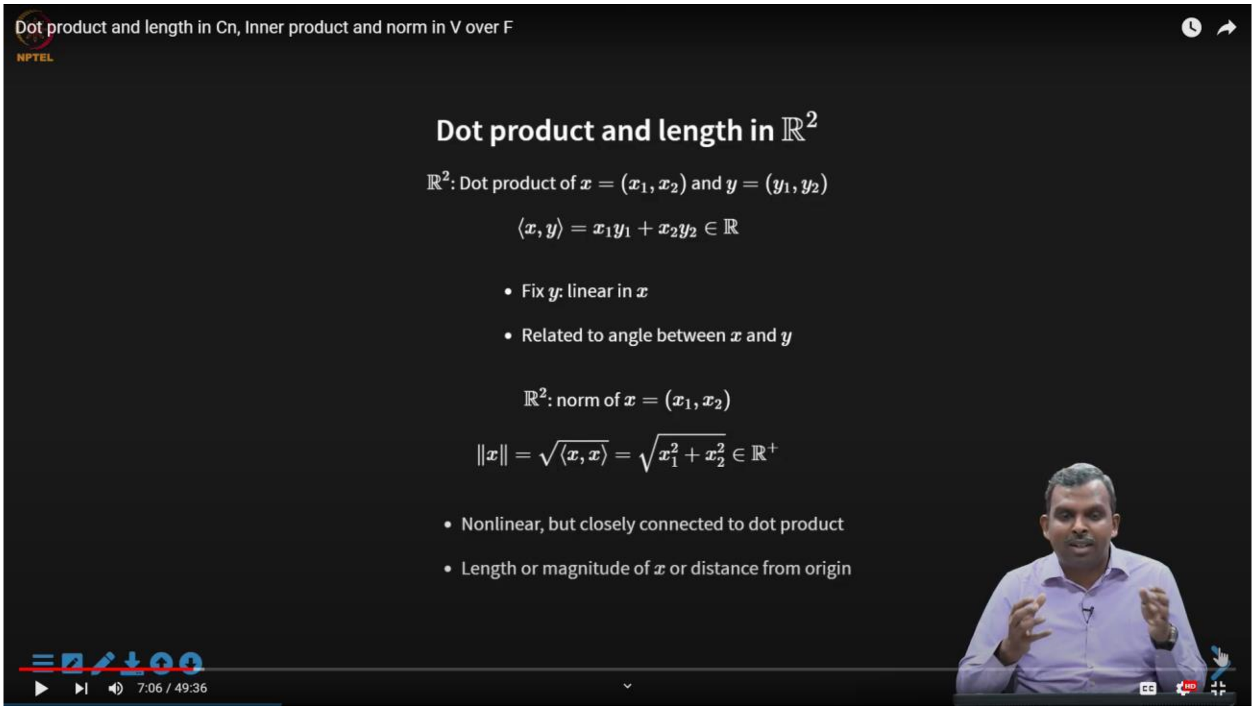

So we also know from our experience from before that dot product is related to the angle between x and y. So on the real plane, if the dot product is very small then the two vectors are in the same direction or similar direction. Maybe not the same, similar direction. So if the dot product is high, I am sorry. If the dot product is very small then the two vectors are sort of orthogonal to each other. And you know the angle is related in that fashion. So let me repeat that. If the two vectors are sort of in the same direction, then their dot product is very large. If the two vectors are orthogonal, then the dot product is very small. And if the two vectors are 180 degrees away, right, opposite, then the dot product becomes negative and large, right? So that's the sort of idea that you have about how the dot product behaves as the two vectors turn around. Now these things can be sort of generalized to n dimensions as well. We will see that. But this intuition about the dot product sort of representing how close together or how similar or dissimilar the two vectors are is important. That's useful to carry over to higher dimensions as well, okay? So there is also a very closely related quantity called the norm which is a function from one vector to, function that takes one vector to a positive real number, non-negative real number, okay? So that's the notion of a norm. How is that defined? The norm of x in $\mathbb{R}^2$ is simply $\sqrt{x_1^2 + x_2^2}$. And you can see that's also the square root of the inner product between x with itself, okay? So these are standard facts. And you can see it goes to. I've used this notation $\mathbb{R}^+$. $\mathbb{R}^+$ simply denotes non-negative real numbers, okay? So 0 and higher, okay? So first thing you notice is the norm is sort of a nonlinear function, okay? So there is some squaring involved. Square root and all. So clearly there is some non-linearity going on. But it is closely related to the dot product. So if the dot product is linear and it is very interesting, norm should also be very interesting. And it represents the length or magnitude of x.

So in general if the norm is large, then the vector is considered large. It's got big magnitude. The norm is small, then the vector is small in some sense, okay? So this is the relative, I mean if you want to compare how big two vectors are, you can just look at their norms and know which is bigger, okay? So if there is a big difference, okay? So this is just a quick recap of what you must already know about dot product and length. So we will see how this generalizes.

We'll first look at the generalization to n-dimensional complex vector space $\mathbb{C}^n$, okay? So in that space, when one also knows how the dot product is defined, there are two vectors x and y. $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$. Their dot product is $(x_1\overline{y_1} + \dots)$, okay? So the conjugate enters the picture here, okay? So it's $(x_1\overline{y_1} + \dots + x_n\overline{y_n})$, okay? So this you see is actually a complex number. It's just a scalar. So once again the dot product maps two vectors in $\mathbb{C}^n$ to one complex scalar, okay? And what is that complex scalar? It's calculated precisely like that. And the conjugate is important, okay? So the conjugate enters the picture. Now if you look at what happens, once again these properties are still true. If you fix y, fix one of the vectors, particularly y, okay? If you fix y, then you have linearity in x, okay? So notice how I am conjugating y and that is why the conjugation doesn't interfere with the linearity, right? So if you fix y, then it's linear in x, okay? So conjugation is in the second argument. So this also generalizes the notion of angle and, you know, in general if the dot product here is large in magnitude, then you expect the vectors to be sort of aligned or, you know, diametrically opposite etc. But of course there's no notion of a sign in complex numbers. So you have to sort of think about how the angles will work, okay? So anyway. So this generalizes the notion of angle. So the dot product also indicates some similarity between the two vectors, okay?

Same way you can also look at norm. Norm for a vector x. Complex vector x. Square root of the absolute value of x_i^2, okay? So notice how the conjugation is very important, okay? So once again this is non-linear. I do not want to skip that point. It is nonlinear and the norm maps, you know, the result of the norm is what you get when you evaluate the norm for a vector. It's a non-negative real number. It's not a complex number, okay? So the inner product is complex. But when you do the norm, when you take the inner product of a vector with itself, you get a, you know, real number. And take square root, you get, positive square root, so you have a positive real number, okay? So the conjugation once again I want to point out in the second argument is crucial. If you did not do that, you will not have a norm, okay? So that will not happen and this is similar extension of length and magnitude to n dimensions, okay? So once again I think this is familiar to most people here. You would have seen it in some course or the other and this is the dot product and length in $\mathbb{C}^n$, okay? The same thing you can do in $\mathbb{R}^n$ also, right? So $\mathbb{R}^n$ is of course contained in $\mathbb{C}^n$ in some sense. So the conjugation will have no effect. So it will be a direct dot product extension.

Notice a couple of things here. $< x, y >$ and $< y, x >$ are not the same in this $\mathbb{C}^n$, right? When you do $< y, x >$, you will have x being conjugated, okay? So maybe I should write that down. So notice here $< y, x >$ will be $y_1\overline{x_1} + \dots + y_n\overline{x_n}$. And it need not be equal to $< x, y >$ in general, okay? So already we see a difference between real and complex, okay? So this is something you keep paying attention to. But if x and y were over $\mathbb{R}^n$, if they were both real, then in that case these two will be equal. $< y, x >$ and $< x, y >$ will be equal. In complex in general they need not be equal, okay? Something to pay attention to, okay? So this is dot product in $\mathbb{C}^n$.

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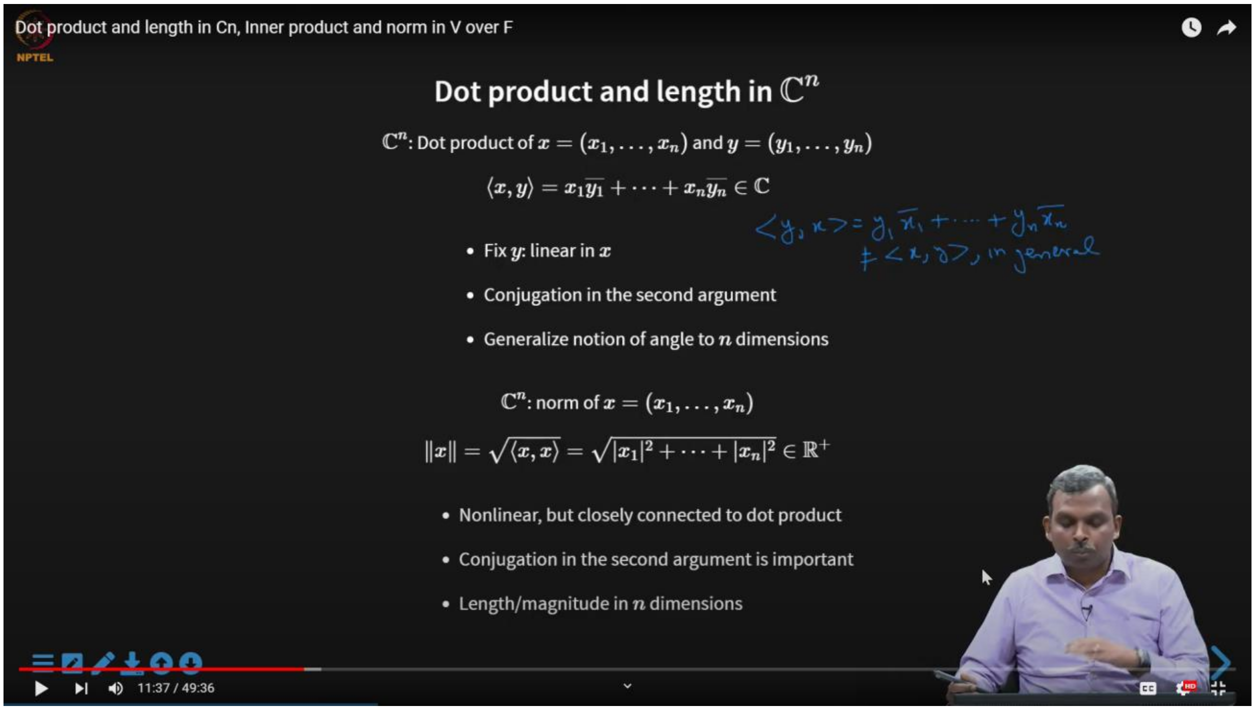

Now what we're going to do is take these ideas and generalize it to abstract vector spaces, okay? So so far in this course, you might have seen that that's one of the themes we looked at $\mathbb{R}^n$, $\mathbb{C}^n$, the familiar vector spaces that you've been studying so far and we abstract it to an abstract vector space over a field, and then we define, you know, just based on the operations what's going on. And we define, extend and we study a lot of properties from that. We'll do the same thing here, okay? Instead of restricting to $\mathbb{R}^n$, $\mathbb{C}^n$, instead of only looking at dot product and length defined in this way, we will abstract and define a general inner product in an abstract vector space V and what norm it leads to etc. etc. So that's how we will proceed in this lecture. Okay. So that leads us to this notion of an inner product. Let's start with a vector space V over a field $\mathbb{F}$ again. $\mathbb{F}$ will be $\mathbb{R}$ or $\mathbb{C}$ the way we define it here, okay? So the inner product on V, on the vector space V is a function that maps two vectors $< u, v >$ in that order. So we will keep them as an ordered pair, okay? So $< u, v >$, to a scalar $< u, v >$. And we will denote it like this, okay? So the scalars are denoted as $< u, v >$, okay? So that bracket, the angle brackets just like before. And that should be a scalar, okay? So the function should map two vectors to one scalar, okay? Additionally it should satisfy all these properties. And you can see all these properties are sort of motivated by the dot product that we studied before, okay? First thing is positivity, okay? If you take an inner product of a vector with itself, you should get something non-zero. Next thing is definiteness. As in when should the inner product of v with itself be equal to zero? That should be true only for $v = 0$, okay? So this is sort of an important restriction also, okay? So non-negativity is very important. And when should it be 0? It should be 0 only if the vector itself is 0, okay? So no other vector, non-zero vector should give you a 0 dot product with itself. So this is an important condition. It should be additive in the first argument $u_1 + u_2$ if you do dot product with v, or inner product with v, you should get $< u_1, v > + < u_2, v >$. It should be homogeneous in the first argument again, okay? So if you put $< \lambda u, v >$, right? You scale u and then you take inner product with v, you should get the same thing as λ coming out $< u, v >$, okay? So this is homogeneity in the first argument. And you should have conjugate symmetry, okay? So this conjugate symmetry you can see already the notion that you are dealing with real or complex comes strongly into the definition, okay? The conjugate symmetry is $< u, v > = \overline{< v, u >}$, okay? So of course there are more abstract definitions where you do not restrict to $\mathbb{R}$ or $\mathbb{C}$ where you may not have such definitions. But then the properties that you can study go down also. So in this case since we are only studying real or complex, we will look at conjugate symmetry, okay? So $< u, v > = \overline{< v, u >}$,. You can see how this is inspired by the dot product definition. You can go back and see the dot product satisfies all of these things. So we keep all those properties when we go to the abstract vector space, okay? So this is the definition. I mean it's quite abstract, you know? The typical abstract definition that we've been seeing in this class.

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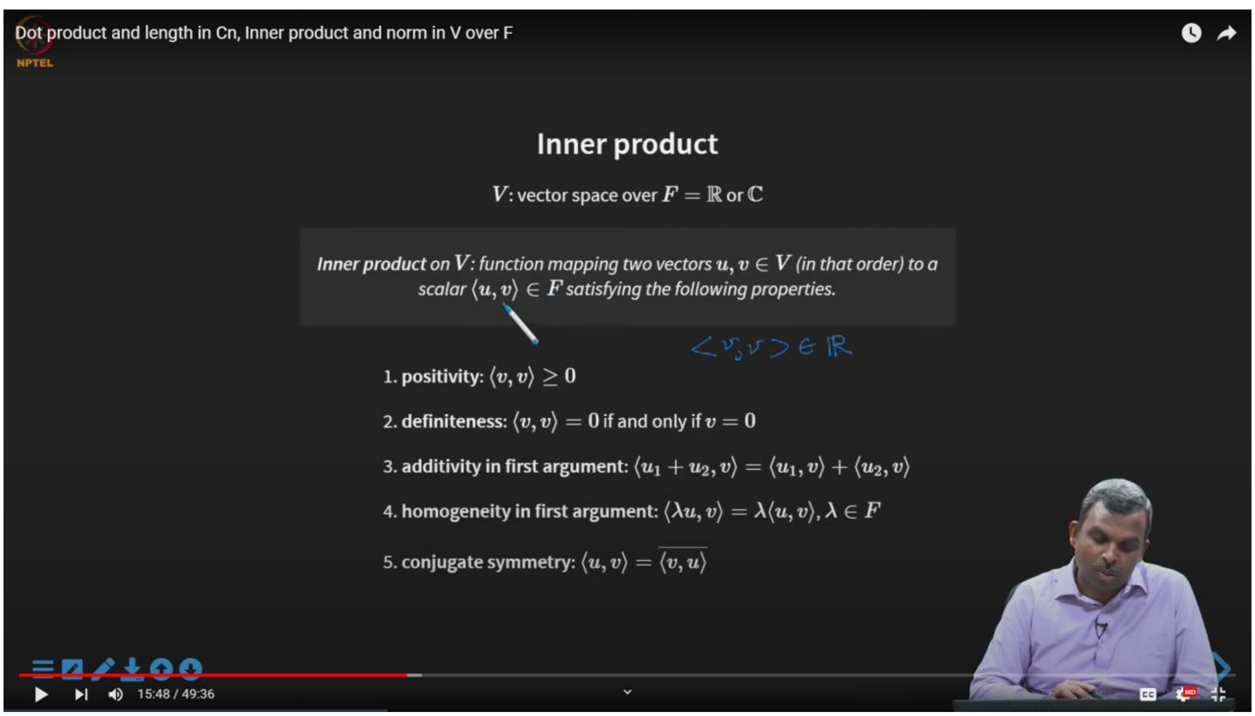

So now if I give you any function that will map two vectors to a scalar, you should be able to apply these rules and decide whether or not it's a valid inner product, okay? So I will put out a function which will take two vectors as input and give out a scalar as output, okay? You have to be able to check whether it's a valid inner product, okay? So that's one of the first things or outcomes that you should have out of this lecture, okay? Given a definition of an inner product, can you apply these definitions, given specification of a function that maps two vectors to one scalar, can you apply these conditions and check whether each of these conditions are true for that definition and whether that's a valid inner product or not, okay? So this is very important. So in the definition, one aspect that you have to pay attention to is that, you know, I have said the inner product should belong to $\mathbb{F}$ but $< v, v >$ I am saying is greater than or equal to 0. So one of the conditions which is very important is $< v, v >$ should be real, okay? So this is a condition that I've not explicitly stated in the definition. But this is very important, okay? So the inner product should be such that $< v, v >$ should be a real number, okay? So this, I mean, even though the $< u, v >$ when u is not v can be complex or something, $< v, v >$ has to be real, okay? So that's the condition sort of implicit in this. I have not stated it down. Maybe that should be noted down.

So let's see a few examples of that, okay? So the first example. I will come back to $\mathbb{R}^n$ and $\mathbb{C}^n$. The familiar dot product in $\mathbb{R}^n$ and $\mathbb{C}^n$, okay? So you can go through and try each one of those conditions and all of them will be satisfied and it becomes a valid inner product, okay? So that's easy enough to see, okay? The second one is slightly different. You see it sort of looks like an inner product, right? So it looks like the dot product. But I have changed something here, okay? So what have I done here? I have done, I am defining a new sort of function which takes two vectors x and y in $\mathbb{R}^2$ to this one scalar. The normal dot product I would have said $x_1 y_1 + x_2 y_2$, but I am putting a $+2$ here, okay? So I am putting $x_1 y_1 + 2 x_2 y_2$. You can go and check all the conditions and all the conditions will be true. You can see, no? Linearity in x, homogeneity in x, you know? $< x, x >$ being non-negative maybe is something, one thing which is slightly difficult to check. So you should check that. $< x, x >$ is $x_1^2 + 2 y_1^2$ and that's greater than or equal to 0. Equality if x_1 is 0, y_1 is 0, okay? So this part you have to check. The second condition, the first and second condition, right? So $< x, x >$ should be non negative and it should be equal to 0 means only for $x_1 = 0$ and $y_1 = 0$, okay? So these two conditions need some checking. Linearity and homogeneity sort of come through. And since it's just $\mathbb{R}^2$, there's no, the conjugate symmetry or something doesn't even apply. So just work out, whether you do $< x, y >$ or $< y, x >$ you get the same answer here, okay? So that also you can check. So all of those properties you have to check one after the other, okay? When you do that, you can conclude that this is a valid inner product, okay? So the point of this is also to show you that the dot product is not the only inner product in $\mathbb{R}^n$, okay? You can have other inner products also, okay? So which have, inherit all the same properties. These are just called weighted inner products. It’s not very special, okay?

So here is another example, okay? So there is a minus and all that. So maybe it’s a bit unusual. Again I am sticking to $\mathbb{R}^2$. I can define in bigger spaces as well but let's just stick to $\mathbb{R}^2$. And I'm asking whether this is a valid inner product or not, okay? So now this needs a lot of careful checking, okay? Yeah. So I mean you have to check every product very closely, every property very closely. I think, let's worry about first positivity of $< x, x >$. So let's just look at that very closely. So if you see that, if you see $< x, x >$, you're going to get $x_1^2 \, - \, 3x_1x_2$ another $-3x_1x_2$ and then $+ \, 20x_2^2$, okay? So this you can sort of write as $(x_1 \, - \, 3x_2)^2$ and then you will have a $11x_2^2$, okay? So this form is quite important, okay? So this is clearly greater than or equal to 0, okay? So once I wrote it down like this, you say it is greater than or equal to 0 and equality if $x_1 \; = \; 3x_2$ and $x_2 \, = \, 0$, isn't it? So that implies, together implies $x_1 \; = \; x_2 \; = \; 0$, okay? So these positivity and definiteness are satisfied. Linearity in x_1 is very clear. Homogeneity in x is also very clear, right? And what about this conjugate symmetry? So basically you need symmetry, right? So if you do $< y, x >$, what do you get? You will get $y_1x_1 \; - \; 3y_1x_2 \; - \; 3y_2x_1 \; + \; 20y_2x_2$ and you see that's exactly equal to $< x, y >$, okay? Notice how this 3 and 3 being the same was crucial for the symmetry, okay? So if you put 3 here and 4 here, it won't work. Likewise here 3, 3, and then the fact that you had 20 here was crucial for the positivity, okay? So if you change these things around, this may not work, okay? So all these conditions are satisfied. So this expression that we have here is once again a valid inner product in $\mathbb{R}^2$, okay? So you see already there is some more interesting stuff that's happening. In general there can be various interesting inner products. So generalizing to inner products is not that bad an idea. You've got some variety there, okay?

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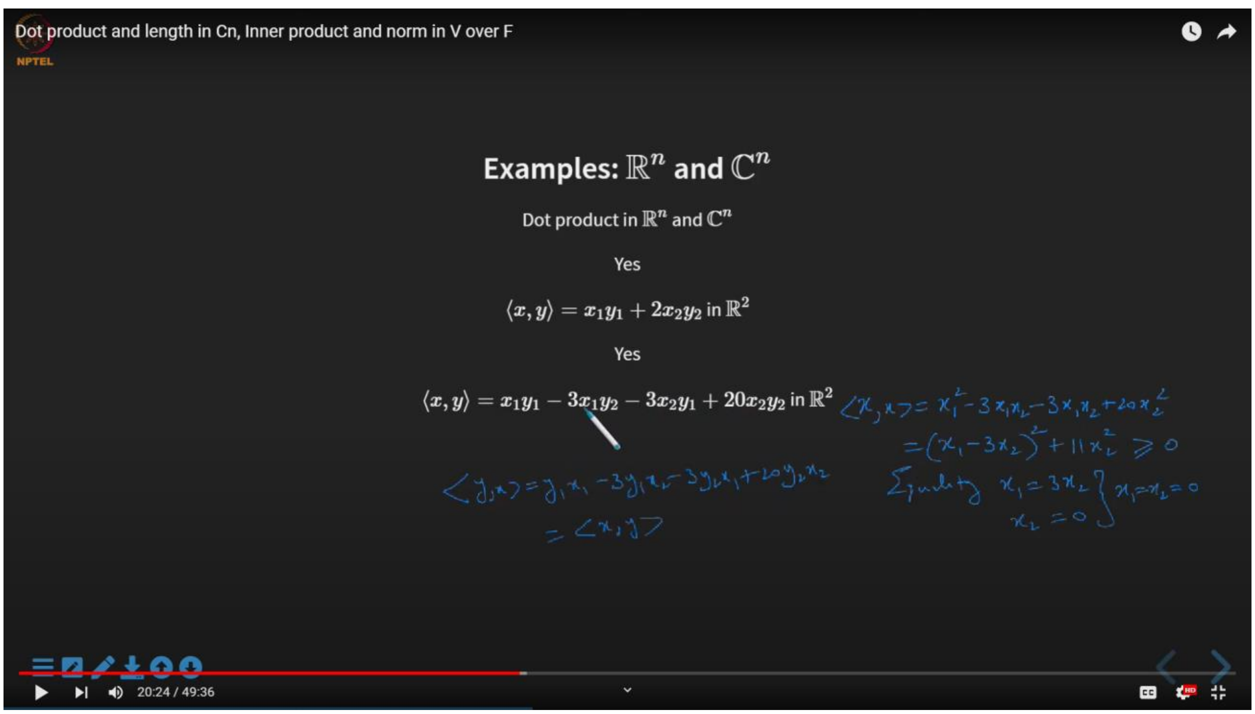

What about this one, okay? So this looks dangerously close to the previous one except that instead of, you know, 20, there I put 9, okay? And you might think this looks quite close but it turns out interestingly this is not a valid inner product, okay? So you can say... I will leave this as an exercise, okay, try on your own and then convince yourself which property gets violated by this, okay? So you can sort of repeat what I did last time and look closely for every property and see which property will get violated if I just leave it like this, okay? So this 9 is sort of a hint for you. You can see why that works out, okay? So these are some examples in ℝ and ℂ. You can also go a little bit further. We've been mostly studying finite dimension spaces, but I've also been pointing out, you know, other types of vector spaces which are more interesting also from applications point of view. Here is a vector space example of functions from, so the interval $[0, 1]$ to the real line, okay? So so many functions out there. So one can define inner products with functions, okay? So here is a definition. Inner product between f and g. I can define as $\int_0^1 f(x)g(x)dx$, okay? So you can see how all the properties will be satisfied. You can check maybe, you know, positivity and maybe the definiteness property needs a little bit of knowledge of calculus and all that but... And also, you know, there is, I mean, what is zero? What is a zero function? Is one there? There are strange definitions with respect to these things so let me not go into great detail there. But let me just say that this is a reasonable definition for inner product. Most of those function properties that we had with suitable extensions and proper calculus and proper understanding of what functions are, when they are 0 and all that with respect to this integration, this will be a proper function. Let me just leave it like that. We are not going to go into great detail there. So with all that, this will be an inner product, okay? So this you can see $\int_0^1 f(x)^2$ if it has to be 0, then $f(x)$ has to be really 0, right? That's integral. So maybe not 0 everywhere but 0, as they say, almost everywhere. So that's not a bad way to think of zero functions, okay? So this is the definition and linearity, homogeneity, symmetry all of that will come through quite easily, no problem, okay? So what about something like this? Here is another definition. $\int_0^1 f(x)g(x)e^{-x}dx$. Again all of the properties are satisfied, okay? So you can go check all of them. Once again that when is a function, when is an integral 0, when is a function 0? That definition usually is made in a sort of hand-wavy fashion. Assuming that that works out okay, this is a good inner product, there's no problem, okay?

And here is another definition, okay? So what have I done here? I've put $f(x)$ and then I've done this $g\left(x - \frac{1}{2}\right)$, okay? So this $(x - \frac{1}{2})$ is sort of a funny function. So one needs to be a bit careful here. So you can say, because the function I'm restricting to 0 to 1, so I will ignore if the $x - \frac{1}{2}$ exceeds 0 or 1 on either side, right? So that's sort of understood here when I do this definition. So $\int_0^1 \left(f(x)g\left(x - \frac{1}{2}\right)dx\right)$, okay? What do you think about this guy? This guy, will it be a valid inner product, okay? I will let you do this once again, I'll leave it as an exercise. It's not too critical for us. You can think of why this would not be an inner product or what function, what property it would violate, okay? So these are some examples. Certainly unusual example. We have not been looking at spaces like this that much in this class. Mostly finite dimensional. But still just to tell you the power of these kind of abstract ideas, it's good to see some more, slightly more different examples, okay? So that's that.

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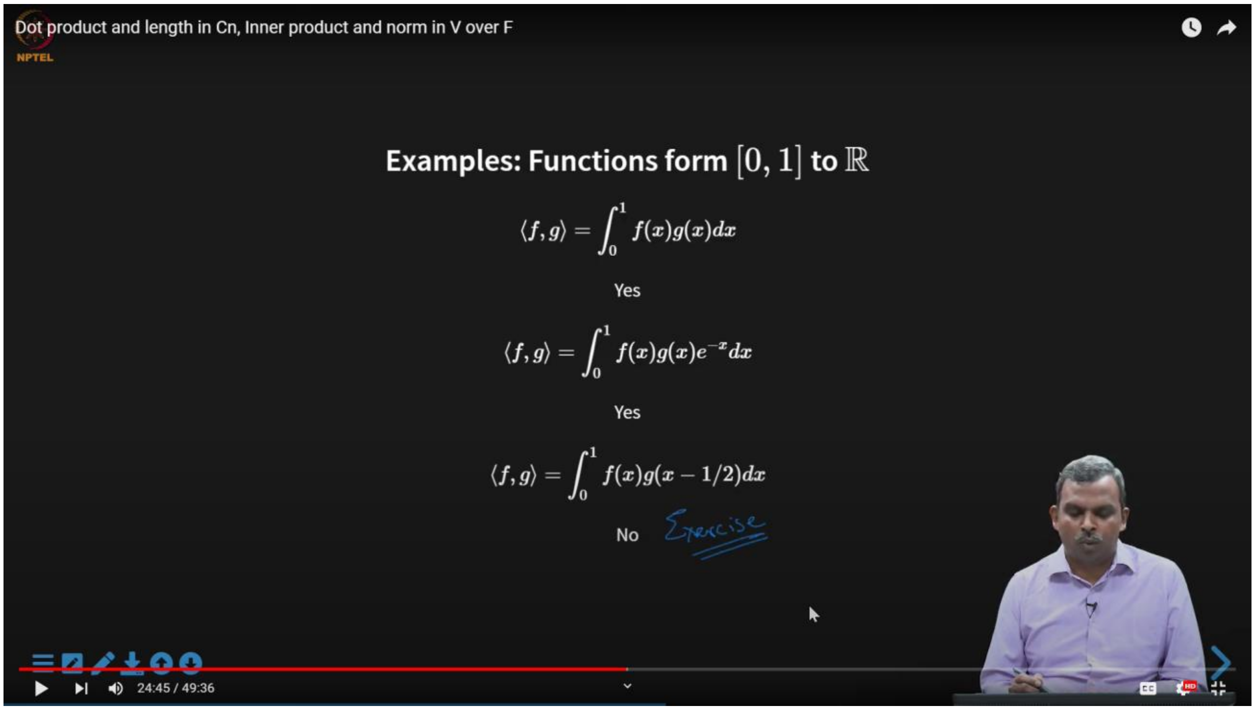

Okay. So armed with all that, let's now, we now have a definition for inner product in the abstract vector space, a whole bunch of properties it satisfies. We saw a few examples, they all look interesting. But now what else can we say? Can we say something precise, more interesting properties about, you know, vectors and inner products? Just understand it a little bit more? So the first thing is just a definition. If you have a vector space over a field and it has a valid inner product on it, it's typical to call it an inner product space, okay? So it's sort of a vague usage. I mean the inner product is actually separate from the vector space and it's just a function from an ordered pair of vectors to scalars but the fact that it allows an inner product and with that inner product if you approach the linear algebra aspects of things like study of operators and all that, it simplifies things tremendously, okay? So it gives you a lot of advantages, okay? So that's what we're going to see step by step slowly. So that's why it's interesting to even, you know, add that inner product to the space and call it not just a vector space but an inner product space. And inner product spaces have special properties that general vector spaces don't have, okay? So we will see that as we go along, okay? So the first few properties that one immediately sees, in fact the first property is very, very crucial. It gives you a nice connection between operators and inner products which we've been seeing before. If you fix u, or maybe I should do it the other way around, okay? Sorry about that. So there is a small bug here. It's not too bad. So if I make this instead of $< u, v >$ if I make it $< v, u >$ then I'm okay, okay? So I fix u and then I do this inner product with $< v, u >$ I get a linear map, okay? So there is no confusion. Once I do $< v, u >$, everything works out very cleanly. All the conditions for a linear map are satisfied. Linear map from where to where? Linear map from V to $\mathbb{F}$, okay? So this is a very, very important notion. This linear map, the fact that the, you know, inner product represents a linear map gives you so many advantages in simplifying our study of operators and all that, okay? So we'll see them as we go along, okay?

Let's also just get rid of a few quick properties. First thing is inner product with zero will give you zero. If you have a zero vector, if you take inner product, you will get 0. This can be proved using the additivity and, you know, then the symmetry, okay? So $< 0, u >$ and $< u, 0 >$ are conjugates of each other. One is 0 the other is 0 as well, okay? The same thing with the additivity in the second argument, okay? So what about the second argument? We've been thinking a lot about the first argument. Since we have additivity in the first argument and then we have the conjugate symmetry, you will also have additivity in the second argument, okay? So this needs a little bit of a proof. So you know $< u, v1 + v2 >$ is conjugate of $< v1 + v2, u >$. And then you apply additivity in the first argument and then you will get conjugate plus conjugate and then you use the conjugate symmetry again, you will get this, okay? So this is a proof. I will leave it once again as an exercise. Please try it, okay? The same thing here also. I will leave this also as an exercise, okay? Try both the proofs. They are quite easy to write down. The homo, the conjugate homogeneity in the second argument, okay? So $< u, \lambda v >$ is $\overline{\lambda}$, the conjugate of λ times $< u, v >$, okay? So again the proof is very similar. You just use the properties, you'll get it, okay? So these properties are useful. And as you keep working with inner space, inner product and, you know, doing algebra with it, this will help you a lot, okay? So basic properties. Mostly intuitive. It's not very difficult to prove.

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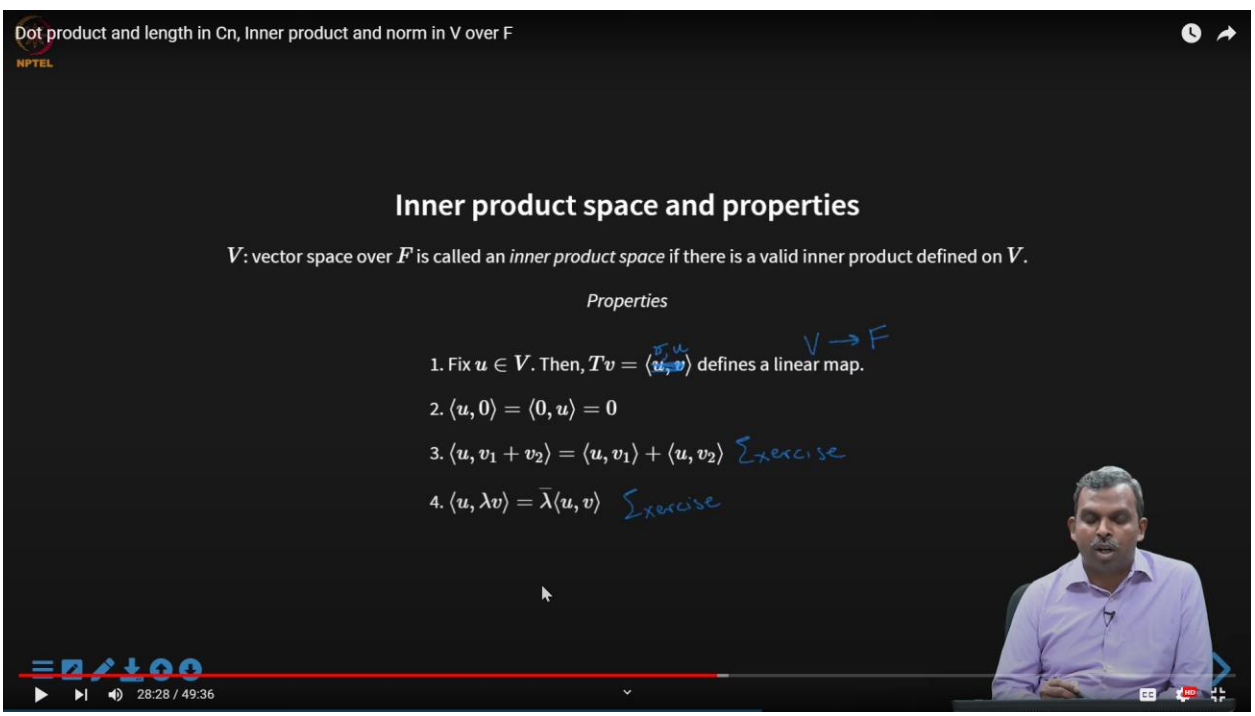

Okay. So the next big definition that helps you quite a bit in classifying operators, understanding how vectors are, the relationship between vectors particularly is this notion of orthogonality. And when you have an inner product properly defined, valid inner product and V is an inner product space, you can define whether two vectors are orthogonal to each other, okay? So if u and v are orthogonal, they are said to be orthogonal, two vectors are said to be orthogonal in an inner product space if their inner product is zero, okay? The inner product that is defined in the inner product space, if two vectors have an inner product of zero, they are said to be orthogonal. So now this word orthogonal is some statement about the relationship between these vectors. We sort of expect them to be different in some sense, okay? With respect to the inner product, as far as the inner product is concerned, these two are sort of away from each other. They give you a very small inner product, okay? So that's the notion, okay? You have to contrast it with $< u, u >$. $< u, u >$ is positive, right? So if u and v are nonzero, $< u, u >$ is positive, $< v, v >$ is positive. So they're all sort of big in some sense. And then $< u, v >$ is zero. So then they're not related, okay? So that's the notion roughly to think of why this orthogonality is important, okay? This is also motivated from the angles that we know with the regular dot product, okay?

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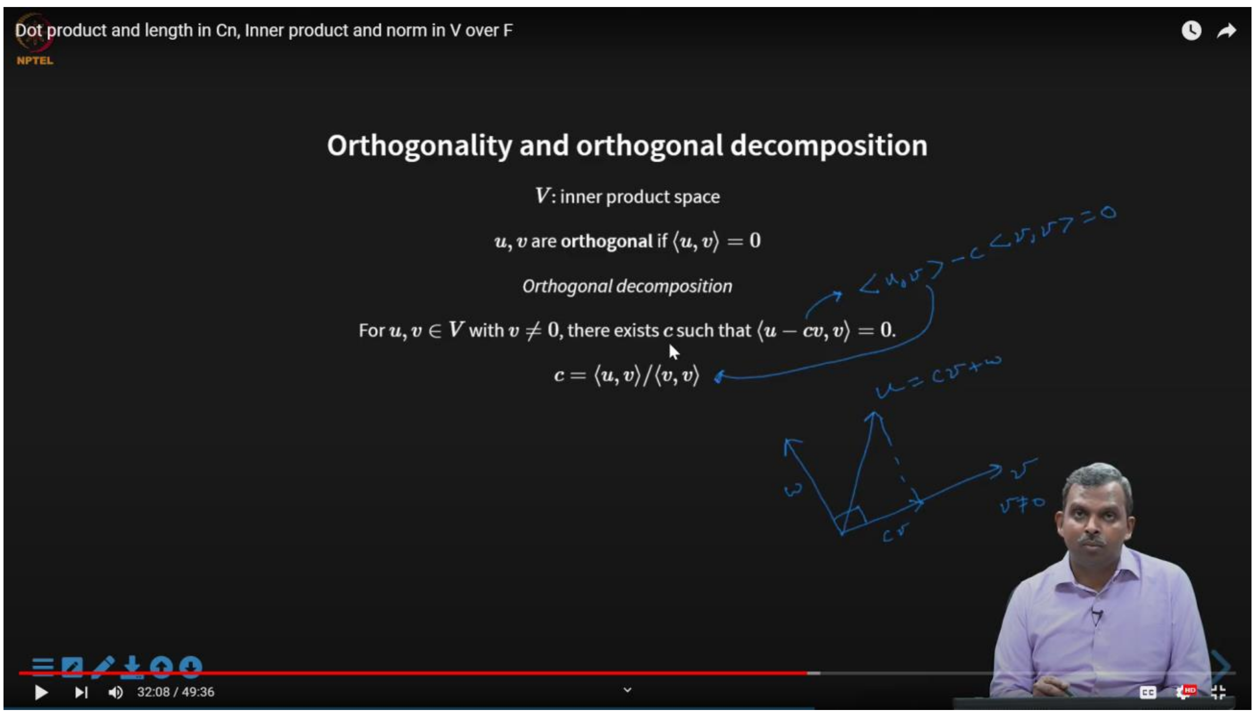

Now what's equally interesting is this notion of an orthogonal decomposition, okay? So you may be very familiar with this kind of pictures that people would draw, right? So you have v as one vector, you have u as another vector, okay? You can write u as some cv plus some w and these two are orthogonal, okay? So $u = cv + w$ and w is orthogonal, okay? So this is a picture, very familiar geometric picture. So when u and v are two, any two vectors and v is non-zero, okay... So v needs to be non-zero. If v is zero, you can't do this. If v is non-zero, you can sort of write u as a part of u which is along v, which is c times v, plus a part of u which is orthogonal to v. So this is always possible for any two vectors. This is huge, this is very, very, very important. The fact that you can sort of relate the two vectors in this very clear, you know, directly proportional, you know, totally dependent part and the completely orthogonal part, okay? So inner product gives you this way to, you know, split a vector into two. When there are, given two vectors, you can relate them so strongly like that, okay? And it's very easy to do, okay? So that's what I've shown here. You can see if you we are in two vectors and v is not zero, there exists a c such that the inner product of $< u - cv, v > = 0$. So it's a trivial thing to check, right? So if you just apply, so what is this guy? You just apply the additivity property, you get $< u, v >$. And the homogeneity property in the first argument $-c < v, v >$ equals zero, okay? And that directly gives you this relationship, okay? $c = < u, v > / < v, v >$, okay? And v is non-zero, you know $< v, v >$ is non-zero. So this is a proper valid construction, okay? So there is always a constant or scalar c, let me not say constant, scalar c such that $< u - cv, v >$ is orthogonal, okay? So you can make it orthogonal and this gives you the orthogonal decomposition, okay? So the same picture is written differently. So u becomes c times, the same cv plus w. And what is w? w is just $u - cv$, okay? So it's not any great thing. w is just $u - cv$. So cv just cancels. So you get $u = u$. The same thing you write like this. But except that you notice that this guy, the first guy is parallel to v, okay? So cv, right, this part. And then w is orthogonal to v, okay? $u - cv$, that picture I drew with the orthogonality, okay? So this is called the orthogonal decomposition, okay?

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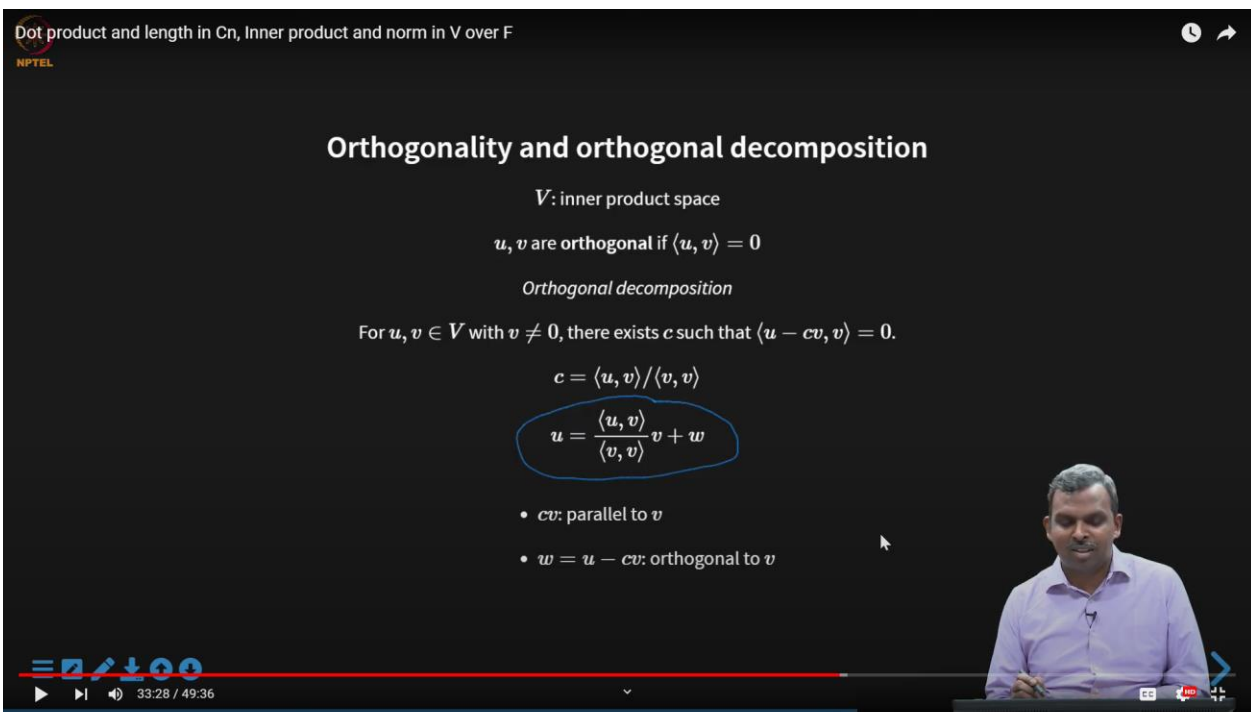

So it's very, very crucial. This is very important, okay? And it plays a big role in many of our applications and intuitions about vectors. So when you want, when you have two vectors and you want to think of them, you know, how close are they to each other, so simply write one as an orthogonal decomposition on the other, okay? At the end you will know what part of it is close, what part of it is far away, not related, etc. And you'll get a very good understanding. It's also very useful in proving so many powerful properties about inner product, okay?

It seems like, you know, we made some very generic definitions. Is it powerful enough? Is it, you know, strong enough? Does it give you good properties? In fact it does and here is one of the very, very important properties about inner products. Once you have an inner product in an inner product space and there are two vectors u, v in V, then this relationship is true, okay? The inner product of $< u, v >^2$ is upper bounded by the product of two inner products. The first one is inner product of u with u. The second one is inner product of v with v, okay? So this is called the Cauchy-Schwarz inequality. It shows up in so many applications, in so many places and gives you a very easy way to estimate things and, you know, just do analysis very cleanly, okay? So it's a very powerful inequality cutting across so many areas of mathematics. It's called the Cauchy-Schwarz inequality and it's valid in any inner product space. Once you have defined an inner product, for every inner product this inequality is true, okay? So this is the power of proving something abstract, right? You make an abstract inner product definition and then you show an inequality that's valid for every specific particular inner product you study. Weighted dot product and all those integral definitions, this, that all of them will satisfy this inequality, okay? So that's very nice.

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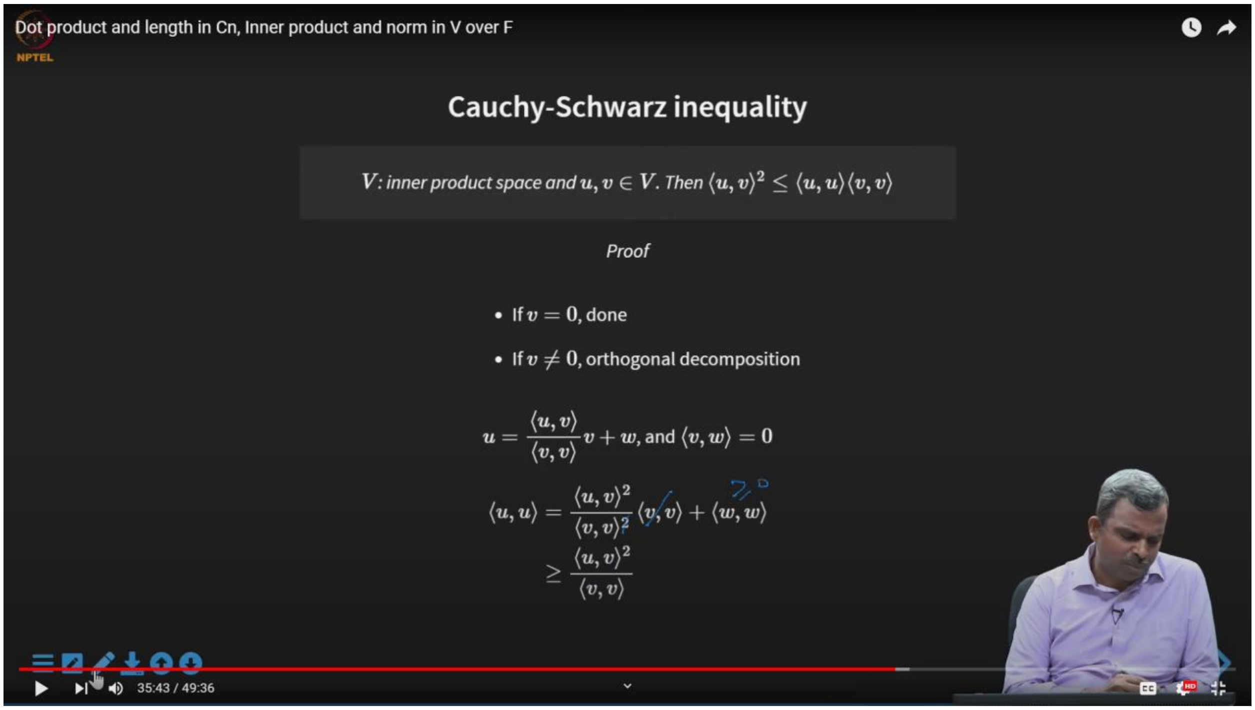

Let's see a very quick proof. I won't spend too much time on it. It's just algebra. You can go through it. But this orthogonal decomposition plays a crucial role, okay? So v is 0, you have equality on both sides. Sort of trivial to see. If v is non-zero, you employ the orthogonal decomposition, okay? So you write u as $cv + w$, okay? And then you just look at $< u, u >$, you will get the component along u and then you will get the component, you know, orthogonal to u and that's it, right? So you won't get anything else, okay? So the term, you know, $< v, w >$, okay, that should also actually show up here. But $< v, w >$ will go up to zero, okay? So that won't show up. And so you'll get one part here. And this $< w, w >$ is positive, right? Okay? Notice this $< v, w >$ is 0. So this becomes equality. And this part is non-negative so you will get this is greater than or equal to this. And there's some cancellation here. So you will get this, okay? And you get the Cauchy-Schwarz inequality. It's a very simple consequence of the orthogonal decomposition and it's really, really powerful. It can give you very non-obvious looking inequalities, okay? So this is Cauchy-Schwarz. Maybe we'll see some examples of how powerful this is later on.

Okay. The next definition is that of norm, okay? So we've seen that the dot product has a very nice extension into this abstract inner product and we saw how to, you know, use it. You know this, we saw two various ways of defining it and already we have seen one nice application of this inner product to get this Cauchy-Schwarz inequality, okay? Now let's see norm. Again we'll start with the vector space over a scalar field $\mathbb{F}$ being $\mathbb{R}$ or $\mathbb{C}$. And the norm on a vector space is a function mapping one vector v to one scalar. And this time we'll use this notation, the usual notation for norm. Two vertical lines on both sides. Double bars and v inside, okay? So this should be a real number, okay? It cannot be complex or anything. It should be a real number and it should satisfy all these properties. So the norm is, like we said, you know, there is this notion of length that we want to extend to an abstract vector space. So we are going to say that the norm should be a real number, okay? Not only that, norm should be non negative, okay? So the first condition is that it should be greater than or equal to 0. The next one is definiteness which we have always been wanting. If it is equal to 0 its if and only if the vector itself is zero. No other vector should have zero norm, okay? And we will want this absolute scalability, okay? So $\|\lambda v\|$ should be norm λ, I mean $|\lambda|||v||$. λ is of course a scalar here. So absolute value of λ will be real. So that's that times v, okay?

And very importantly triangle inequality, okay? So this is a sort of a distinguishing feature of the norms that we'll study here. The triangle inequality is the $||u + v||$, when you add, should be less than or equal to $||u|| + ||v||$, okay? So this is the familiar concept that $u + v$, u and v together should form a triangle, right? So u and then v, and then $u + v$, they should form a triangle. And if they should form a triangle, then the $||u + v||$ better be less than or equal to $||u|| + ||v||$. Otherwise it will not form a triangle, right? So that's something that the triangle inequality ensures that you do in the norm. So you don't associate something with the vector which doesn't really represent a length in that sense, okay? So that's nice to see. And there are of course extensions of these definitions by dropping some of the conditions etc. But these conditions are particularly interesting and important to study. So anytime somebody gives you a function from a vector to a positive real number, you should be able to check these conditions and figure out if it's a valid norm or not, okay? So just like we did for the inner product, we had a bunch of definitions and somebody gives you a function, you should plug in those definitions and check whether all of them are valid and say yes, this is a valid inner product. Like that, like that, you should be able to do for a norm, okay? So if somebody gives you a function from a vector to the real numbers, positive real numbers, you should be able to check each of these conditions and say yes, this is a valid norm, this is not a valid norm etc. okay? So that is one important outcome.

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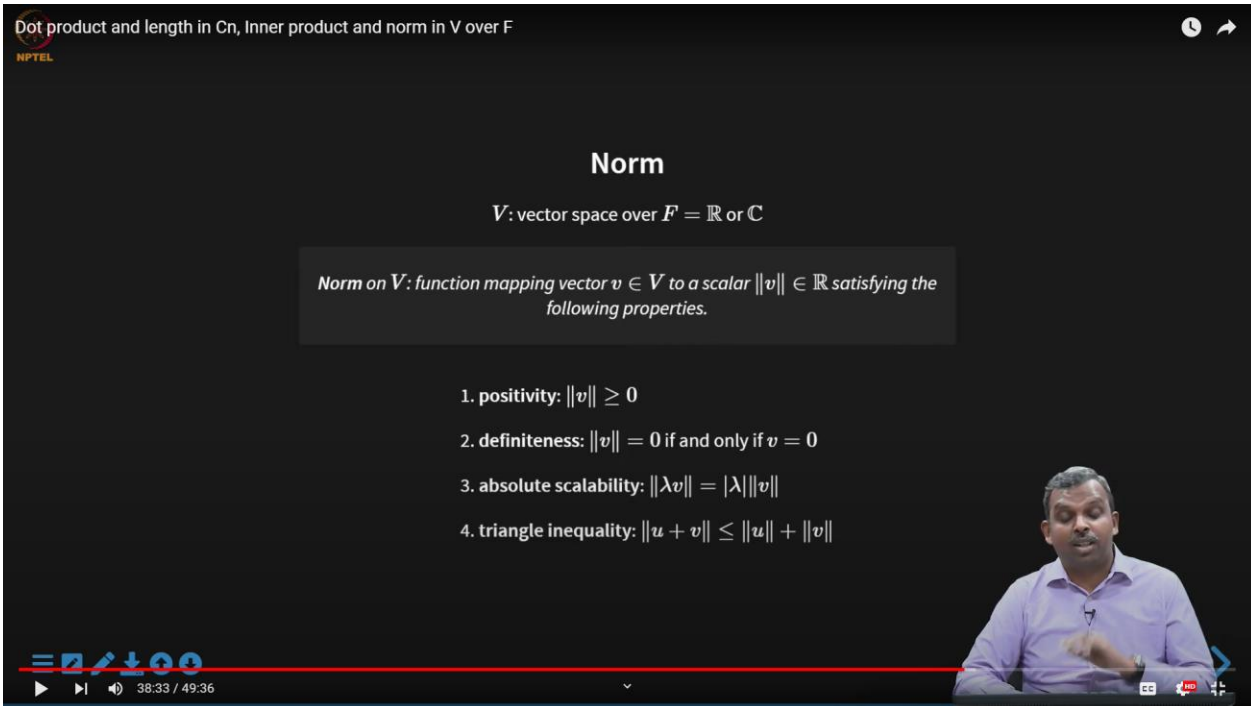

So let us see a few quick examples to see when something is a norm, when something is not a norm etc. The most famous and popular norm is the Euclidean norm or the two-norm. We'll see the secret of this word two soon enough. Or square norm or square root norm, a lot of, lot of names people give for this norm. It's very, very useful, very popular. It gives you a good sense of how big a vector is in a relative way also. It gives you a very good handle on the vector. So this is a common definition for, you know... There are small variations on this. You can do a one norm. It is also called a Manhattan norm or taxicab norm, there's various other names for it. One norms are also very popular. One norm for a vector is: there's no square root and squaring, it's simply absolute value of each guy added to, each coordinate added up, okay? So that is the one norm, okay? So you can check that the one norm satisfies all the conditions that we listed before, okay? Positivity, definiteness, the scaling property and also the triangle inequality. Triangle inequality might be a little tricky, but you know that the triangle inequality is satisfied by each coordinate separately and you can add it up and you can check, okay? The other norm is called the infinity norm, okay? So where are all these numbers coming from? One norm, two norm, infinity norm... You'll see soon enough. It's also called the max norm for an obvious reason. And it's simply the maximum of the coordinates. Absolute value maximum of the coordinates is called the max norm or the infinity norm. So this is also a norm. You can satisfy, so you can check all the conditions. Even triangle inequality is easy to check, everything is easy to check here. All of these things become norms.

So it turns out all these norms are a special case of what's called a p-norm, okay? And p has to be greater than or equal to one, okay? And the p-norm is defined in this fashion. p-norm of a vector x is absolute value of each coordinate raised to the power p added up, and the whole thing to the power 1/p, okay? So you can see why each of these things is the same. And the infinity norm you may not be able to quickly see, but the infinity norm will give you the max, right? Because as p becomes larger and larger and larger, I mean infinity is, meaning of course p going very large. And p goes very large, only the maximum absolute value is going to matter, everything else will get decimated, okay? As p increases and then since anyway you're doing a 1/p, that will sort of cancel and you'll get the max, okay? So infinity goes to this. So all these three norms are special cases of this. And in fact it turns out for any p this is a valid norm, okay? So this, proving this is a slightly difficult exercise. Most of the other properties you will be able to prove. Triangle inequality is a little bit tricky, okay? So you will need some slightly advanced inequalities, particularly what is called the Minkowski inequality. You can look it up on the internet, you will see that that's needed for showing that the p-norm satisfies the triangle inequality, okay? So that's a bit of a tricky definition. So all these are norms and they are valid norms and they are useful in vector spaces, okay?

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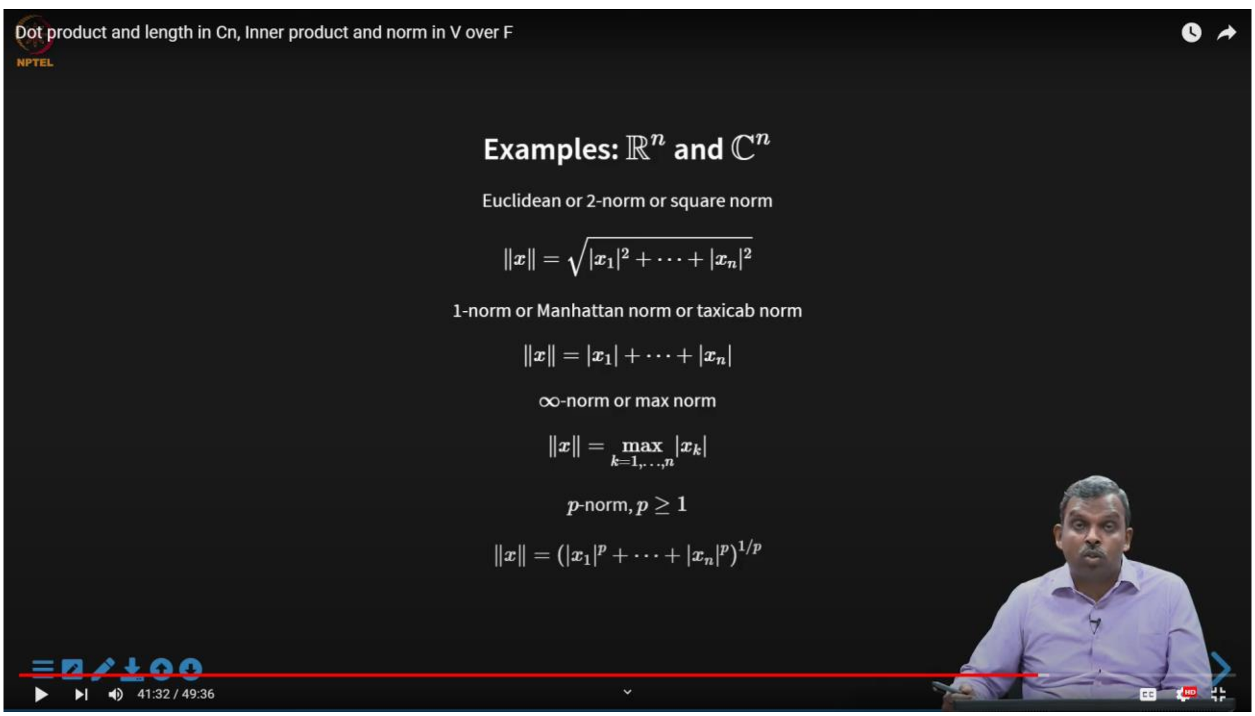

So for us the most interesting norm is a norm that comes from an inner product, okay? So we have seen this before in the case of dot product and length. Supposing you have an inner product space. It turns out square root of the inner product $< v, v >$ is a norm. It's a valid norm, okay? So once you have this, this is a nice definition to have, right? So square root of $< v, v >$. We can even for instance simplify the way in which we wrote down the Cauchy-Schwarz inequality. In fact you can simplify the way in which you wrote down the orthogonal decomposition, right? u, v are in V, then you can write u to be equal to, you know, $< u, v > v/ ||v||^2 + w$, right? And so this w is, you know, $u - cv$, okay? So you can see this. This kind of, this norm gives you at least a simplification to write. Previously we were writing $< v, v >$. So now wherever you have $< v, v >$ you can simply write $||v||^2$. Same thing you can do to the Cauchy-Schwarz inequality. How did we write the Cauchy-Schwarz inequality? We wrote $< v, u >^2$ is less than or equal to $< u, u >$ and then $< v, v >$, right? So now this is $||u||^2$, this is $||v||^2$. So absolute value of the inner product $< u, v >$ is less than or equal to $||u||||v||$. This is a rewriting of Cauchy-Schwarz just in terms of this norm. Checking that this is a norm is a very easy exercise, okay? All the properties will come out to be true except for the triangle inequality, right? So most of the other properties are easy. The homogeneity, the, you know, positivity, definiteness, all of them are very easy for this property except for the triangular inequality. So let's just see how to quickly check the triangle inequality, okay?

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The triangle inequality, the Cauchy-Schwarz inequality will play a role, okay? For triangle inequality to be true. Here is a proof. I'll just walk you through the proof very quickly. You can look at it closely later. You just look at $||u + v||^2$, okay? So that's sort of the starting point, right? You want to show $||u + v||$ is less than something, we look at square, okay? So that will be inner product $< u + v, u + v >$. And then you can expand it out using the additivity properties. And then you look at $< u, v >$ and $< v, u >$. One is the conjugate of the other. And when you add the two, you get two times real part and that's upper bounded by two times absolute value, okay? And once you have absolute value you use Cauchy-Schwarz, you get $< u, v >$ and you get $(|u| + |v|)^2$ and triangle inequality is satisfied, okay? So this Cauchy-Schwarz is again a sort of a restatement of, or equivalent, or sort of leads very crucially to this root of $< v, v >$ being a norm. So all these things are sort of interrelated in that nice way, okay?

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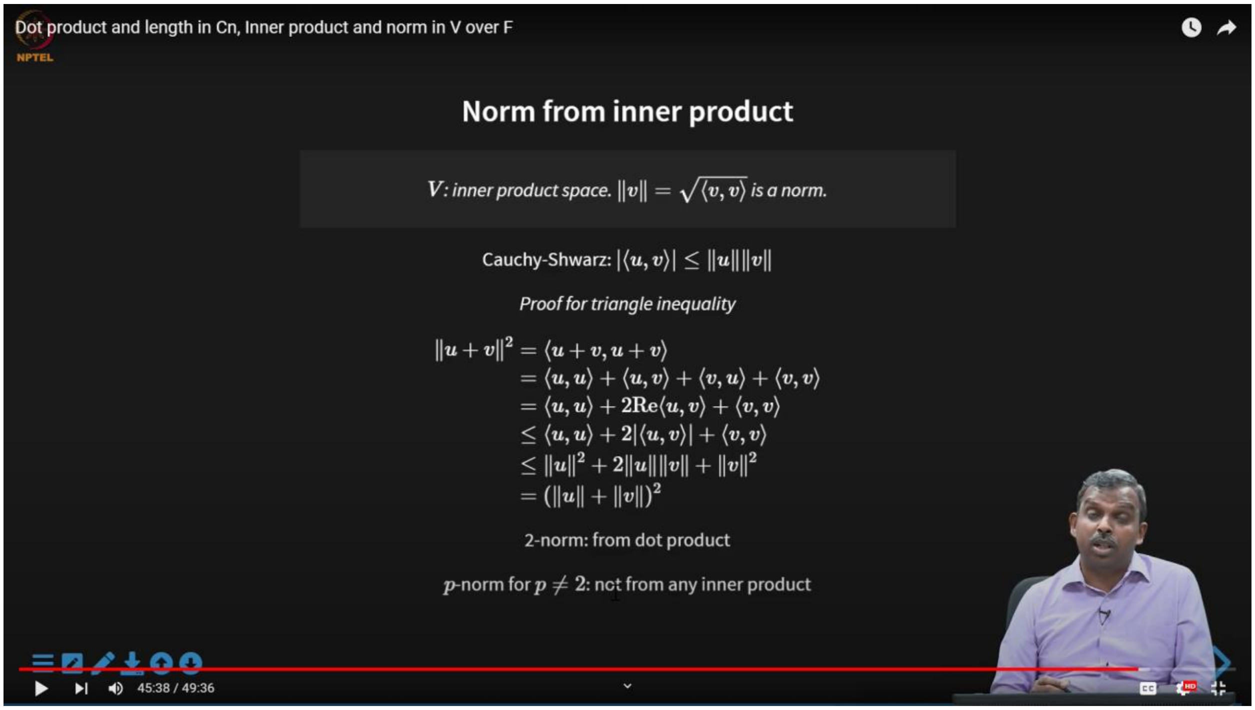

So this is a norm and this norm is said to come from the inner product, okay? So if you define a norm in this fashion, square root of $< v, v >$, this is a norm that comes from the inner product. Norm comes from inner product, okay? So these kinds of norms are very nice. We will mostly study these kinds of norms, okay? So you can see two norm comes from the dot product, right? So the familiar dot product or the, you know the dot product in complex space if you want. So it turns out the p norm for p not equal to 2 is not from any inner product, okay? So these are sort of slightly interesting problems to think about. So the two norm is special for this reason. So the two norm comes from a dot product. So you also have an inner product associated with it and it's very nice to deal with. All these other norms, they also have a lot of applications but they are not from an inner product so they have very different properties from the two norm, okay? So this is something interesting. So it's important to know that not all norms come from inner products, okay? So there can be norms which are not from inner products but mostly at least in this class we'll study the two norm and that's enough for us.

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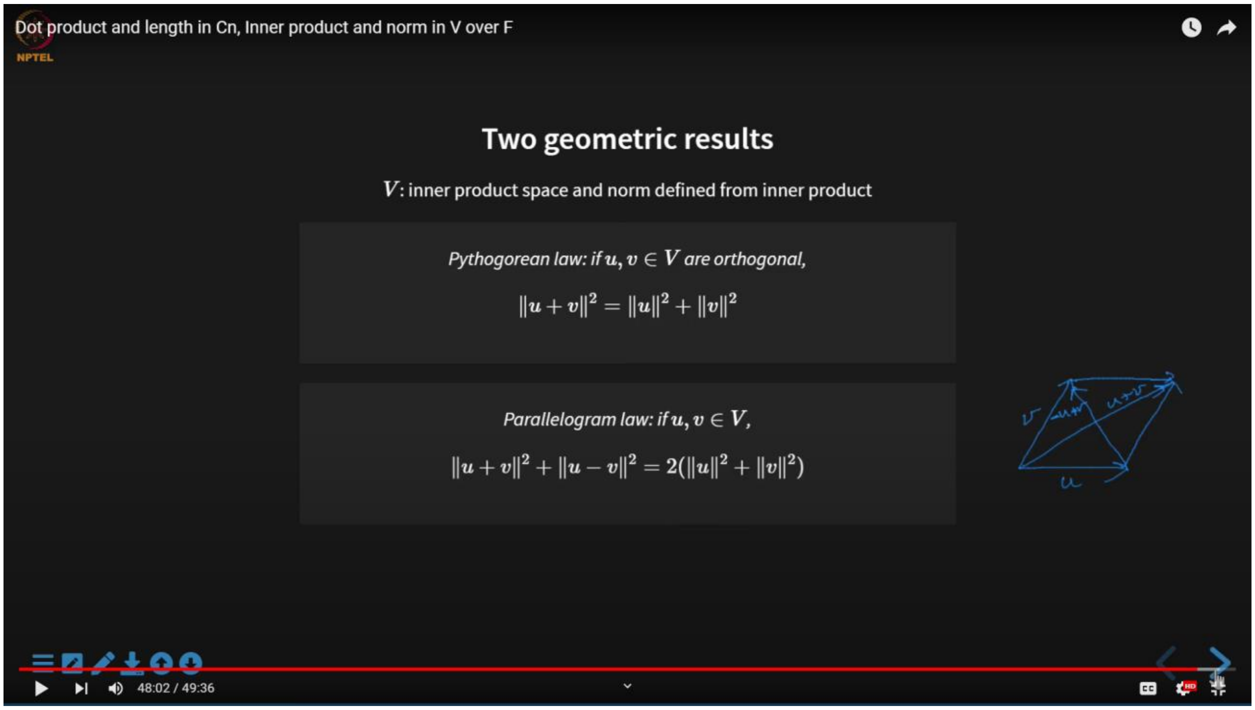

So a couple of geometric results. A couple of very powerful and neat geometric results that you have in inner product spaces. If you have an inner product space and the norm defined from an inner product, this is how, this is a setting in which we'll work. The first law is the Pythagorean law. If u and v are orthogonal, then $||u + v||^2 = ||u||^2 + ||v||^2$. It's a very familiar restatement. Very easy to prove, you know? You can go back to the way in which I showed triangle inequality. You'll see $< u + v, u + v >$. $< u, v >$ will be zero. $< v, u >$ will be zero. So it will just become $||u||^2 + ||v||^2$. It's quite easy to prove. The next law is what's called the Parallelogram law, okay? So this is something very interesting. I'll tell you what's very interesting about it in the next point. But at least this is easy to prove. If you have a norm defined from the inner product, okay, if $u, v \in V$ then this is true, right? It's just a basic expansion, right? You do $< u + v, u + v >$, u and v need not be orthogonal by the way. u and v can be anything. This is parallelogram because, you know, you can see where the parallelogram is coming from. Maybe I should just draw this picture for you. if you have u and if you have v, right, and then, you know, you can think of this parallelogram, you will have a, you know, $u + v$ will be on one diagonal, $u - v$ will be on the other diagonal, right? You can sort of think of $u + v$ being here and then $u - v$ being here. Or $v - u$ being here. So this is sort of related to the parallelogram. These vectors show up in the parallelogram, right? So I might be making some mistake here. It might be $v - u$ as opposed to $u - v$, whatever, okay? So you can see... Is that correct? Oh yeah, this should be $-v + u$, okay? But anyway. So these vectors show up in the parallelogram, okay? So this expression is an expression, it's a very familiar geometric result, with respect to a parallelogram, okay? So the $||u + v||^2 + ||u - v||^2$ is this. So you can see why this is true. I mean this just, it will cancel, all the cross terms and you will get this, okay? So this is the, sort of a right triangle law, Pythagorean law and the parallelogram law which gives you, relates the norms of sums and differences to the individual norms, okay? So these are nice, nice geometric results that you are once again getting.

What's interesting is this little exercise that, I mean I will leave it as an exercise to you, it's a very interesting result. A norm is from an inner product if and only if it satisfies the parallelogram law, okay? So if you have a norm which satisfies the parallelogram law, it is from an inner product. Otherwise it's not, okay? So this is one way, it's easy to prove. Already we have shown one way, right? If it is a norm from the inner product, it satisfies the parallelogram law. If it satisfies the parallelogram law, it turns out you can define an inner product using the parallelogram law and it would satisfy all the properties of the inner product, okay? So that's the idea. So this is a crucial relationship in that sense, okay? So if you're given a norm, you want to find whether it's from an inner product or not, apply the parallelogram law, you'll get the answer, okay?

So that's the end of this lecture. Hopefully you got a glimpse of how this notion of dot product and length is generalized through inner product and norm to arbitrary vector spaces, abstract vector spaces that we have been defining. We saw some familiar things on how to check whether a given function described in a proper way is an inner product or not, or a norm or not. More importantly we saw this Cauchy-Schwarz inequality. I cannot stress the importance of it enough. It's very, very useful and powerful. And we saw some simple rules that seem to be true with norms and inner products and all that and how these things work together in a very nice way. We haven't yet seen how to use inner product and norm to understand linear operators, right? So that is one of the important things. We will see that in the ensuing lectures. Thank you.