Applied Linear Algebra

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Week 07
Linear Functionals, Orthogonal Complements

Hello and welcome to this lecture. We're going to deal with two topics in this lecture. These are closely associated with inner products and they, you know, sort of tie up with subspaces and all these other things that we've been studying so far. So these are, one is linear functional. It's actually a special case of a linear map and it has a very interesting and simple connection to inner products. So that's something that we will see. And the other one is orthogonal complements. Once again, you might be familiar from what you studied earlier about this in description of planes in $\mathbb{R}^3$, right? So you can describe a plane either by providing two vectors on that plane, or you can provide a vector that is normal to the plane, right? So this normal description of a plane is something that you might have studied. So orthogonal complement is sort of related to that. So what is this orthogonal complement? And inner product of course plays a big role in that as well. So these two we'll study in this lecture, okay? So let us get started.

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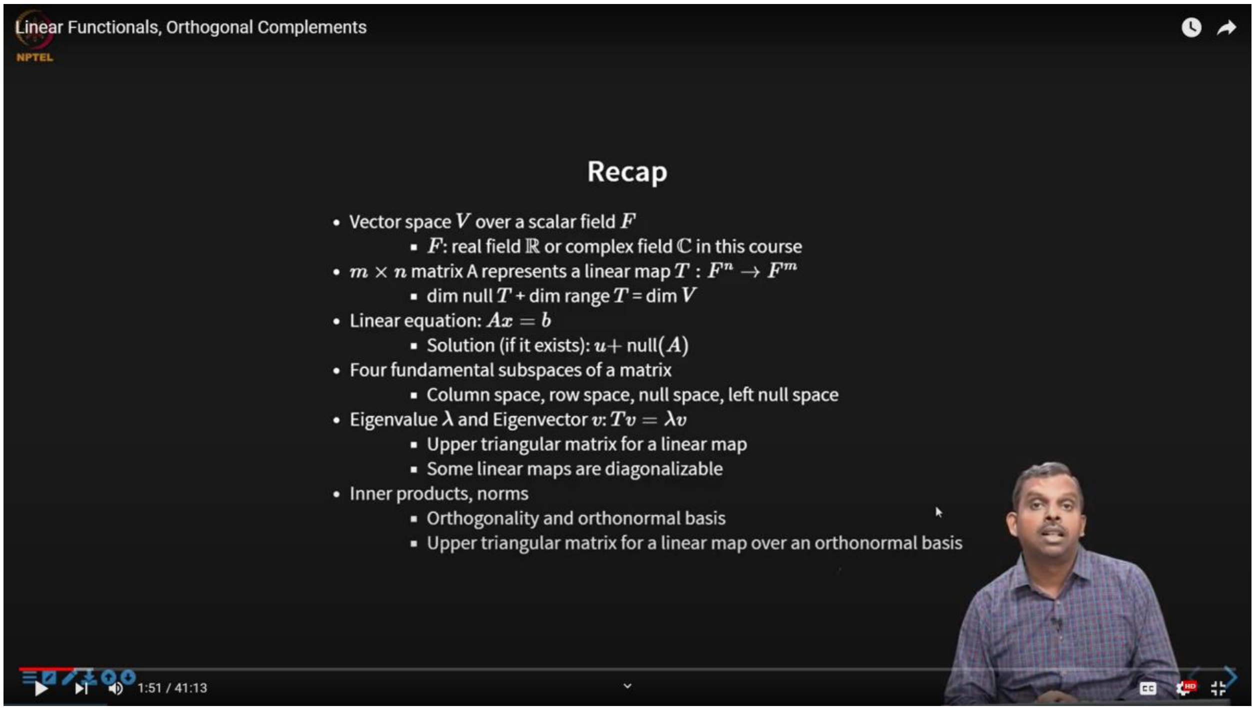

Quick recap. We have been looking as usual at vector spaces over scalar field. Real or complex. We saw, we've seen matrices are very useful to represent these linear maps and work with them. There’s these nice results about dimension of null and range and four fundamental subspaces and their relationship, how they are useful in solving linear equations, this whole, you know, invariant subspace, one dimensional invariant subspace leading to the definition of eigenvalues, eigenvectors and how, you know, you can have an upper triangular matrix representation in a suitable basis for any linear map. That's something we've seen. We also saw that some linear maps are diagonal. So that's also something that's very interesting. And the latest topic we are studying this week is inner products and norms and how orthogonality and orthonormal basis simplify a lot of descriptions and we'll continue on in that mode today. So we'll see, inner product, orthogonality, helps us in simplifying some of the notions and connections between these four fundamental subspaces and linear maps and all that, okay? And we saw one very nice result that in fact in an inner product space, when you have a proper inner product defined in the vector space, you can have an orthonormal basis with respect to which any linear map... Okay I should say it in reverse. For any linear map, there exists an orthonormal basis such that the matrix representing that linear map in that basis becomes upper triangular. So that is very nice to know. It is good to have orthonormal bases because we know coordinates are easy to find and all that, okay? So that's a quick recap. Let's study now two topics in this lecture. Linear functionals and orthogonal complements which further illustrate the usefulness of having this inner product, okay?

What are linear functionals? A linear functional is a linear map, except that the, where the result of the linear functional is always a scalar, okay? So it maps vector to scalar. That's called a functional, okay? A functional is usually like that. You have a vector or a more complicated object. From that if you do a function and you take yourself to a scalar, a very simple object, usually people call it a functional. So linear functional is nothing but a linear map except that from V you go to $\mathbb{F}$ the scalar field itself, right? So that is called a linear functional. So all the properties we studied for linear maps apply for linear functionals as well, right? Except that, you know, the range is actually a scalar. So it's got dimension one or zero. So it's very simple in some sense. So maybe we can expect a simple description, okay? So let's look at a few examples to get ourselves started. So if you look at linear map ϕ, so this is, it's common to use a notation like ϕ as opposed to T for a linear functional. If you look at $\mathbb{R}^3$ to $\mathbb{R}$, here is an example of a linear functional. You can see it's a linear map. It works out in some obvious way. So, in fact, in general if you go from $\mathbb{F}^n \to \mathbb{F}$, any linear map would be like $c_1 x_1$ to $c_n x_n$. It's very easy to see, you know, linear maps correspond to matrices. The matrix of a linear functional is going to be a $1 \times n$ matrix, right? So if you go from $\mathbb{F}^n \to \mathbb{F}$, the matrix is a $1 \times n$ matrix. So clearly the map itself is represented by something like this, right? So it has to be like that, okay? All linear functions will be in this form. It is quite easy to see. But maybe potentially, if you look at a slightly different type of linear functional which takes, say polynomials of degree less than or equal to two to the real line, okay, so here is a very interesting little definition where we use this integration, right? So we use, you take a polynomial of degree less than or equal to 2, and then you do $\int_0^1 p(x)cos(\pi x)\ dx$ .Now this $cos(\pi x)$ is not a polynomial, right? It seems to be some other object. But overall this ϕ is still a linear functional, right? You can use any definition, the definition for the linear map. You see the additivity works because integration is also linear and scalar multiplication also works, right? But this $cos(\pi x)$ is a bit disturbing, right? It's not something that's part of my linear algebra, the field or the vector space that I'm dealing with here. It seems like something else. But nevertheless the overall ϕ is a linear map. So can I say something in this case?

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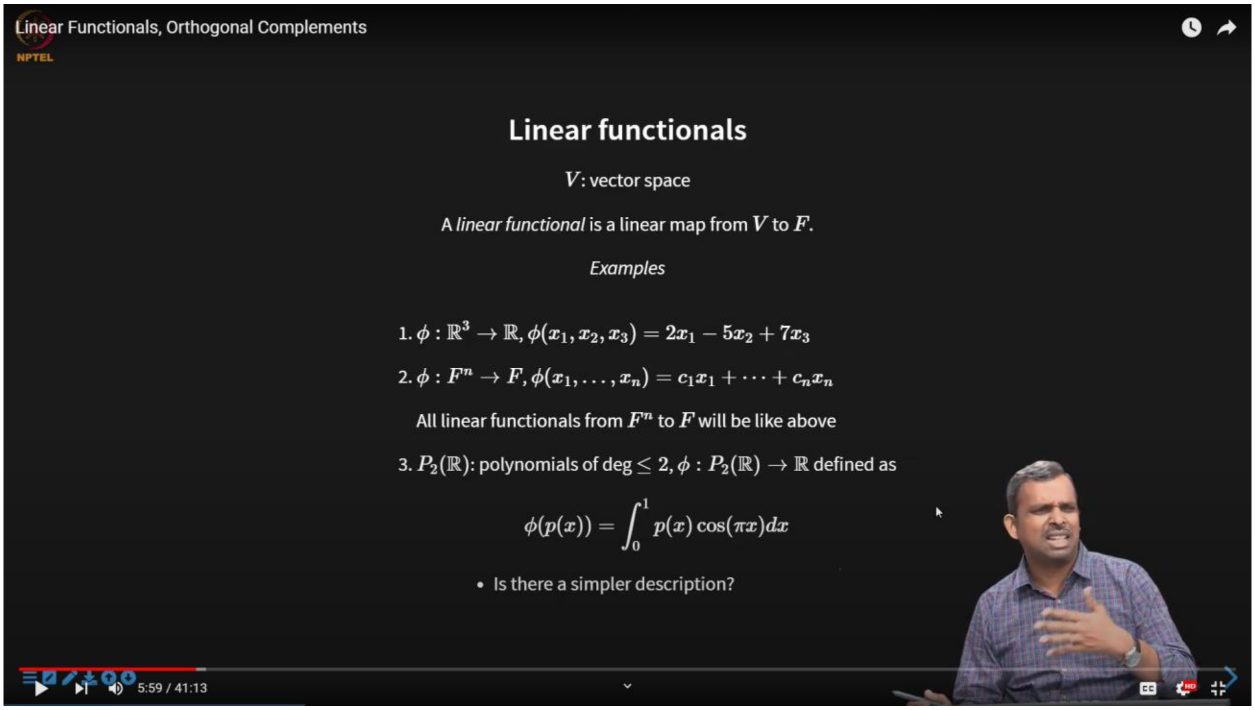

So is it still that, is there still a simple description in terms of my own vector space even if the function involved is cos or something else which is outside, okay? So things like this can happen sometimes. You may get a linear map from a vector space to the scalar field but it may involve things outside of your vector space, okay? So something else may happen and may look very strange and different to you. But I am still in this vector space, right? So something simple must be there for describing such things. And it turns out that's also true. And once you have an inner product space, such things can be very, very clearly written down, okay? So here is a, so if your V is not just a vector space, it's an inner product space, there's a very interesting example of a linear functional. It's a very simple linear functional. I have spoken about this when I introduced inner product as something that connects in a product to the world of linear maps and linear algebra, right? So if you fix a particular vector u and define a ϕ as the inner product $< v, u >$, then you have a linear functional, right? So it is quite easy to see. All the properties can be verified directly. So this is a linear functional. It's a very interesting linear functional. The reason is, we can ask a very interesting question: can there be in an inner product space any other linear functional which is not like this, right? So this looks like a very typical or easy example of a linear functional. Can there be a linear functional which is not like this, right? So that is an interesting question one can ask.

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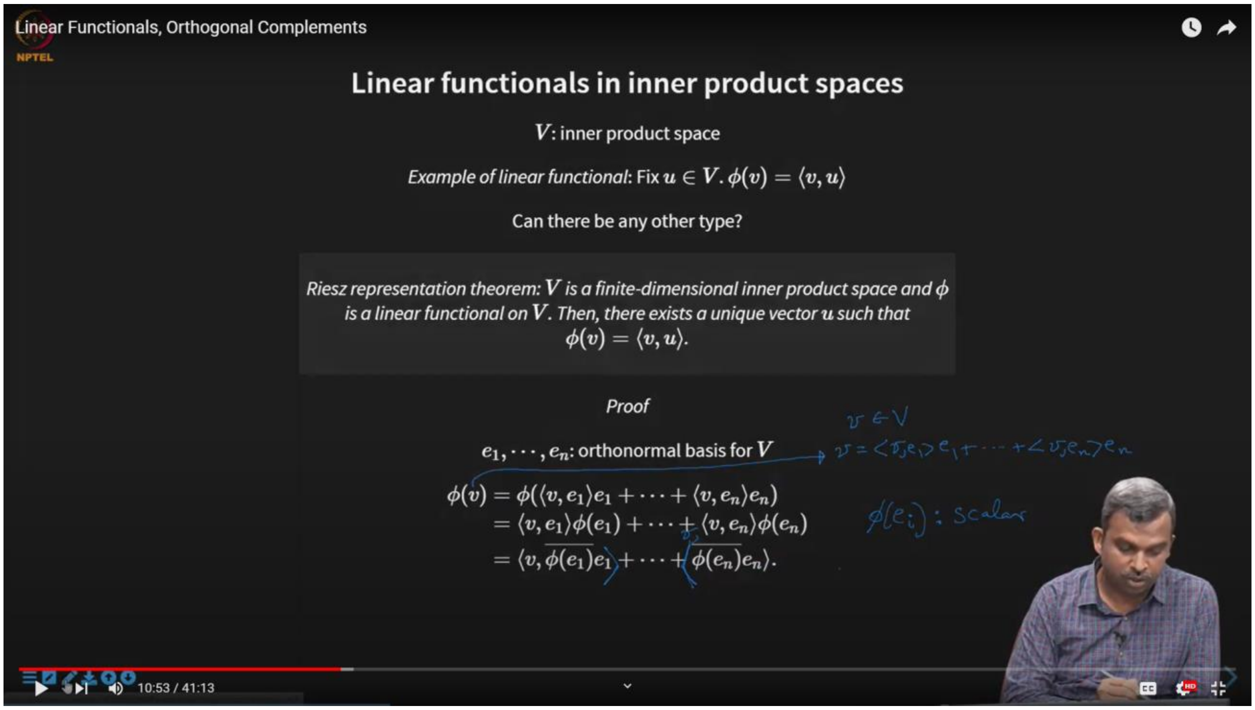

And it turns out there is this wonderful result which is called Riesz' representation theorem. Riesz representation theorem as it's called. If you have a finite dimensional inner product space and if somebody gives you a linear functional on V, it could be whatever linear functional, it could involve the elements of your inner product space or maybe some other element like the previous example we saw, whatever, just as long as it's a linear functional, it turns out there exists a unique vector in your V, in your own, the inner product space that you started with such that that linear functional becomes the inner product $< v, u >$, okay? So in other words, this inner product $< v, u >$ is the only linear functional that's interesting in a finite dimensional inner product space. So that's the Riesz representation theorem. It's probably intuitive. It's very easy to see. But let's write down a proof. The proof involves orthonormality and the properties of inner product and all that, okay? So you start with an orthonormal basis for V, okay? And then you see $\phi(v)$, right? Now once you have an orthonormal basis, any vector $v \in V$ can be written as $< v, e_1 > e_1 + \ldots + < v, e_n > e_n$, right? So this is true. So this is orthonormal. So you can write like this. So this is what I have done here. So instead of v, you plug in this guy v equals this, right? So instead of v, you put that in. Now you use the property of the linear functional, right? So ϕ is going to sort of go inside the linear combination. $< v, e_1 >$ is a scalar, e_1 is the vector. So ϕ will just apply on that e_1, the scalar will come out, right? So this is a linear map, so it's under linear combinations, the ϕ will go in, right? So you have $\phi(e_1), \ldots, \phi(e_n)$. Now notice this $\phi(e_i)$ is a scalar, okay? Okay? In general complex, okay? It could be, if it is real, it's real. But otherwise it's complex, right? So it's a scalar. Now what happens when an inner product multiplies, is multiplied by scalar? That scalar can be taken inside, right? It could be taken inside either to the first argument in which case it would just enter in without anything else or it could be taken inside into the second argument in which case it would be conjugated, okay? So I forgot to put the bracket here. So this is important, okay? So I am taking it into the second argument, there's a reason why I think the second argument is interesting here. So I am taking it into the second argument. So I have to conjugate, right? So I am putting a conjugate there, right? So this is so far so good. So you see how the, you know, orthogonality and the inner product are nicely playing to convert any linear map into an inner product like situation, okay? So there is a bug here. I apologize for that. This is actually, okay, so let me just write it properly. You should have a $< v, \overline{\phi(e_n)} > e_n$ here, right? So you can see how the same thing has been written here. And then I can combine all these guys, right? So that's what's been done I think in the way I wrote it. So this is for each term, the ϕ goes in and then I sum up all the remaining guys, right? So when I do that, I get that, okay? So you can see that that works out quite okay. So hopefully you see that it's quite clear I think, okay? So that becomes the whole thing.

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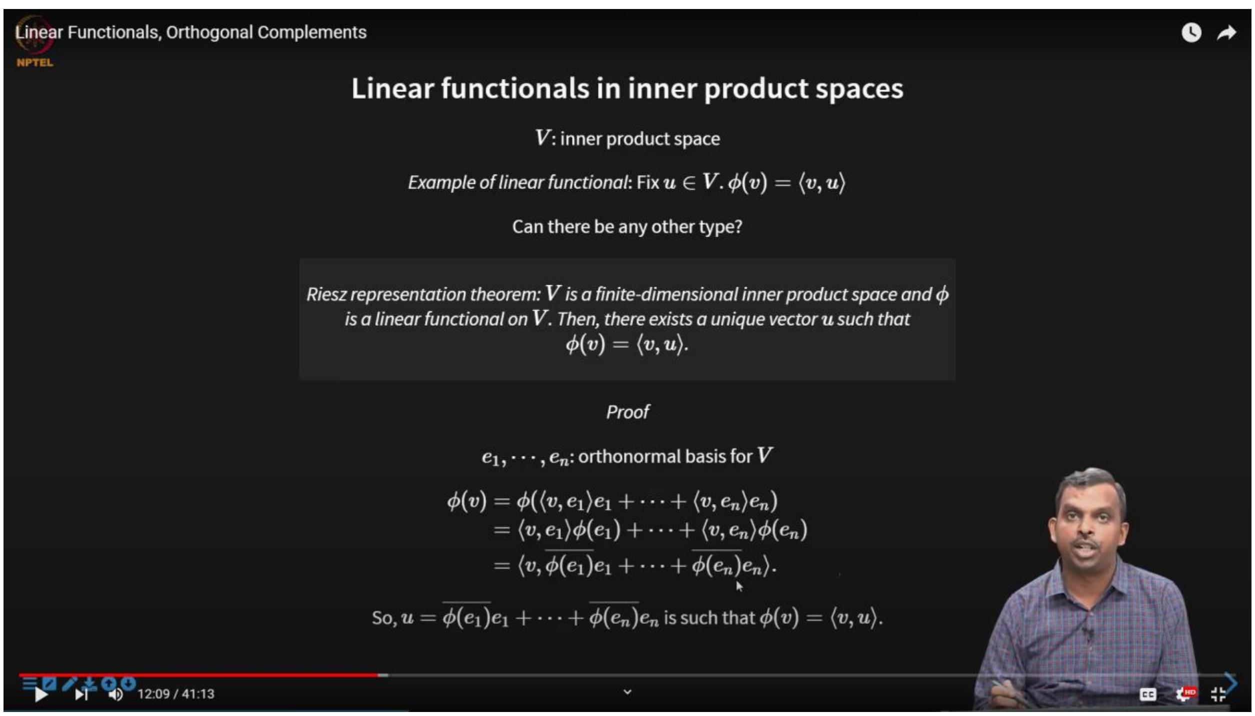

So then once you do that, you use the additivity in the second argument. So this becomes inner product of v comma this, okay? So I just showed you the intermediate step there. Each term ϕ goes in and becomes conjugate. And then you add up all of those. v is the same, so the second one becomes additive. So if you put u as this guy, right, whatever the second one is, $\overline{\phi(e_1)}e1 + \ldots$) then $\phi(v)$ becomes $< v, u >$, okay? So somebody gives you some linear functional $\phi(v)$, you can always go in and find the u such that that $\phi(v)$ becomes inner product $< v, u >$, okay? So that is interesting. So this statement also makes something about uniqueness of this vector u. I will leave that as an exercise. Go ahead and try and show it. The book also has the proof for uniqueness. Usually uniqueness is always done by contradiction. You assume, you know, there are two different vectors u_1 and u_2 and then show finally that u_1 has to be equal to u_2. It's quite easy in this case. You can do that, okay? So this is reassuring to know. So any linear functional in an inner product space which is finite dimensional you will have only the inner product showing up. You can always find the u so that this will happen, okay?

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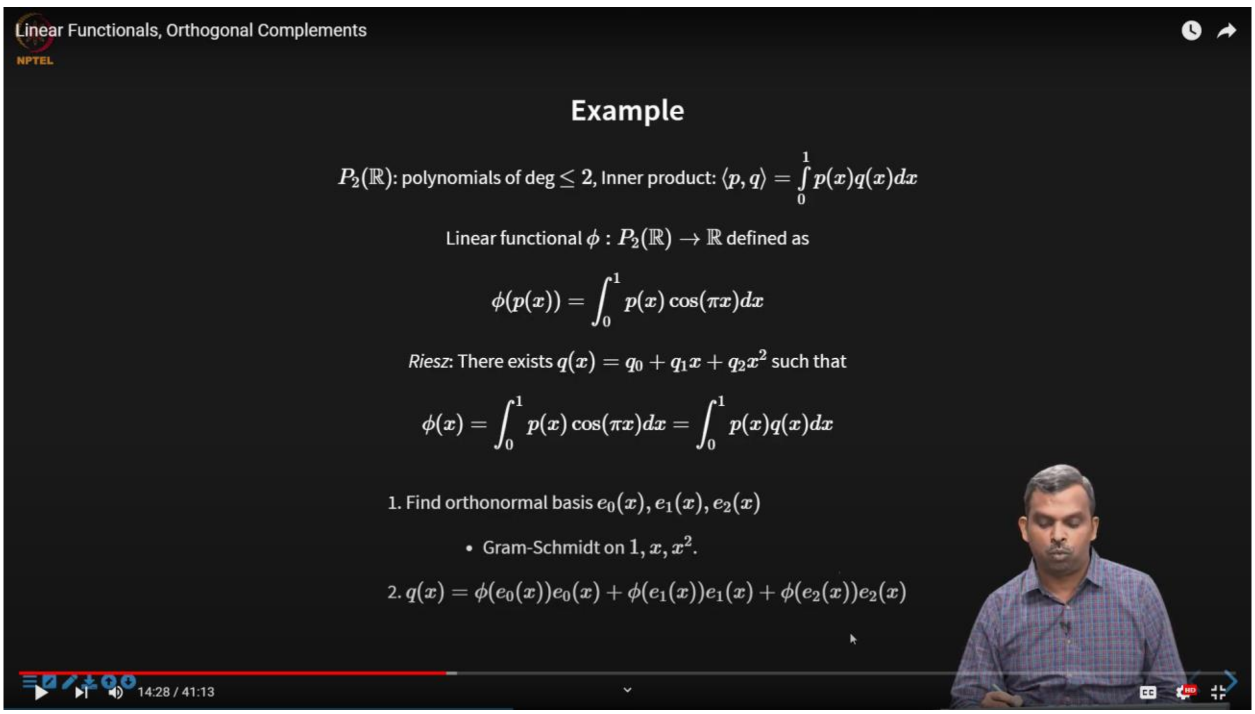

So let us go and use it in our example that we had here. We had polynomials of degree less than or equal to 2 and we use this inner product $< p, q >$ is $\int_0^1 p(x)q(x)dx$ and let us say we look at this linear functional which is defined with respect to this $cos(\pi x)$ showing up. This example is done in your book as well, you can take a look. But let me just quickly illustrate what is going on. So what does the Riesz tell you? Riesz tells you there exists a $q(x)$ which is a vector in your space. What are vectors in my space? These are polynomials of degree less than or equal to 2. There exists something such that this $\phi(x)$ which is $\int_0^1 p(x)cos(\pi x)dx$ has to be actually equal to $\int_0^1 p(x)q(x)dx$. So this is the main Riesz result. Now the question is: how do you find this $q(x)$? You use the elements from the proof. So you fix a, first find an orthonormal basis for this $P_2(\mathbb{R})$.

This needs a little bit of work. For instance, you could take $\{1, x, x^2\}$ which is an obvious basis and run Gram-Schmidt on it. Once you run Gram-Schmidt method, you will get $\{e_0, e_1, e_2\}$. e_0 will be 1 itself. e_1 will be a slightly modified version of x and 1. And then you will have x^2, okay? So this can be done. You can find an orthonormal basis. Once you find an orthonormal basis, $q(x)$ is simply, you know, just from the proof before. ϕ applied on the orthonormal basis. So that will involve an integration, right? With $xcos(x)$, $x^2cos(x)$, like that. And then e_0 itself. You evaluate this polynomial, you will get q, okay? So this is the basic idea. I am not doing it in great detail. Your book has the details here if you want to see it. And that's true, okay? So this is a nice result to see, right? So $\int_0^1 p(x)cos(\pi x)dx$ which is really not a polynomial is actually the same as $\int_0^1 p(x)$(a polynomial of degree two). In this case it will be equal to two as well, okay? So that's the example worked out. And that's a summary of linear functionals. It's this, I don't want to say too much more about linear functionals. There's nothing more to say, right? Once you have the Riesz representation theorem, it is simply the inner product, okay? So all these inner products together make linear functionals. We'll come back and look at these things later on when we talk about adjoints and those definitions. This will be very interesting to look at as well, okay? Good.

So let us move on to the next topic of this lecture which is orthogonal complement, okay? So it is very interesting to see how, you know, orthogonality and inner products can be used to define interesting subspaces of a vector space, okay? So we have a vector space V which is an inner product space, okay? And we consider a subset of that vector space which is some U. I'll show you a simple example. U could be a point. You could be a set of points, you could be a subspace itself, you could be the entire V, okay? So any subset. The orthogonal complement of U we will denote it as $U^\perp$. This perpendicular symbol is usually abbreviated and pronounced as perp. So this is $U^\perp$, is the set of vectors which is orthogonal to every vector in the subset U, okay? So that's the definition. It's written down in set theoretic notation below. $U^\perp$ is the set of all v in the vector space V such that the inner product $< v, u > = 0$ for every $u \in U$, okay? So for everything it has to be true, not just any one, okay? So that's important. That's the definition. It's easy enough to see the definition. So let's see a few examples and we'll use our familiar $\mathbb{R}^2$. First is $u = 0$, okay? Supposing I take only the 0 vector in $\mathbb{R}^2$. What would be $U^\perp$, okay? So I'll go through and write down all the four and then explain how this would go, okay? So you can see for $u = 0$, it's quite easy to see. $U^\perp$, any vector is going to be orthogonal to 0, right? So $U^\perp$ would be the entire vector space. What if u is (x_1, y_1) where maybe (x_1, y_1) is not 0, right? So what if u is (x_1, y_1), what would be $U^\perp$, okay? So if you think about it, you're going to write $U^\perp$ as the set of all v equals, let's say (x, y), such that the inner product $< v, u >$... So u is just one point, right? So this is for every u. But there's only one point in U. So this would be $xx_1 + yy_1 = 0$, okay? What is this? What is this description? This is a line through the origin, isn't it? This is a line through origin and it is perpendicular to the line joining (x_1, y_1). So if you were to draw, you know, this point, and you have let us say (x_1, y_1) here. $U^\perp$ is, you draw this line through this, then $U^\perp$ would be the perpendicular line, okay? So this would be 90 degrees and this guy would be $U^\perp$, isn't it? So that's nice enough to see, okay? So you take one point and ask which is the set of all, one vector and ask which is the set of all vectors which is orthogonal, they will lie on the perpendicular line. It's easy to see. What if U itself is a line through the origin? The same example extends, right? So this dotted line would be a line through the origin. Then $U\,perp$ would again be a vector which is perpendicular to it, right? So, because the line through the origin, there is only really one vector in it, right? It’s one dimensional. So there is nothing more to worry about. So you just make sure it’s perpendicular to that, it will be perpendicular to the whole thing, okay?

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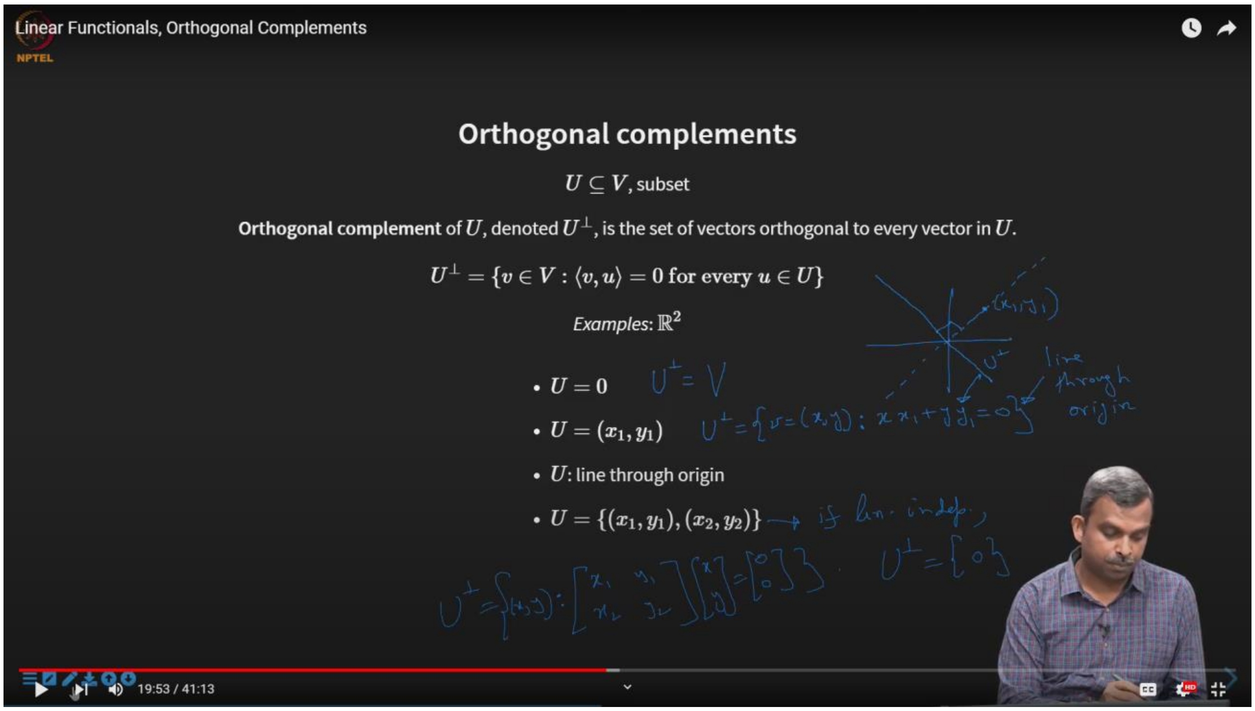

So what if you take two points, okay? So here there are two cases. These two points could lie on a line through the origin. If they lie on a line through the origin, the same line through the origin, then $U^\perp$ will be the line perpendicular to it. But in case they don't, in case $(x_1, y_1), (x_2, y_2)$ are linearly independent, then what will happen? So if linearly independent, then $U^\perp$ is only zero, right? So there is nothing else that will be orthogonal to both this and this, right? So think about why that is true. So if you have, there cannot be one vector which is orthogonal to both of these guys. So that would be, you know, you can write it down. So supposing you say $(x_1, y_1), (x_2, y_2)$. $U^\perp$ is the set of all (x, y) such that this is zero, isn't it? So you can write like this, okay? So this is a good way of capturing what happens here, right? So (x, y) has to be orthogonal to (x_1, y_1). (x, y) also needs to be orthogonal to (x_2, y_2), okay? So this is a nice way of writing it. So this trick of writing it in terms of, you know, the points that you want and the equation that you want, if you go to two points it will be like this... So now everything depends on the rank of $[x_1\ y_1;\ x_2\ y_2]$. If this is actually linearly dependent, then you can have a line. If this is like all zero, then you will have the whole V. If it is linearly independent then $U^{\perp}$ will be zero. There is no non-zero (x, y) which will be in the $U^{\perp}$, okay? So you see lots of interesting connections are establishing, getting established between equations and subspaces and all that and orthogonal complements. Already it's beginning to look interesting. So you can see here, these connections to null space here. All of that we'll explore and sort of formalize in the next few slides, okay? So $U = V$ is another example. So if $U = V$, then $U^{\perp}$ is just the zero, nothing else is there, okay?

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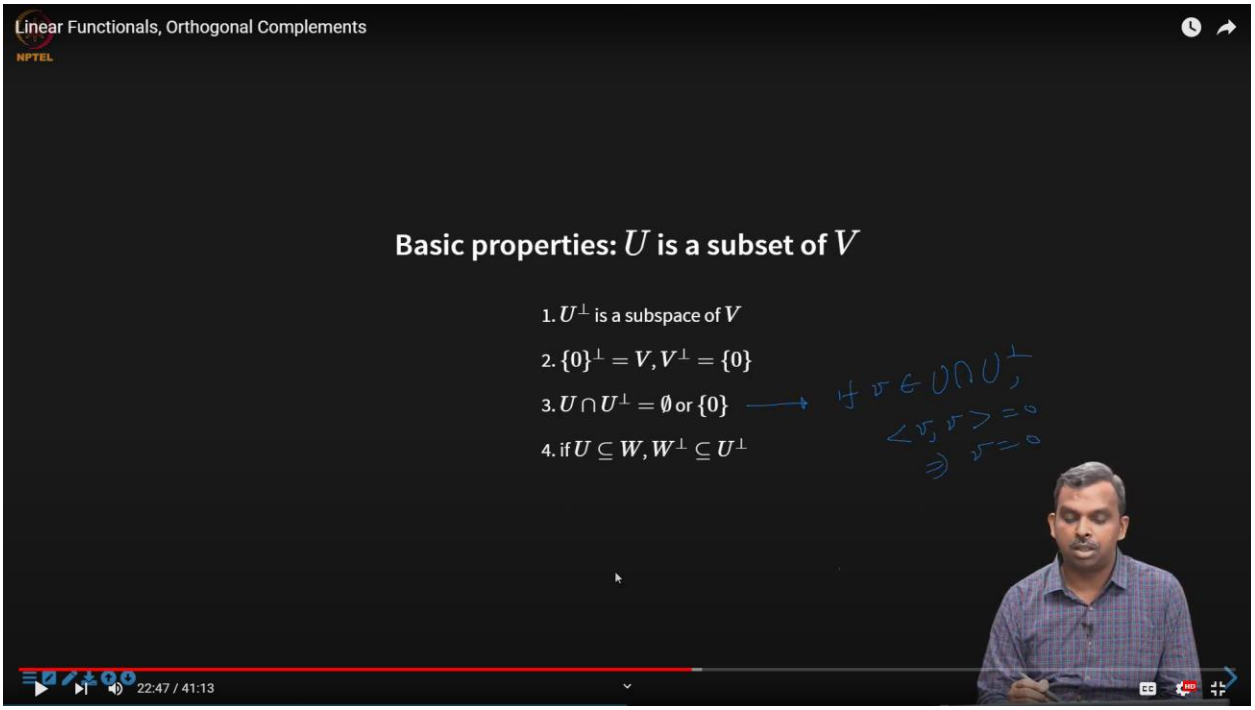

Okay. So let's see a quick few basic properties. If U is a subset of V, the first result is $U^{\perp}$ is a subspace of V, okay? So all the examples we saw before, you saw that whatever U was, the $U^{\perp}$ ended up being a subspace, okay? So that's not too difficult to prove. I'll leave the proof as an exercise. You can check that, you know, the points of U perp are closed under linear combination, it's quite easy to see, right? So if any vector has to be orthogonal to all the points in U, if you take a linear combination, that's still going to be orthogonal to all the points in U, okay? There's no problem there. And zero perp is going to be V. $V^{\perp}$ is going to be zero, okay? And next is the interesting little result. The third point I think is very interesting. So $U \cap U^{\perp}$ has to be either empty or it has to be zero, okay? So you can see why? So for instance in some of the cases we saw U was (x_1, y_1). And if (x_1, y_1) is not 0 and $U^{\perp}$ was a subspace, there can be no real intersection, right? So that point is maybe easy to see. If you see, if v belongs to $U \cap U^{\perp}$, okay? So if $U \cap U^{\perp}$ is empty, then there is nothing to do, okay? If there is a v that belongs to it, then what should happen, right? $v \in U$ and $v \in U^{\perp}$ also. So v should be orthogonal to v itself which implies $v = 0$, okay? So if there is any vector in $U \cap U^{\perp}$, it has to be zero or there could be no vector at all in which case it will be empty, okay? So these are the only two possibilities for $U \cap U^{\perp}$. Particularly interesting cases, you know? If U is a subspace then U will have zero also and $U \cap U^{\perp}$ will be zero. If U is not a subspace, then... U should have zero, right? That's the only criteria. If U has 0, then $U \cap U^{\perp}$ will be 0. If U does not have 0 then it will be empty, okay? So that's a nice result to have. Another result which you can quickly show is: if U is contained in W, if W is a superset of U, then the, you know, the orthogonal complement of $W^{\perp}$ will contain U, right? See if U is contained, something that belongs to $W^{\perp}$, right, will also be perpendicular to every point in U, right? So it should be contained in $U^{\perp}$. So the containment in the orthogonal complement will be reverse of the containment here. If U is a smaller set, $U^{\perp}$ will be a larger set, okay? So U is contained in W. $U^{\perp}$ will contain $W^{\perp}$, okay? So it will go reverse because the conditions are working that way, okay? Some basic properties, some basic intuition to build up on orthogonal complements.

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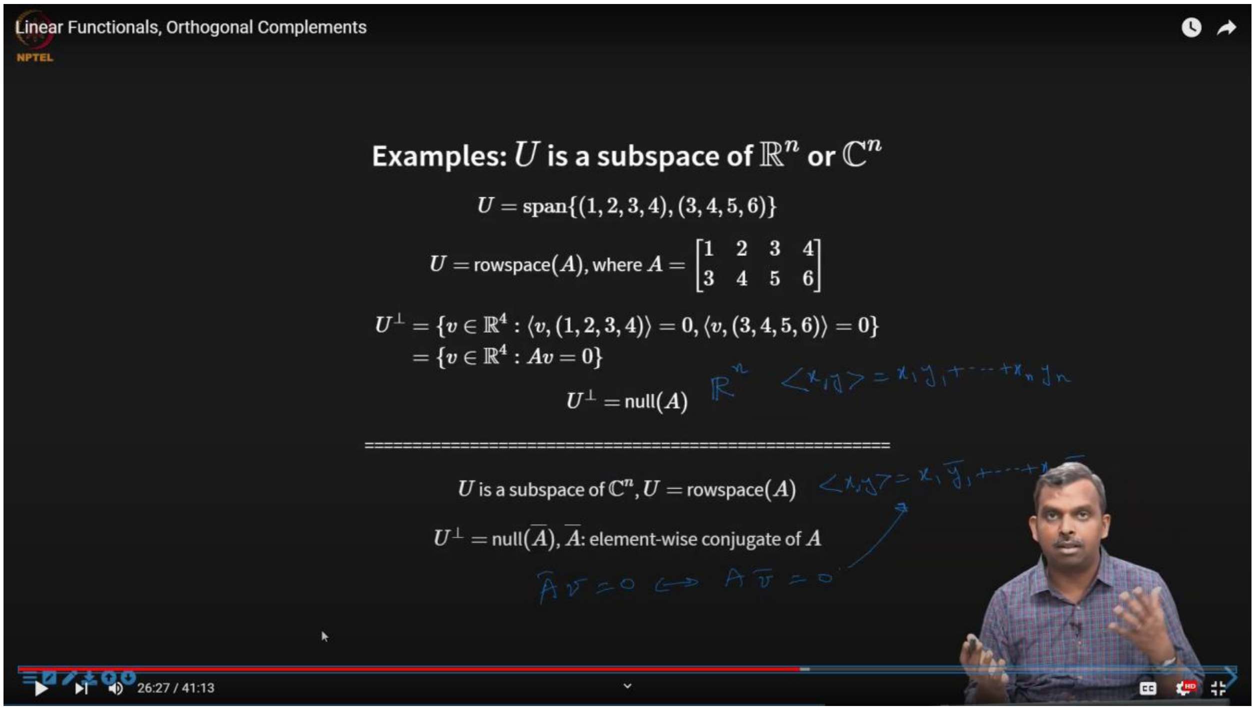

Okay. The next actual interesting case is when U becomes a subspace itself, right? So far when we looked at U, you kept U as a subset, you got some interesting properties. When U becomes a subspace itself, then it becomes much more interesting, okay? So lots of interesting connections you can make between the subspaces that we know of and the complement, okay? So let us say U is a subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$. I am taking a specific type of $\mathbb{F}^n$, the usual coordinate vector space to see what we can say, okay? So let, in this case we will take an example where U is the span of these two, right? This is how we specify subspaces, right? We specify the basis or a spanning set, okay? Here is a spanning set. So one could write this in terms of rows of a matrix A or columns. I mean, if you do not like rows, you can write columns as well. I am writing it down in rows. There is a specific reason why, okay? So U becomes the row space of A and A is this, okay? So you can see this nice connection comes about. Any subspace I can do this, right? So what is $U^{\perp}$ now? The set of all vectors in $\mathbb{R}^4$ such that the inner product of v and $(1, 2, 3, 4)$ is zero. Inner product of v and $(3, 4, 5, 6)$ is zero. These two you can combine just like that I did before in the previous example and write it's the set of all v. So now notice. See, remember, since U is a subspace, it's enough if I check orthogonality of v with the two spanning set vectors. If v is orthogonal to each of the spanning set vectors, then v will be orthogonal to any linear combination of them, so I don't have to go and check for orthogonality with every point of U, it's enough if you check for orthogonality with the spanning set of U. So that is a big simplification when U becomes a subspace. So $U^{\perp}$ simply becomes just this definition and this can be rewritten, right? So this inner product v comma this you can write it as $Av = 0$. So this is like a shorthand notation and you quickly see the connection to the null space, right? So $U^{\perp}$ when U is the row space of A is nothing but the null of A, okay? So this is a good connection for $\mathbb{R}^n$, right? So this is $\mathbb{R}^n$, okay? So this is in $\mathbb{R}^n$, I should say that very clearly. This is for $\mathbb{R}^n$, okay? Remember that.

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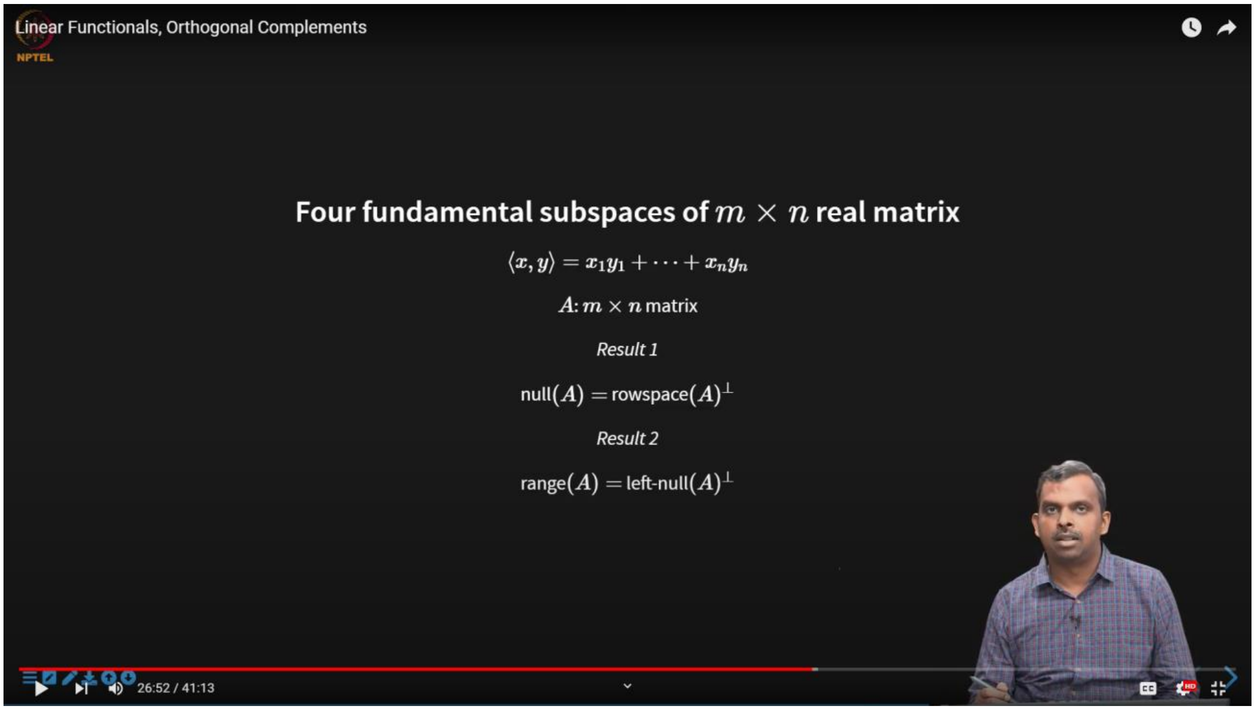

Now if U is a subspace of $\mathbb{C}^n$ where you expect complex numbers and you write U as row space of A, so the problem in complex is: In complex, okay... So this is for $\mathbb{R}^n$ where inner product $< x, y >$ is defined as $x_1 y_1 + \ldots + x_n y_n$. In complex, what are you going to do? In inner product we'll have this other definition, right? So $x_1 \overline{y_1} + \ldots + x_n \overline{y_n}$, okay? So this conjugate enters the picture here, right? So if you define A and make U as the row space of A, $U^{\perp}$ in this case will become null of $(\bar{A})$, okay? So what is $\bar{A}$? You take conjugate of A, okay? So instead of putting A, you have to put $\bar{A}$, okay? So what will happen now? If you have $\bar{A}v = 0$, okay, so that is the same as $A\bar{v} = 0$ and that will give you the inner product, right? So the proper inner product will come in, right? So in complex also there is a connection with the null space except that you will do an element wise conjugate, okay? So we see that this complement and the four fundamental spaces of matrices are connected and one can write it down. Particularly for the real case, real case is very easy where you have the inner product, null of A is the row space of $A^{\perp}$ and you can also have for range, right? So you define the left null and range will be the orthogonal complement of left null, okay? So this is a good result to see. So we see this nice connection. And what you were doing when you found null space is actually finding the orthogonal complement of the row space, okay? So this is very useful in some cases.

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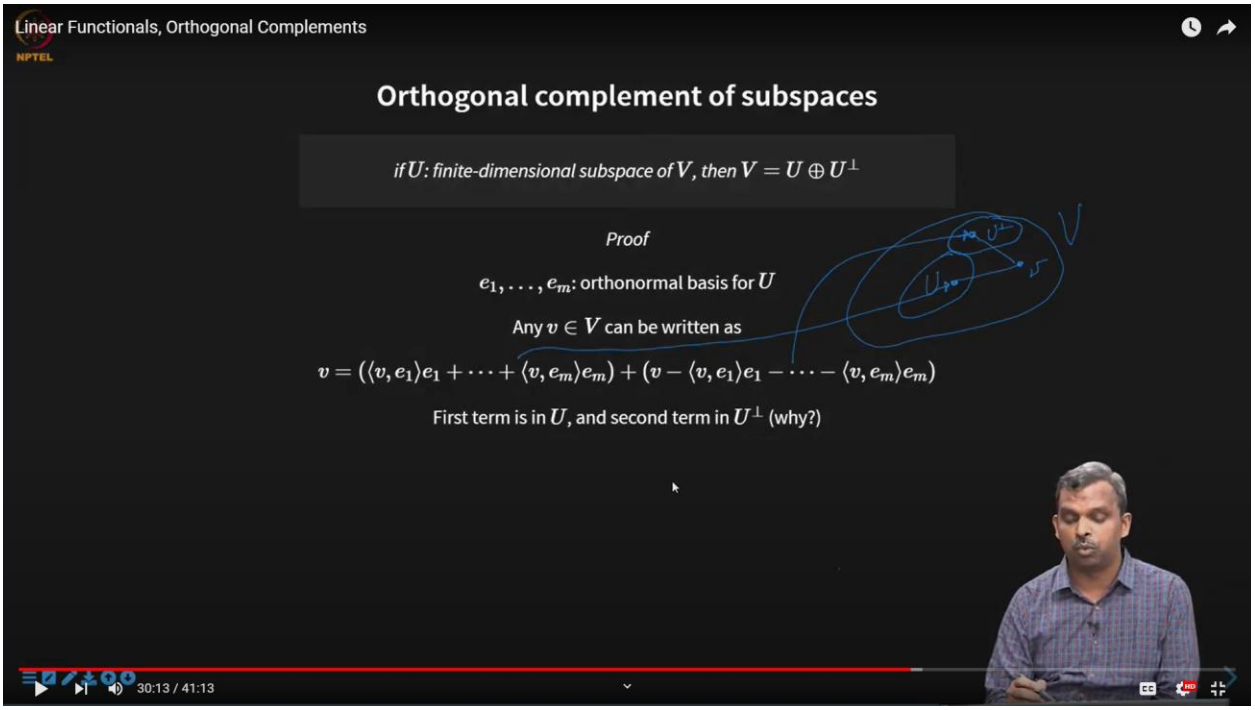

So there are few more results which are very, very interesting. Here is the first of them. So we know that if you have a subspace U, there is some W such that V becomes the direct sum of U and W. We know that that's true, okay? So it turns out $U^{\perp}$ is a candidate for W, okay? So you have a sub, you have an inner product space V, you have a subspace U. You find $U^{\perp}$, then V becomes the direct sum of U and $U^{\perp}$. Why direct sum makes sense? We know already $U \cap U^{\perp}$ has to be only 0, right? So that we know. So if I take a sum, it will become direct sum. That's okay. But then why should the direct sum be equal to V? That is the question. And this is a proof that you can write. It's a very easy argument. You start now with an orthonormal basis for U, that's where the starting point lies. Not just a basis but an orthonormal basis for U. Because we are in an inner product space. Any v now I can write in this fashion, okay? So it's sort of like the orthogonal decomposition we saw before. You know v, any v... Now v need not be in U. If v were to be in U then only this first part will come, second part won't come. So when it is general, I sort of do the first part, okay? And then write everything as v minus that, okay? It's just v equals v written like this, right? So nothing major here going on. But interestingly the first term ends up being in U, okay? And the second term ends up being in $U^{\perp}$, okay? I'll ask you to prove this. It's not very hard. You can show for the second term, if you take a dot product of the second term with e_1 for instance, what will happen? The first term will give you $< v, e_1 >$. This term will give you $-< v, e_1 >$, right? When you are taking dot product with e_1, all the other guys will disappear because they are all, it's an orthonormal basis, right? So you get $< v, e_1 > - < v, e_1 >$ and that will go to 0, okay? So this is a bit interesting. This is sort of like an orthogonal decomposition of v with respect to U and this is possible for any subspace U.

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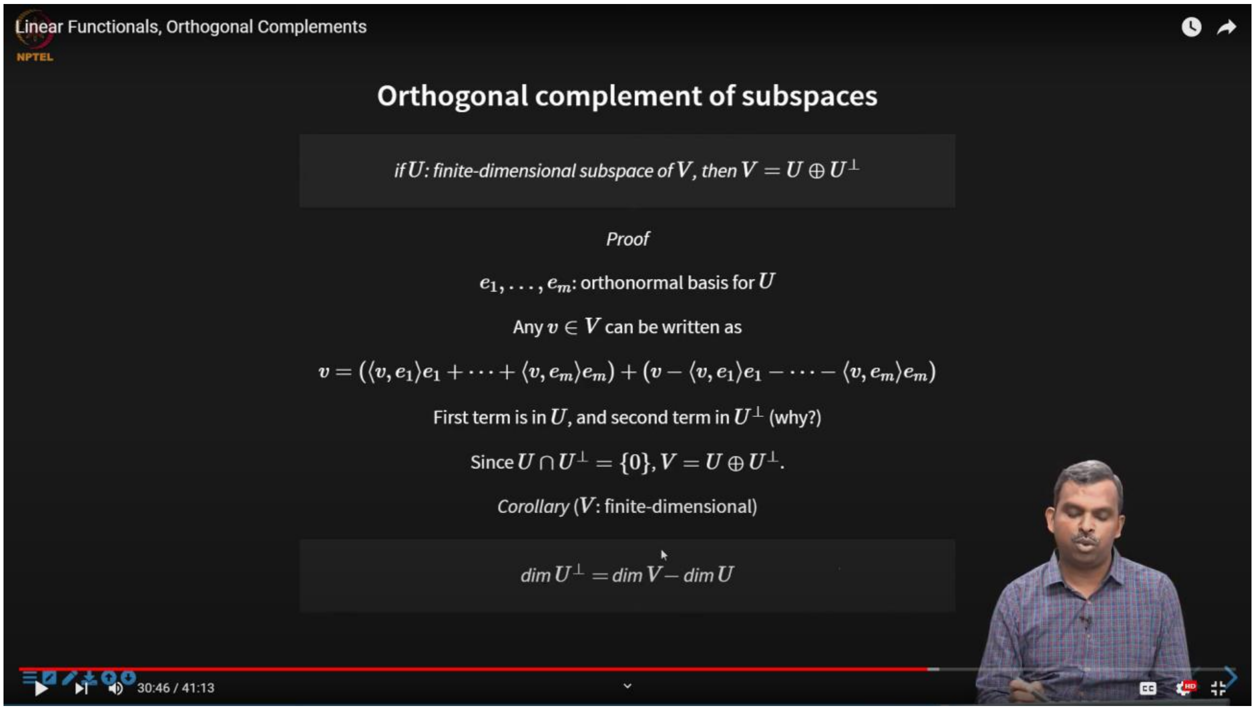

So this part is very important and interesting. We'll come back to it in later lectures. This orthonormal basis is very powerful in this fashion. For any subspace, even if you have a vector outside the subspace, you can sort of write it as something that belongs to the subspace plus something that is orthogonal to the subspace, okay? So you can sort of picture this. You have U, you have the whole V. If you have a v outside of that, you know you can, there is something here. And then something that is perpendicular, right? And this is this guy. And this guy is in... Right? So let me just write $U^{\perp}$ like this. This is $U^{\perp}$ and this belongs to U, okay? So any V that is true. So there is something that is here, something that is here. And this plus this would give you v, okay? So you can write down a proof of it. So this is very important to know, okay? So and that's it, we are done, right? So any v can be written as something that belongs to U and something that belongs to $U^{\perp}$. So, you know V is $U \oplus U^{\perp}$, okay? So that's nice to know. So orthogonal complement also has this powerful thing that it has a direct sum with U to give you the entire vector space, okay? So if you, I mean one can quickly write a corollary here. If V is finite dimensional, this result immediately tells you the dimension of $U^{\perp}$. It is simply, you know, $\dim V - \dim U$. You could have also proved it using other methods. But anyway. So this is quite easy to see. Okay. There's another interesting result. When U is a subspace, you take $U^{\perp}$, right? $U^{\perp}$ will also be a subspace, right? If you U is a subspace, when, whatever U, even if U is a subset, $U^{\perp}$ is a subspace. And then you can do a perp again. It turns out if U is a subspace, you will get back U, okay? This if, when you do complement twice, orthogonal complement of the orthogonal complement gets back to the original subspace, okay? I'm not going to go through the proof. It's a bit technical, it's just showing two sets are equal. An element here is element there. Except that you have to really watch out for what it means to say, you know, perp of perp and then look at it very carefully. It's, the proof in your book is pretty nice, please take a look at it. I'm not going to repeat it in this class. But this is interesting to know, okay?

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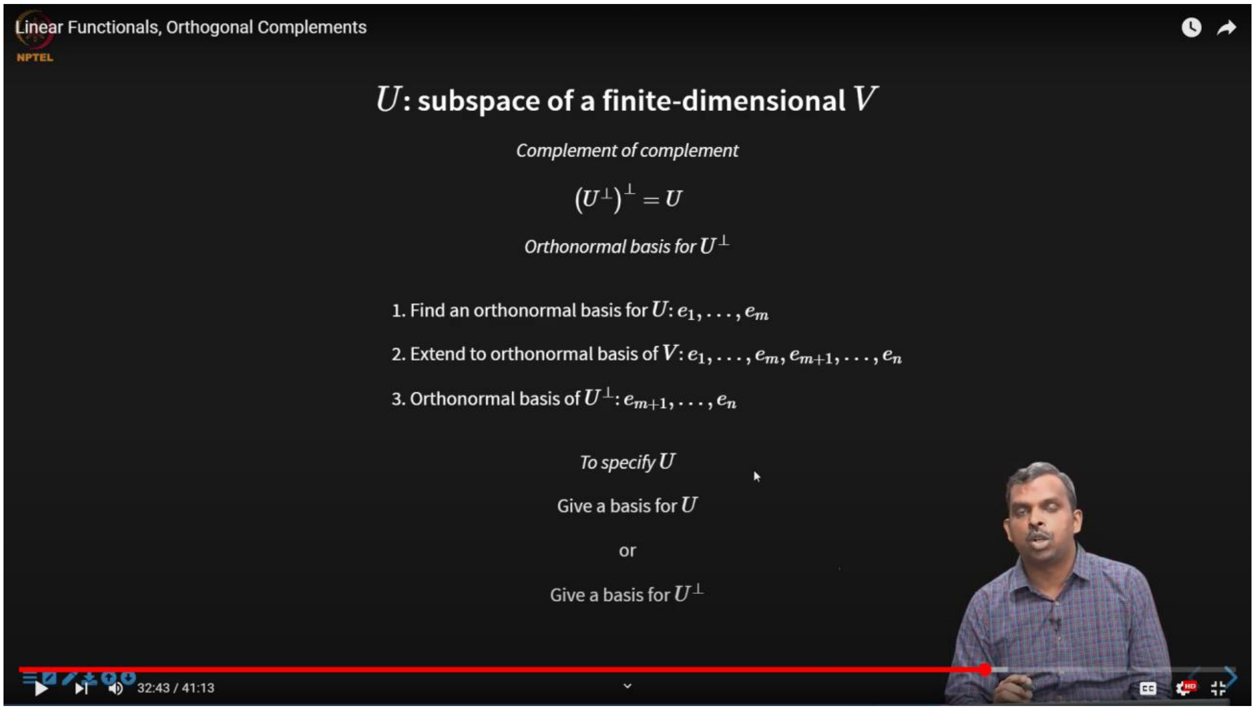

So how do you find an orthonormal basis for $U^{\perp}$? Is there an easy method? Yeah, you can use extension. The orthonormal basis extension, you find an orthonormal basis for U, you extend to an orthonormal basis for V and simply take the remaining guys, okay? So it's easy enough to see. So this is quite straightforward to do. Or you find a, you know, simple basis for $U^{\perp}$ and apply Gram-Schmidt, you could do that also. So many methods out there. So what is interesting because of these results is that you can specify U by either specifying the spanning set for U, as in give a basis for U. Or you can give a basis for $U^\perp$. Both of these sort of determine each other, right? If you give a basis for $U^\perp$ and you define a matrix with the basis is coming up one row after the other... What will be U? If you define $U^\perp$ as a row space of something? U is simply the null space of that, right? So both of these are very valid definitions. And this is where the normal definition of a plane comes from, right? To specify a plane through the origin in $\mathbb{R}^3$, you can either specify the plane, vectors in the plane, or you specify just one vector orthogonal to it. It uniquely specifies the plane, okay? So that is the idea behind these things, okay? So this is quite powerful. You can use it depending on some problems. For some problems, specifying the spanning set is easy. For some problems, specifying a basis for $U^\perp$ is easy, okay? So for instance, if you want to test membership in a subspace, right, it's easy to check if it's orthogonal to $U^\perp$. I mean the basis for $U^\perp$ is better there, right? So with the spanning set, you have to keep solving some equation or something, right? So it's not that easy, okay?

Okay. So I'll close this lecture with sums and intersections of subspaces. We have done this before also. We looked at how to compute basis for the sum of two subspaces, basis for the intersection of two subspaces and one can also use these orthogonal complements and ideas from there to help us with this, okay? So there are other ways of doing it. But orthogonal complements give you an interesting approach to this problem of sums and intersection of subspaces also and it gives us some practice to think of orthogonal complements, okay? Sum is easy with the spanning set, okay? You have a basis for U, basis for W. If you want to look at $U + W$, you already have a spanning set with you, right? Spanning set is $\{u_1, \dots, u_k, w_1, \dots, w_l\}$. You put them one below the other. You do elementary row operations. Simplify. You find the basis, right? So you can find $U + W$ in a straightforward way. So what do you do for intersection of U and W? One sort of textbook method, easy method is this result. $(U + W)^\perp$ is actually $(U^\perp) \cap (W^\perp)$, okay? So this result one can use to do intersection. So let me just first prove this result and then show you how this works. So proving this result, usually, when you want to show two sets are the same, you want to show if some vector is on the left hand side, it is also on the right hand side. And some vector is on the right hand side, it is also on the left hand side. So if v belongs to $(U + W)^\perp$, right... So remember $U + W$. U is contained in $U + W$. W is also contained in $U + W$. So clearly, you know, so if this belongs to $(U + W)^\perp$, that same vector will belong to $U^\perp$ and $W^\perp$, okay? So it belongs in the right hand side, okay? So this sort of shows, this shows LHS is a subset of RHS, okay? So the next step is to show RHS is a subset of LHS. So for that let's define $U^\perp$. $U^\perp$ is the set of all vectors v which are orthogonal to all the u_is, right? So u_is are the basis for U, okay? They would be orthogonal to all the u_is. What is $W^\perp$? Set of all v which are orthogonal to all the w_i, okay? What is the intersection of these two? Set of all v which is orthogonal to all the u_i and orthogonal to all the w_i, okay? Is that alright? So this is the definition. Now what is that? At that point we are done, right? So because $U^\perp$ is this. Now, so this is the same as, you know the spanning set, okay? So this is... Okay? v is orthogonal to spanning set of $(U + W)$, okay? That is the definition, right? So this guy becomes exactly $(U + W)^\perp$, okay? So any vector here belongs to $(U + W)^\perp$. So this sort of shows, you know, RHS is contained in LHS, okay? May be even equal, you know? You can see how this is true. So this is a nice little result to have. So if you want to do intersection of two subspaces, there is a connection between intersection and sums, okay? So you can do a corollary here using the complement of complement idea. Instead of U, you put $U^{\perp}$. W, you put $U^{\perp}$. You see that $U \cap W$ is $(U^{\perp} + W^{\perp})^{\perp}$, okay? So this is a result one can use to find intersect. There are other methods also. I am just giving you a method based on orthogonal complements. So this kind of connection is very interesting.

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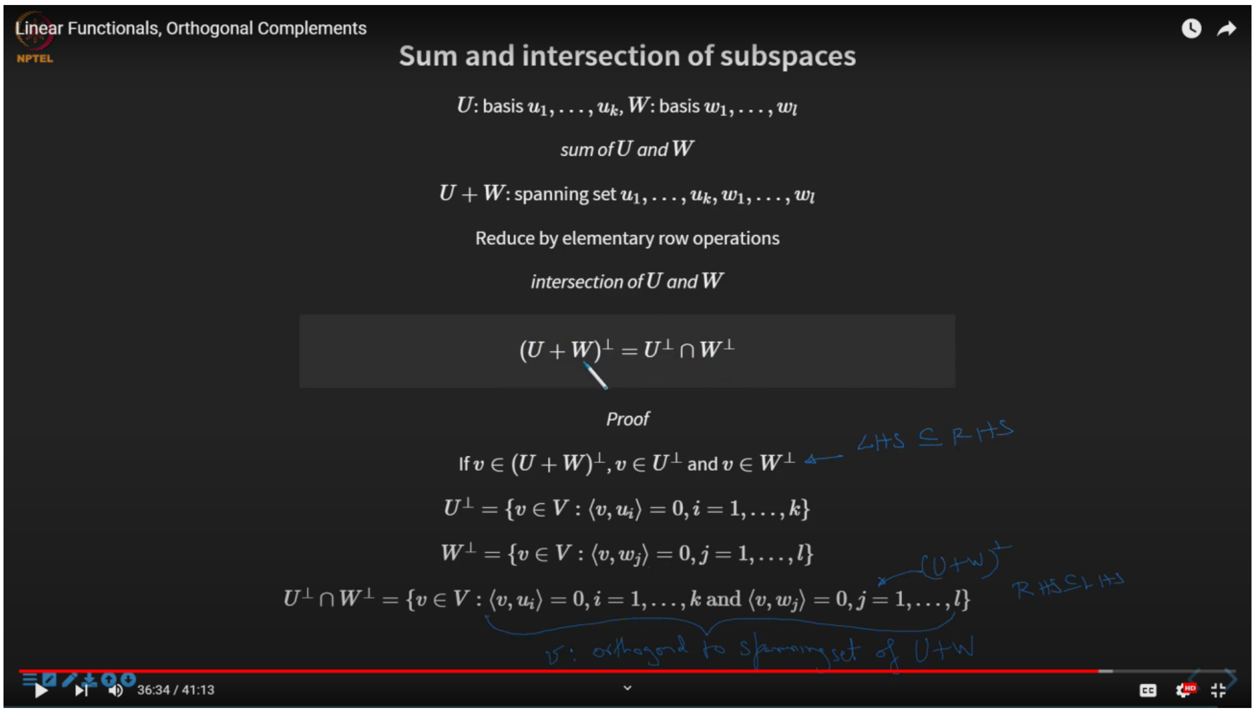

So let me show you a quick example of this and close this lecture, okay? So here is a, here's two subspaces I have specified using spanning sets, right? So U is row space of $[1\ 2\ 3\ 4;\ 3\ 4\ 5\ 6]$ and W is row space of something else. So supposing you want to find $U + W$. This is good enough, right? So you want to find $U + W$, it's nothing but all these four put one below the other. It's easy to do, right? So maybe I should write that down. $U + W$, the spanning set description is very good. Simply row space. Row space of $[1\ 2\ 3\ 4;\ 3\ 4\ 5\ 6;\ 1\ 1\ 1\ 1;\ 1\ -1\ 1\ -1]$, okay? So that is easy enough to write. Now $U \cap W$, what do you do for $U \cap W$? It is good to think in terms of the orthogonal complement. So you find this basis for the orthogonal complement which is nothing but this, you have to do row elimination and find the null space of this. So U simply becomes the null space of this guy, okay? So once again what did I do here? So I found, so U is the row space of this, okay? And the $U^{\perp}$, right? What is $U^{\perp}$? $U^{\perp}$ is going to be null of this itself, right? Maybe I am just going around in circles here. So let me now just write it. Yeah, so this is not wrong. Except that maybe you don't quite see it. So $U^{\perp}$ is basically null of $[1\ 2\ 3\ 4;\ 3\ 4\ 5\ 6]$, okay? And you can find the basis of the null, okay? You do the, you know, the usual row elimination and find the basis of the null, you will find that to be $(1\ -2\ 1\ 0)$ and $(2\ -3\ 0\ 1)$, okay? So there are other basis vectors as well. But this is the basis for the null, okay? This is the basis for the null space. So $U^{\perp}$ is nothing but the row space of this, right? So $U^{\perp}$ is the row space of $[1\ -2\ 1\ 0;\ 2\ -3\ 0\ 1]$. So I can write U in other words as the null of this same matrix, right? So it's all, I mean, I think in this one little thing, I have specified so many connections. So instead of writing U as row space, I can write U as the null space of this guy. So same thing with W. Instead of writing it as row space of this, I can write null space of this, okay?

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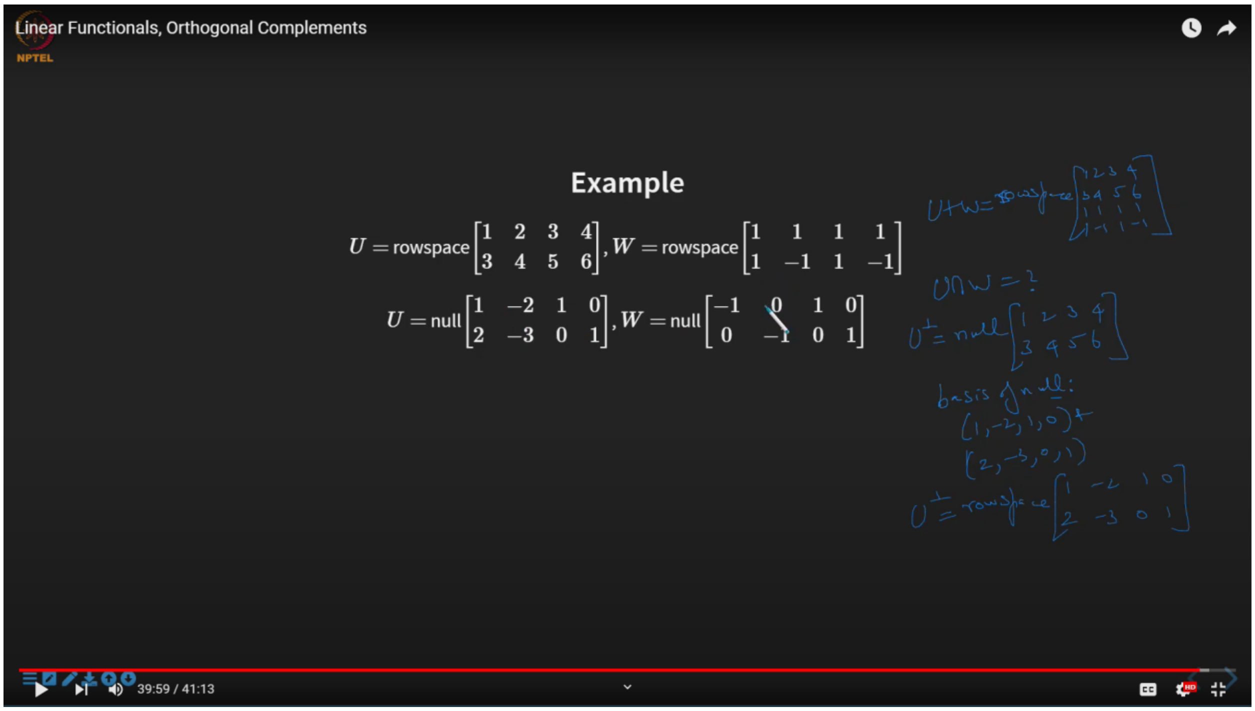

So think about what I did here, right? So I have U, I know $U^{\perp}$ is null of that same matrix. I find the basis of the null. So $U^{\perp}$ becomes row space of this guy. So now what would be U? U will be null of the same guy, that's what I wrote here, okay? Same thing I did here, okay? So hopefully you get this. This is a nice little connection that you can do here between row spaces, null spaces. So once you write U and W as null spaces of two things, $U \cap W$ is very easy, right? U is everything orthogonal to the rows of this. W is everything orthogonal to the rows of this. So what is $U \cap W$? It needs to be orthogonal to all four of them. So you put one below the other, so you get null of this and now you do row elimination here to get it into standard form, you get null space of this. And from there you can go back to the row space, okay? So you find, you know, this is, what is null of this? You have a basis $(1, 1, 1, 1)$. So $U \cap W$ is row space of this. Is that okay? So these two row spaces have an intersection which is just $(1, 1, 1, 1)$. It's a dimension one intersection, okay? So this kind of comfort in moving from, you know, spanning set to the orthogonal basis and orthogonal basis back to the spanning set can help you quite a bit in problems like this. Finding intersection, finding some things. And this is very interesting to look at as well, okay? So that is the end of this lecture. Thank you very much.