Applied Linear Algebra
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Week 9
Properties of Adjoint of a Linear Map

Hello and welcome to this lecture. We're going to continue what we studied in the previous lecture where we defined adjoint of a linear map. So in this lecture we are going to start looking at properties of adjoint and this adjoint will play quite an important role in all that we are going to study later in this, in the remainder of this course, classifying operators etc. But right now we are still looking at linear maps, the more general linear maps and what does it mean to work with the adjoint, what kind of properties it satisfies and all that, okay? So let's get started.

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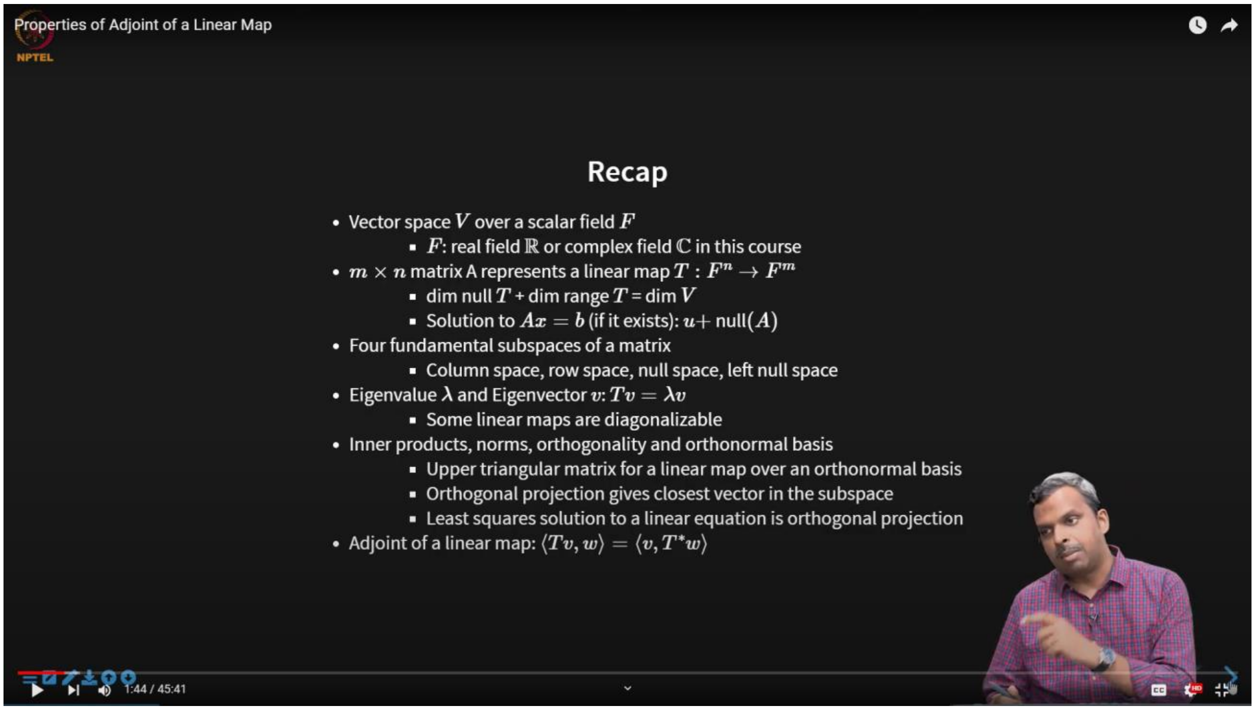

A quick recap. I think the first part I'll quickly go through. And we're really looking at adjoint right now. We are in inner product spaces, we are considering orthogonality, orthonormal basis and that's resulted in quite a few simplifications. In particular the definition for adjoint of a linear map has this very nice characterization in terms of what it does with the inner product, right? So you have a vector v which is taken to Tv by a linear map T. And then if you have a w which is another vector in W in the range of… Not in the range of T but in the other, second subspace into, the second vector space into which you are going, that one. And if you look at the inner product of Tv and w, it turns out there is always a T^*, this one adjoint operator which can take w to the original vector space V. And on that the inner product is sort of preserved. $< v, T^*w >$ is the same as the $< Tv, w >$ itself, okay? So that picture that you have with the adjoint is very interesting and this gives a lot of nice properties for the adjoint, okay? So let's see what those properties are.

So to clearly explain these properties, we'll look at three different finite inner product spaces. They are all over the same field $\mathbb{R}$ or $\mathbb{C}$. And the first property is quite easy to see why this should be true. We will prove all of these, at least one or two I'll show you the proof. The method is very similar in proof but these are all sort of intuitive, given the linearity of the whole thing, all this should work out, right? So if you look at the sum of two operators, S and T are two operators from V to W and you look at the operator… I should say linear map, okay, so let me not say operator. S and T are two linear maps from V to W and you look at the linear map $S + T$, okay? And then you ask what will be its adjoint, okay? It turns out that is equal to $S^* + T^*$, okay? So this is linearity for the adjoint. Additivity for the adjoint. The second one shows a slightly different property with respect to homogeneity. If you take an operator, a linear map T and you scale it by λ, and you look for the adjoint, it turns out it is $\bar{\lambda}$, there is that conjugate, pay attention to that, there is conjugate here, it's a small little bar on top of lambda, but that makes a difference. So $\bar{\lambda}$ times the adjoint of T, okay? For any λ. So of course if you are in a real space, then this $\bar{\lambda}$ will become λ, there's no problem there. But if you are in a complex vector space, then this $\bar{\lambda}$ will matter, okay? And then if you take the adjoint of T and ask what is its adjoint, okay, so you can do… I mean T^* is after all a linear map from W to V. Of course you can ask for the adjoint of that operator. It turns out you go back to the original operator T, okay? Original, I keep saying operator, to this original linear map, okay? So in the more general world of, in the linear map which goes from V to W and not necessarily an operator which goes from V to V, okay?

Identity operator. Now we are an operator. It is an operator. It goes from V to V or W to W or anything. The adjoint of the identity is identity itself. These are all easy to see, you know? Identity does nothing. So the inner product should also do nothing, okay? Inner product will be preserved just by itself. And here's an interesting property. If you have two operators S and T which you can compose, okay, you can compose in this fashion, S and T… So notice what's going on. T takes you from V to W and S will take you from W to U so ST is well defined, okay? So you can look at ST which will be an operator from V to U, isn't it? I keep saying operator, but it's a linear map, okay? So ST is a linear map from V to U, okay? Now you can ask definitely the question of what is the adjoint of ST. It turns out that is equal to another composition here but in the reverse direction, okay? You can do adjoint of T and then… $S(T)$. So, as in, $S^*(T^*)$, okay? So this is the relationship, the order reverses, but the adjoint works and you can see this composition is well defined, right? S^*, S adjoint is from U to W, okay? And T adjoint is from W to V. So T^*S^* actually from U to V, okay? ST is from V to U and T^*S^* is from

U to V and that is correct, okay? That is the correct way in which the adjoint will work, all of that works out, okay?

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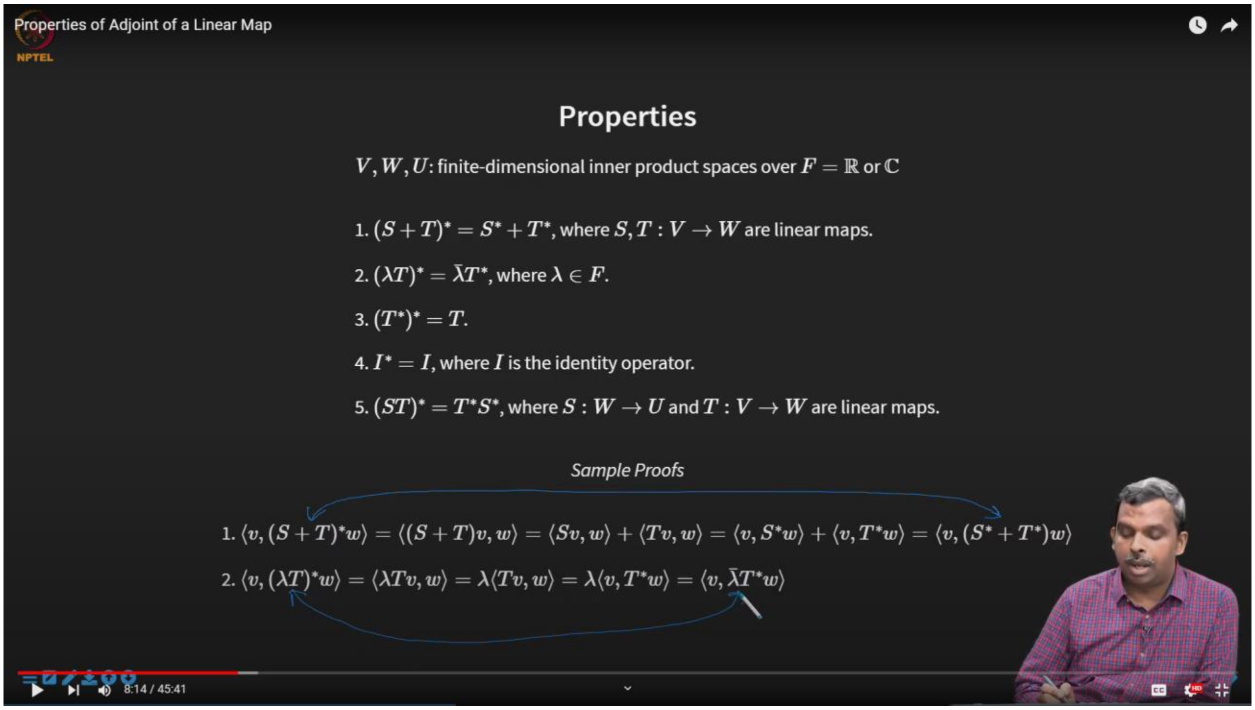

So I am not going to prove all of these, I will prove maybe, I think I have written down proof for one and two. I'll just quickly walk you through it. The philosophy is always the same. The proof for this adjoint property always uses that inner product relationship, right? So I have this $(S + T)^*$, what property do I know it satisfies? An inner product with v with the $(S + T)^* w$ is always going to be equal to the inner product of $(S + T)v$ and w, right? So this is the starting point, okay? Then once you come to this world where there is no adjoint, you use all the properties you have. Or even if there is adjoint, split out, you use all the properties of the inner product, right? $(S + T)v$ is $Sv + Tv$, so since inner product is additive in the first argument, you will get $< Sv, w > + < Tv, w >$ and then you use that adjoint property on each of these things, right? Individually. Now that you come to individual, you can use the adjoint property there. So this first $< Sv, w >$ is $< v, S^* w >$. $< Tv, w >$ is $< v, T^* w >$. Now you use the additivity in the second argument and you will get $< v, (S^* + T^*)w >$. So if you look at the first and the last now, okay, you compare the first and the last, you have your equality, okay? So $(S + T)^*$ is $S^* + T^*$. Is that okay? So you have that whole relationship, okay? So just go through this proof and convince yourself that every step was valid, that I have not made any mistake and check for yourself that this whole thing is true.

So now when, once you have a relationship like this, this $(S+T)^*$ operator should be equal to $S^* + T^*$, right? So think about why that is true also. You can sort of prove that to be true. You can bring it to this side and show that that operator has to be zero, okay? So it's not too bad, it's true for all v so this has to be true, okay? Okay. So now look at the next one. The second is the proof for the claim 2 here. And here again we are looking at $(\lambda T)^* w$. I mean whole adjoint. So we start with this inner product $< v, (\lambda T)^* w>$, then you use the property here. So this is going to be equal to the inner product $< \lambda T v, w >$ and you can pull the λ out and you get $< Tv, w >$. And $< Tv, w >$ you know is $< v, T^* w >$ and then you take λ into the second argument. When you take it into the second argument, you have to do $\bar{\lambda}$, right? So that's how we define the inner product. So you get $(\lambda T)^* w$ okay? So this now again you can compare these two guys and you quickly identify that $(\lambda T)^*$ is equal to $\bar{\lambda} T^*$, okay? So that's the proof. The other three I will leave it as an exercise, you have to write very similar proofs, everything will work out in the exact same way and you will be able to identify, the you know, what happens when you do adjoint of, you know some modified linear map, how does it relate to the adjoint of the original linear maps, okay? So you can do all this. So these are good properties to keep in mind. So this additivity, homogeneity, sort of conjugate homogeneity properties and repeated adjoint, what happens, all of this is easy to... You should remember this because it will come handy quite often.

Okay. So more properties. So adjoint we know is a linear map, right? It's a, if T is a map from V to W, T^* is a linear map from W to V. So of course it has the null space, range space all these other routine properties that we have for a linear map, okay? And it turns out those spaces are related to the similar spaces for the original operator T, okay? And those relationships are captured here, okay? So this orthogonal complement will enter the picture and you can see it plays a big role here. Look at the first result for instance. It says… So it's good when you look at a result like this to also think of what vector space each of these things are a subspace of. And, I mean, just think about that because sometimes it can be a bit confusing, so we should be clear on that. So $null(T^*)$ is actually a subspace of W, right? So range T is also a subspace of W. So $null(T^*)$, null of the adjoint of T which is a subspace of W is equal to, apparently, we will prove this, but this is a property here, it's equal to the orthogonal complement of (range of T). So clearly this is also a subspace of W so it's quite believable that this might be true. And one can see, we'll prove that this is exactly true. So the null of adjoint is equal to the orthogonal complement of the range, okay? So these are all, so this adjoint and the original operator are tied together in a very strong way through this inner product and so it's sort of believable that these things happen, okay? And there is a very precise relationship here, right? So null, remember null is something that makes the subspace which is orthogonal to, you know, to T in some way, right? So it's not difficult to imagine that this kind of property ends up being true, okay? So we will see how this works out. We will prove this, you will see it.

And similar properties are true. So in fact I mean once you see one is true, you know one and three are related, right? What is three? How do you get three from one? Simply put T equals T^*, okay? If you put $T = T^*$ in one you will get three, okay? Isn't it? So three is not a new result.

Three is simply a, you know corollary to one, okay? And in fact even two or maybe four, even two is sort of related to… Maybe not two but, you know, so all these guys are related. So you can do, yeah one and four. If you look at one and four, these are also the same thing, right? See if you take complement on both sides, if you put T equals T^*, from one you get three and if you take, if you put complement on both sides, you take complement on both sides, you simply get range T equals $(null\ T^*)^{\perp}$, right? So you have taken complement on both sides, $((range\ T)^{\perp})^{\perp}$ will become range T itself and you will get $(null\ T^*)^{\perp}$. So four is actually simply a restatement of one. Remember three was a restatement of one, four is also a restatement of one. Now once you have four, you simply put $T = T^*$, you will get two, okay? So, right? If you put $T = T^*$ in four, you get two. So these one, two, three, four, even though they all look very different it's just only one relationship, it's enough you remember one of them, whatever one you prefer, your favourite one, you remember one of them, everything else is obtained by either complementing or putting $T = T^*$, okay? So as long as you remember that, this is okay. All right?

And here is this other nice relationship. The range of T and range of T^* have the same dimension, okay? So this is another thing that we will see. This is actually, you know, if you take one of these results and use the Fundamental Theorem, I think two, you can take two for instance and use the Fundamental Theorem along with the dimension, this property, you will get the answer. So it's not very hard to prove this result, but it's good to, you know, take any one of these results. You'll get it. So it's just a question of using this but maybe a slightly non-obvious property that range of T and range of T^* have the same dimension. We will relate it to something else also later on. But this is interesting to see. So these are nice connections between null space, range space and various dimensions for T and T^*, okay? So notice this. So range of T and range of T^* have the same dimension, okay? So it's an interesting relationship. So let's do a quick proof for these things. I will prove the first one, okay? The first one is what's being proved here, okay? Proof is for the first result and all the others will quickly follow. You will see it is quite easy to see, okay? So to show this, I will start with a vector which is in null of T^*, okay? T^* is, remember, is the adjoint. It is an operator, it is a linear map from W to V. And null of T^*. w is a vector in null of this linear map. Which means what? That's true if and only if, I keep doing the sequence of if and only ifs, T^*w is zero, right? Right? That's sort of the definition. So this T^*w zero you can also write in an inner product form. So this inner product form is very important. This is true if and only if $< v, T^*w > = 0\ \forall v \in V$, okay? It's not too difficult to imagine. So you say you take a w, you know, T^*w itself becomes zero. So clearly $< v, T^*w >$ will be equal to zero for all $v \in V$, right? So this is okay. Now once you have this, look at what happens when you use the property of the adjoint. Here is where the property of the adjoint comes in. So crucial step, right? Up to this it's all trivial, $< v, T^*w > = 0$, yeah I mean of course if this is zero, it just works out. If it is zero for all v, then T^*w also has to be zero, right? So we know how the inner product works. That's how it is. So this, this switch is what's critical. Notice what happens $< v, T^*w >$ by the property of adjoint becomes $< Tv, w >$, okay? So this w belongs to null space of T^* if and only if w inner product with any vector in the range of T, Tv right, Tv for all v in V

is what? Any vector in the range of T, right? This has to be zero. All right? So this is where the crucial thing comes in. And if this has to be 0, then clearly, you know, w belongs to (range $T)^{\perp}$, right? So every vector in the range of T is orthogonal to w if $w \in null(T^*)$. And it's all if and only if you can go the other way also, okay? So w belongs to T^* if and only if $w \in$ (range $T)^{\perp}$. And notice how this nice relationship came about because of this property of the adjoint, okay? So this is important to note, okay? So clean little proof. But, you know, go and think about what it is, maybe draw a couple of pictures if you like to understand where this came from. But this is true. So $null(T^*)$ is : (range $T)^{\perp}$, okay?

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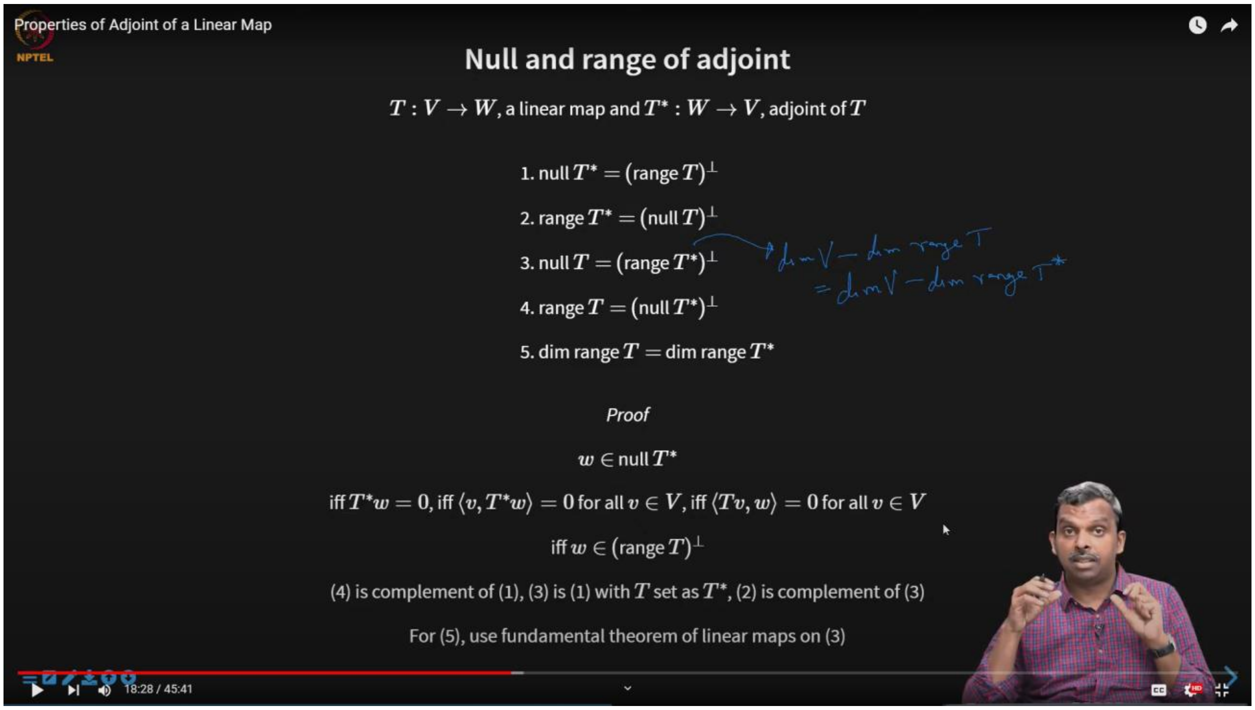

So the other proofs can quite easily be done. I think I illustrated the next part. How do you get, once you get 1, how do you get, you know, 4 from 1? Which is just the complement. 3 is setting T as T^* and 2 is the complement of 3, yeah. So 2 is the complement of 3. You can also see 2 as setting T equals T^* in 4, okay? Both of them are possible. So others are easy to prove once you prove 1, okay? For 5 I have said use Fundamental Theorem of linear maps on 3. Yeah. You can use it on 3 if you like, you can use it on 2 if you like, you can use it on one of these two and you will get it. So let me maybe show you how that is happening here. So I have said 3. So if you use this, so this tells you $null(T)$, $null(T)$ is a subspace of V, right? So $\dim V - \dim(\text{range } T)$, right? So that is dimension of $null(T)$. That equals range(T^*). So range(T^*), remember $T^*: W \to V$. So range T^* is a subset of V. So the dual of range T^* will have again dimension which is this, okay? So you can cancel dimension of V and you will get $\dim \text{range } T = \dim \text{range } T^*$, okay?

So looking at this relationship you may be tempted to think that $\dim null\ T$ is also equal to $\dim null\ T^*$. That need not be true. You can see that none of these results will give you that, it won't work out because $\dim V$ and $\dim W$ can be different, okay? That is a problem. If $\dim V$ were equal to $\dim W$, then you can say other things as well, but in general that need not be true. But whatever be the case, range of T and range of T^* will have the same dimension, okay? So that's the proof that we have done, okay? So these are good properties. So if you notice the previous slide, we saw properties on, you know, manipulating up linear maps and what happens to the adjoint. And now null spaces, range spaces. There's some powerful results for adjoint and T. So adjoint and T are sort of strongly connected, they are very similar to each other in some fundamental ways.

Okay. So now what is the connection between matrix representation of linear maps and adjoint, okay? What is the matrix of the adjoint. How is it connected to matrix of the original linear map, okay? So this might be a question that's of great importance and it turns out there is a very simple and nice answer. If you have matrix representation with orthonormal basis for both V and W, the adjoint of a linear map T is, the matrix representation is simply given by conjugate transpose, okay? So it's a very simple answer and it also has a bearing on many of the previous results we discussed. So it's nice to look at, okay? So I've captured that here. But the orthonormal basis is very very very important. Usually we use standard basis. So it's always true. But if you're not using an orthonormal basis, then this result of conjugate transpose is not true, okay? So remember that, okay? So let's see how that is developed. So once again we have two finite dimensional inner product spaces and an operator T, a linear map $T: V \to W$ and we have two orthonormal bases, one for V and one for W, okay? I have used n and m as the dimension, okay? The matrix of T with respect to B_v and B_w is, we will denote it as $M(T, B_v, B_w)$ just to say V has basis B_v, W has basis B_w and then the matrix M, this represents this operator T with respect to those two bases, okay? So here's the big result. The matrix for the adjoint of T with B_w as the basis of V, remember T^* will go from W to V okay, with B_w as a basis for W and B_v as the basis for $V\ldots$ So notice the reverse thing there that will also happen. That is simply equal to the conjugate transpose of the original matrix we had for T itself. So what is conjugate transpose of a matrix? You transpose it, make every row column, do the row column interchange and you conjugate each element, do a complex conjugate of each element. Of course, if your field is real then the conjugation will have no impact, only the transpose matters. But if the field is complex, the conjugation will have a change in the entries, okay? So this is something to keep in mind.

Okay. So proof is not too complex. I have written it down here. I will quickly walk you through it and I'll let you think about it. It's quite an easy proof. If you look at the $(i,j)^{th}$ entry of the matrix $M(T, B_v, B_w)$, okay, what do I do? So notice what is $M(T, B_v, B_w)$. So if you look at the column, you have $[Te_1, \ldots, Te_j, \ldots, Te_n]$, okay? So each column the j^{th} column is Te_j. Coordinates in what basis? In B_w, right? So you find Te_j and find its coordinates in the basis B_w and write it as the j^{th} column. So what will be the i^{th} entry? Remember i^{th} row, if i^{th} row were here, what will be the i^th^ entry here? It is the coordinate of Te_j for f_i isn't it? Now f_i is, this f is an orthonormal basis. So what is that coordinate? Inner product $< Te_j, f_i >$ okay? So you can see that just neatly works out as $< Te_j, f_i >$. So notice how this orthonormality is very very crucial. Otherwise you won't get this inner product formula for the entry there, okay? So this is important. So now you use all that you know about adjoint, okay? So $< Te_j, f_i >$ is the same as inner product $< e_j, T^* f_i >$, right? And then you do the conjugation. It's $\left(< T^* f_i, e_j >\right)^*$ whole conjugate, okay?

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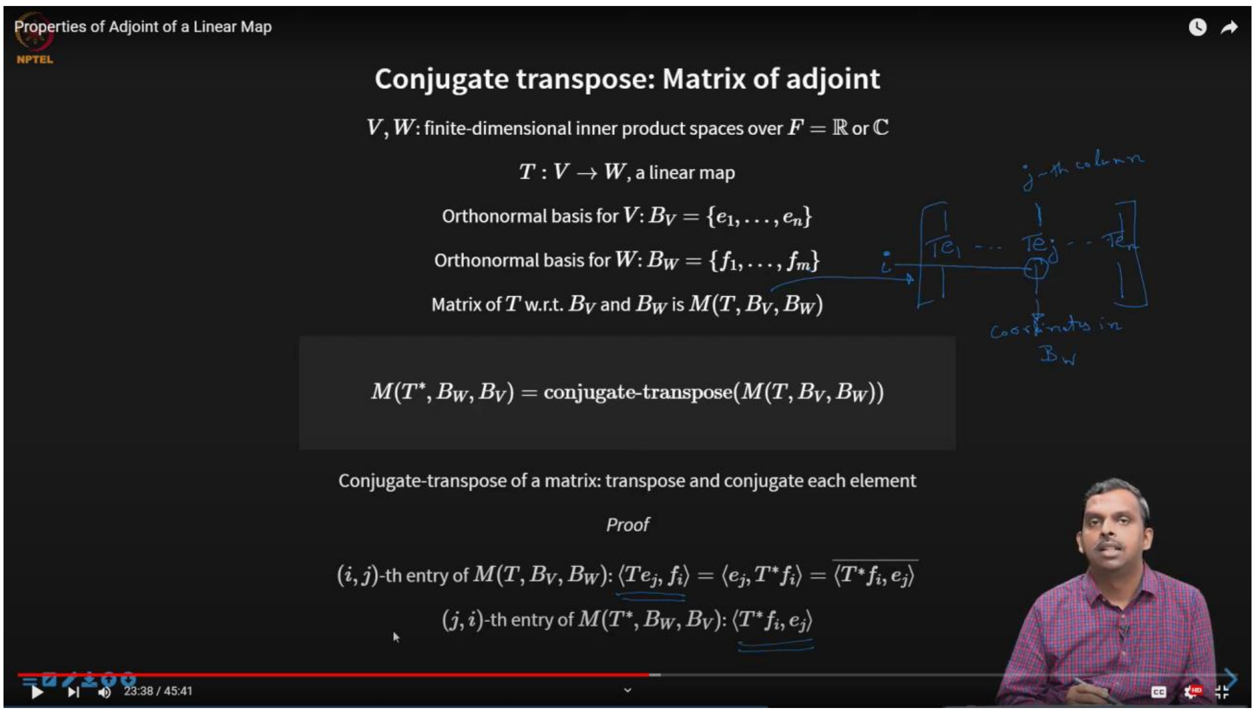

All right. Now what about the $(j,i)^{th}$ entry of $M(T^*, B_w, B_v)$, it is the same thing, okay? So this thing is the exact same thing, right? See (T^*, B_w, B_v) will be $T^* f_i$. I am looking at $(j,i)^{th}$ entry, right? So $T^* f_i$ inner product with each and so you see the (i,j) becomes (j,i), this entry becomes its conjugate. So that is where you get, the conjugate transpose, okay? So simple proof, but nevertheless it's a good idea, a good thing to go through this proof once again. You will understand the mechanics of it a little bit better, okay? So transpose, conjugate transpose and this adjoint operation are very very closely and intimately connected. And if you go back and interpret the previous result, we had a null and range space in terms of the four fundamental subspaces of the matrix, this notion of transpose will become very clear to you, connection between transpose and adjoint will become clear to you. So let's do that, let's look at adjoint and four fundamental subspaces of a matrix. We'll pick a matrix A which is $m \times n$ over $\mathbb{R}$. I'll take $\mathbb{R}$ just for simplicity, or $\mathbb{C}$ also there are similar statements which are true. Let's say it represents a linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ with respect to standard basis. Notice I am saying standard basis. This is very important. If you change the basis here, all that I say will not be true, okay? Very important to keep in mind, okay?

So T^* now I know is a linear map from $\mathbb{R}^m \to \mathbb{R}^n$ and it is represented by the conjugate transpose A or $\overline{A^T}$, I can put a notation like that. But since entries of A are real, you can say the conjugation is irrelevant. So it's simply A^T. Once you have that, you can go back and look at how these, you know, spaces are related and you will see the very natural thing that is happening with respect to the matrix, right? Range of T^* is the column space of A^T and that is nothing but the row space of A and clearly that is the null of $A^\perp$ right? The dual of the orthogonal complement of the null of A, okay? Now null of T^* is the null of A^T. That is left null of A and that is the $(\text{column space of } A)^\perp$ whole complement, okay? So the same things, the same thing we had on, you know, range of T and null of T^* and all those connections clearly plays out and gives you a very nice grasp if you look at it in terms of four fundamental subspaces of the matrix, okay? It's A and A^T, rows become columns. If you look at null, the null will be a complement for the row space, left null will be a complement for the column space, that's it, okay?

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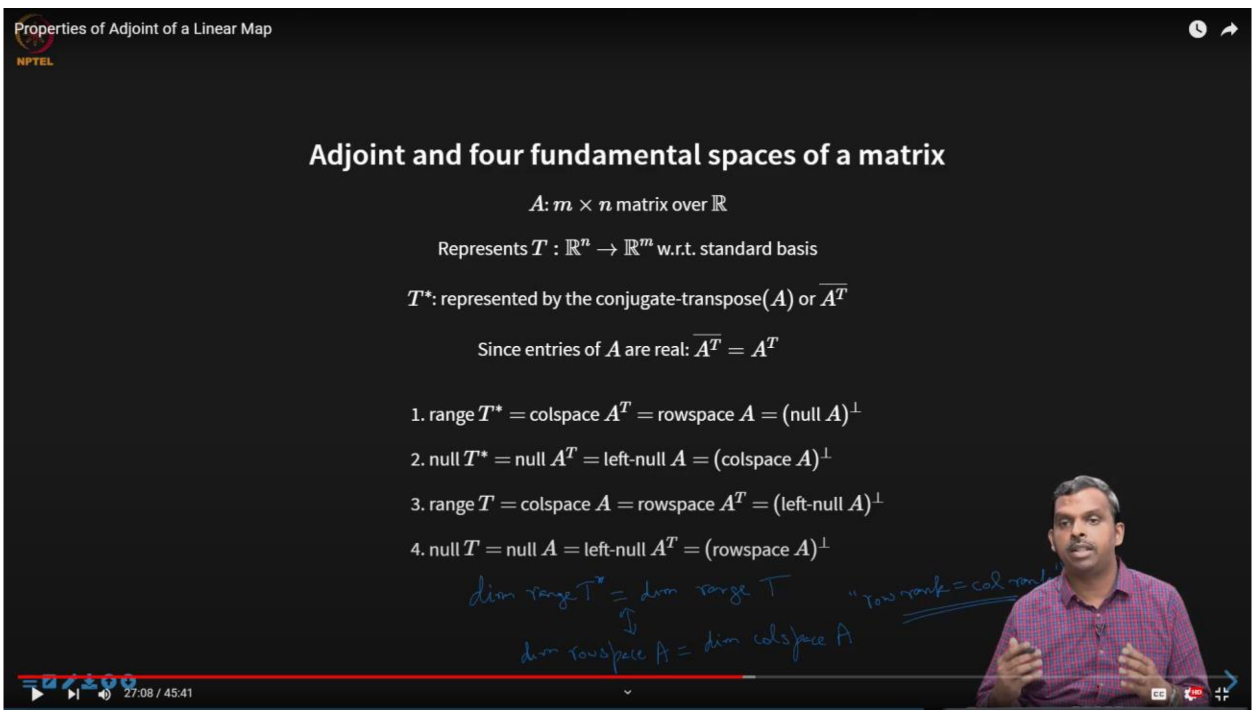

And what about this relationship, that dimension of range of T should be equal to dimension of range T^*? That is nothing but row rank equals column rank, right? Range of T^* is nothing but the row space, range of T is the column space, both of them have the same dimension. Row rank equals column rank and that gets reformulated as dimension of range T equals dimension of range T^*, okay? I think maybe that point I have not written down, maybe it's worth writing down. Dimension of range T^* equals dimension of range T is exactly equivalent to dimension of column space… Okay, maybe I should write row space first. Row space of A equals dimension of column space of A which is nothing but row rank equals column rank, okay? The familiar result that we proved using another trick in the previous, one of the previous lectures you can see when you define adjoint in a very clear fashion, okay? So this sort of gives you a feel for what adjoint is, its connections with the matrices we have been talking about. Notice this important thing is the orthonormal basis. This adjoint thing works very well only when you work with orthonormal bases. If you change basis you have to do something more to find the adjoint, it's not very obvious or easy, okay? So keep that in mind.

Okay. So now let us move on and study slightly more advanced notions for the adjoint. Like I said, the adjoint plays a very important role and to do all that, we will need to look at what happens when two operators are composed. In particular, what happens to the null spaces. We've been looking at it in some exercises and applications we have been talking about. If you have a product AB, what is null of AB in as a function… I mean, what is the connection between null of AB and null of B, null of A etc. etc. So let us formalize that first and that will play a key role when we study more advanced things with adjoint also, okay? So let us say, we are back to linear maps now, you have a linear map $T: V \to W$ and another linear map $S: W \to U$. Now ST is a composition of the two maps and it will take you from V to W, right? That's also a linear map. We know that. Now the crucial thing we want to study is how is the null space of ST connected to the null space of S and null space of T, that's what I want to do, okay? So here is the crucial result. I think I have alluded to this in earlier discussions, but let me just formally write it down. Let us define a new subspace. This is a subspace of W, okay? Remember ST takes you from V to U, but it goes through W, right? And in that W in the middle I am going to define a subspace. What is this subspace? It is the range of T, which clearly is a subspace of W, intersect with null of S, okay?

So range of T is a subspace of W, null of S is again a subspace of W, okay? So clearly their intersection will again be a subspace. So that subspace I am calling as N_w, okay? So some N_w, okay? This subspace is very important, this will play a crucial role in how to determine null of ST, okay? So in particular here is the relationship, okay? The dimension of null of ST equals dimension of null of T plus dimension of this N_w, okay? So this is the important relationship, okay? And sort of makes sense in so many different ways. I will draw a picture soon enough to clarify this. But let us go through and try and prove this first in some fashion and we'll understand. So maybe I should draw a picture right way before we… Or maybe I should wait for the proof to come out and then I'll draw it, okay? So, so the first step in the proof is to figure out how this N_w enters the picture, right? So you will see how this N_w enters the picture here. Supposing I give you a vector v which is in the null space of ST, okay? What do I know? STv equals zero, right? And that is if and only if $Tv \in null(S)$, okay? So this is the crucial relationship here. Tv has to be, so $STv = 0$ means Tv, right, after you did Tv has to be in the null of S, right? Only then $STv = 0$. These are all basic definitions. Now remember Tv is in the null of S but Tv clearly is also in range of T, right? Tv by definition is in range of T, it also has to be in null of S. So if v is such that v is in null of ST, then Tv must be in range of T, obviously but it should also be in null of S. So what does that mean? Tv is in the intersection of these two. So Tv has to be in N_w, okay? So this is how this N_w enters the picture and it plays a very important role, okay? So I will draw a picture soon, but let us just bring up the rest of the proof and then I will draw the picture.

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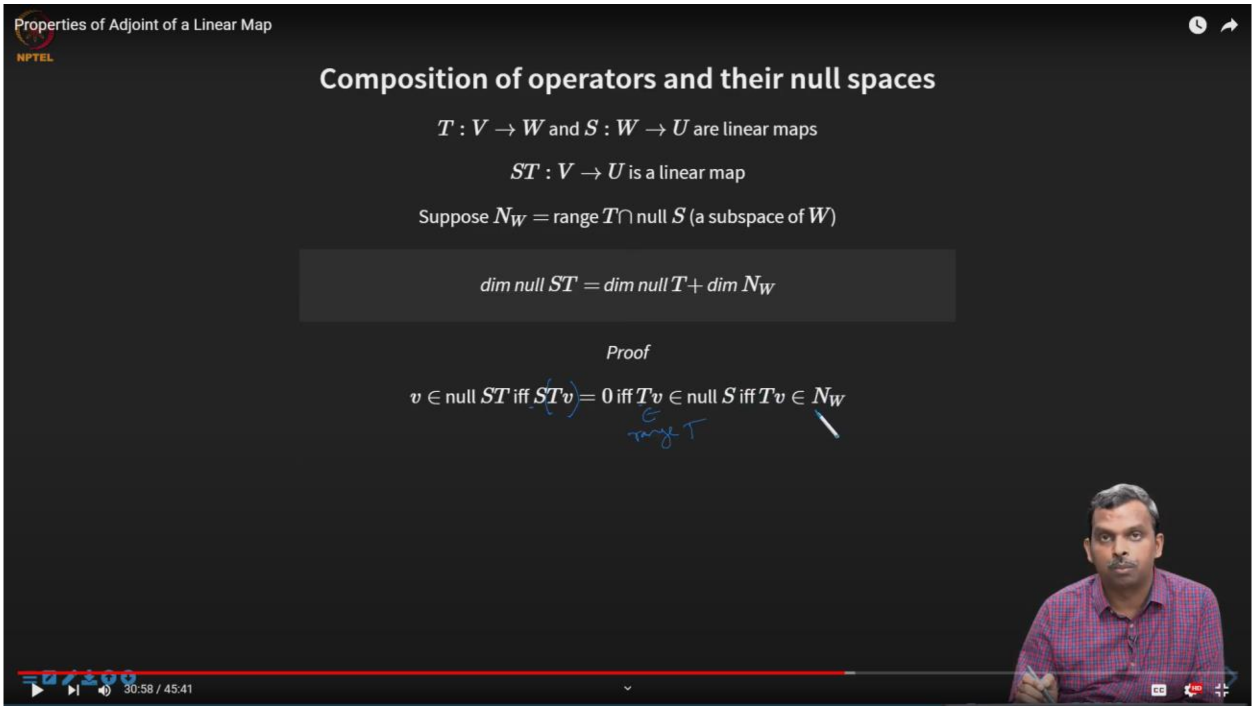

So from this argument, we clearly see that null of ST is the set of all v such that $Tv \in N_w$, okay? So this is crucial. There is nothing else in the null of ST, right? If v is in null of ST, this is all if and only if, if and only if, if and only if. It all works out like this. Is that okay? So think about it. Null of ST is the set of all v such that $Tv \in N_w$. From this it comes out, okay? So we will start thinking of basis for null of ST. See, for looking at dimension we need to look at bases. I think at this point maybe I should draw the picture. It will be helpful I think. So this picture has three subspaces. There is V, and then there is W. I will draw it a bit bigger so that a lot of action happens in W. And then you have U, isn't it? So V is the first vector space. This T operates from V to U, and then you have S operating from W to U, V to W, T operates from V to W and S operates from W to U. You have these two subspaces here, interesting ones, there is this range of T and let us say there is this null of S, okay? And then you have this nice little intersection here which maybe I will draw in some green. This is N_w, okay? You will also have other interesting subspaces, right? You will have null of T, will be here, okay? And you know how this operator works, right? So under T what is going to happen? The whole V will go to range T and null will go to just one point here, right? So this zero will be somewhere here and this whole null will go to that point, okay? So keep that in mind, okay? Hopefully you can see that. And then what happens when I go from, you know, W to U? Remember once I do ST, I start with a v here, I will only come to range of T, whatever is outside of range of T is irrelevant to me and this S when it operates I can restrict it to range of T, right? And then this range of T will take me to something here, okay? This will be a subset of range of S, okay? Which is where you will go. And this guy will take me to the 0, isn't it? This N_w will take me to 0 when I restrict it to this, okay? So when I want to look at null of ST, I am looking at the subspace of V which takes me to N_w, right? Subspace of V which takes me to N_w. So what will that be? So now if you look at any other point here, some other point here, what will that come from? That will come from a translate of null of T, right? So there will be a translate here, okay? So maybe I can draw it like this, okay? So that will take you to this any other point, isn't it? So there will be similar translates. I am just drawing it grossly out of proportion, but you can still see for every point here there will be a translate here, isn't it? So all of these guys will take you to this subspace N_w in W and I have to just characterize that. What is this subspace here, what is this null of ST, which is the set of all v such that Tv takes you to Nw.

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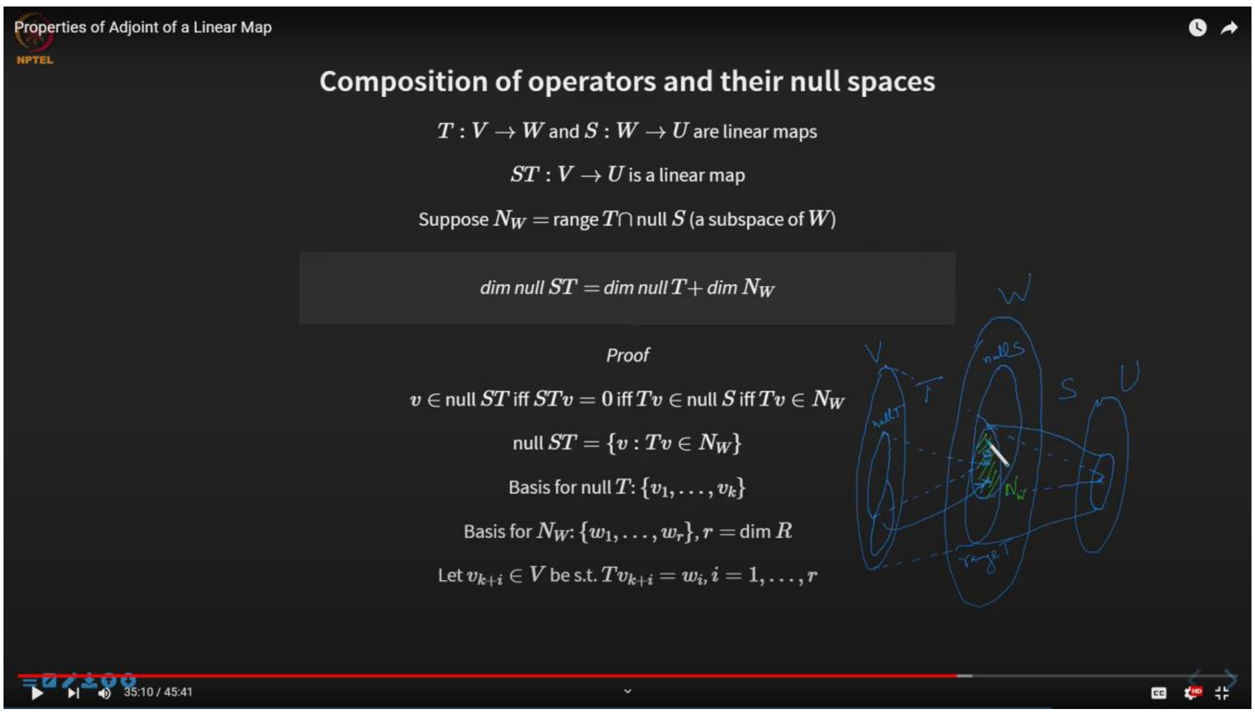

So how do I do this? This is not very hard if you visualize this a little bit. I am not going to be too detailed on this, I'll just provide the final answer in some sense, okay? So first thing you will have is a basis for null T, okay? Everything in null T will definitely take you inside the N_w. It will take you to 0, there's no problem. What about other things? For that let me look at this very closely. I'll come up with the basis for N_w, okay? I'll call it $\{w_1, \dots, w_r\}$, okay? r could be the dimension of, okay, I've said R here, it's not R, sorry, it's N_w, sorry about that, okay? So this is the basis and I will try to find solutions in V for this equation $Tv = w_i$, okay? So what is that solution? It is v_{k+i}. Just one solution let us say, okay? So you take v_{k+i} such that $Tv_{k+i} = w_i$. This is true for 1 to r. Now why do I do, how do I know that the solution definitely exists? How do I know that the v_{k+1} will definitely be there? It's, you know, w_i actually belongs to the range of T, okay? So definitely there should be at least one v_{k+i}. So you take your favourite v_{k+i}, does not matter what it is, you take that. So you collect all these vectors in v, just one vector in v which will take me to the basis vector of N_w, okay? Now my claim, here is the claim, okay? So maybe I should do this here also, this is N_w. This is the claim, the basis for null of ST is simply the original, the basis for null T, v_1 to v_k and then this new v_{k+i} that I have, v_{k+1} to v_{k+r}. That's the basis for null T, okay? This needs a little bit of proving and I am not going to prove that for you in detail in this lecture. I will let you think about it.

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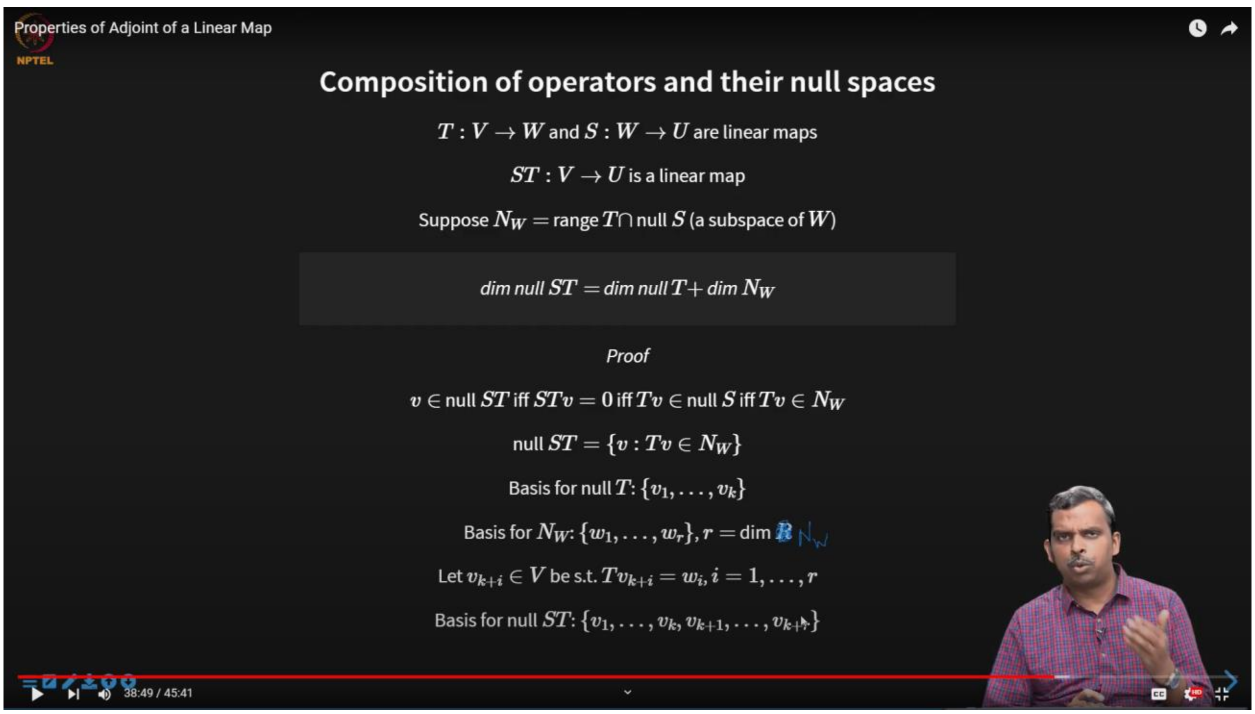

But it's sort of intuitive from the picture I drew in the previous slide. You can sort of figure out why this should be true. You take a basis vector for, basis for N_w, figure out solutions for the equation Tv equals the basis vector and add it to the null vectors, null basis. You should get the basis for null of ST, okay? So I will leave out some technical aspects of the proof to you. If you are interested you can fill it out. Otherwise also it's okay. But the most important result is this, okay? This N_w is naturally entering the picture in W and the dimension of null of ST is dimension of null of T plus dimension of N_w. And the basis is formed in this fashion. And from here you can see, you know, dimension of null of T comes in and dimension of N_w comes in after that, okay? So that picture is very important. You have V here and you have this N_w here and 0 in U, right, comes only through this N_w, right? So and then that's the connection. And how do you get to N_w? You can either come from the null space or you can come from other parts of V which will take you inside N_w. And how do you solve for the set of all v that takes you to N_w? You can take a basis for N_w, find individual solutions and put them together, you will get a basis for the subspace that you are interested in, okay? So that part needs a little bit of proving and I'll let you do the proving yourself, okay? So this picture is very important and this will play a key role in several of our proofs later on. And it's also good to know that what happens when you compose two linear maps, how do you think of the null space. It gives you a lot of interesting ideas, okay?

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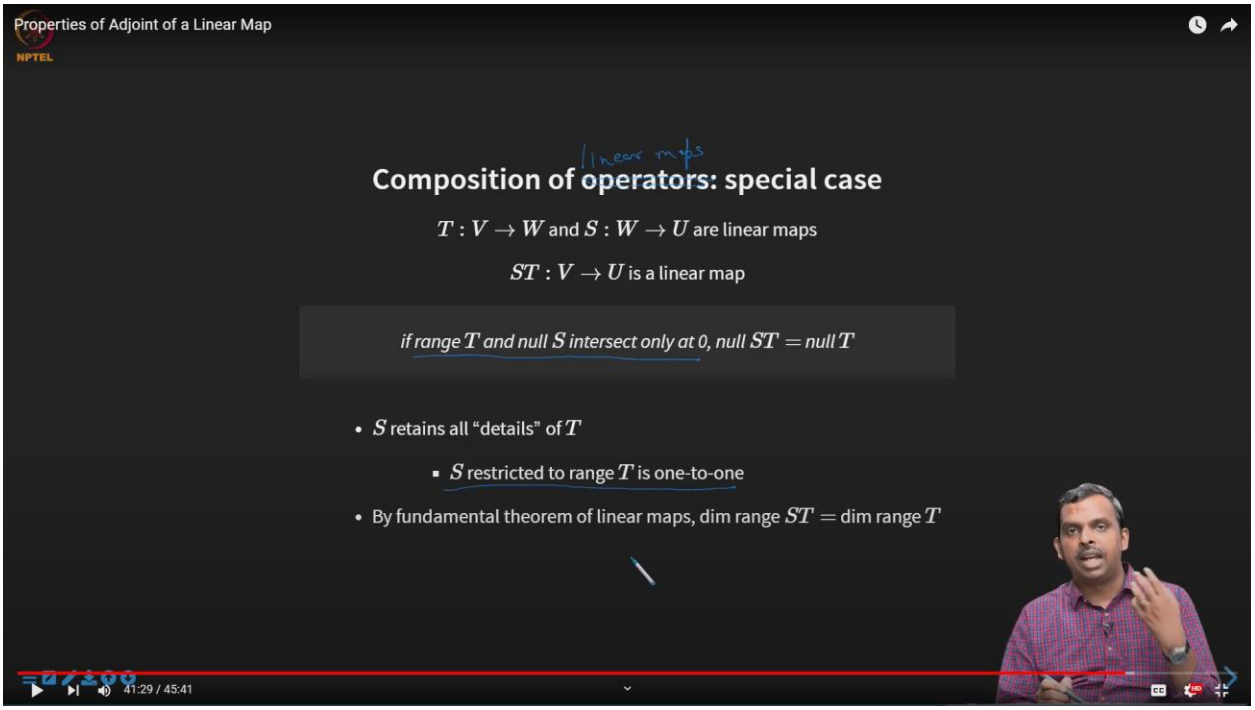

So there is a special case of this which is very important, okay? When you compose two operators, as before, there is an operator from V to W, there's an operator from W to U, you compose it, ST, you get operator, you get a linear map from V to U. I keep saying operator operator operator. So it's, again it's compositions of linear maps, it's not really operator, okay? So you may want to replace this with linear maps, okay? So I keep saying this operator all the time but you know, keep a flexible definition in your mind. When I say operator, it could be a linear map also, okay? So that's okay. So anyway, what is this special case? This is the special case, okay? So a very special situation is when range of T and null of S do not intersect, okay? So in the previous picture we saw that range of T and null of S intersected in a big way. The special case is when they intersect only at zero, okay? So they are sort of disjoint mostly except for that zero, okay? In that case, null of ST equals null of T, okay? So there is nothing else, right? N_w is only zero, the only way you go to zero in U through S is through that zero in W, there is nothing else in W which can take you to zero in U. So you better be in the null space at T, okay? So that happens when that intersection is zero. So it's sort of like, you know, S and T are sort of nicely related to each other. As in, S does not diminish T in any way. T takes you from V to W and S does not further diminish the dimensions by operation, okay? So for instance, the simple thing to think of is: suppose S is an invertible operator. right? U and W are the same. If S is an invertible operator, clearly ST and T will have the same null space, right? It's invertible. But you don't need S to be an invertible operator, S could be a linear map but its null space could intersect trivially with the range of T. In that case also you have the same effect of the invertible operator like thing, except that it's only specific for this particular T, right? So if you pick your T and S carefully to match, or not, sort of mismatch in the range and null, you can almost have that invertibility effect with ST, okay? So that is the idea here. And once you have null of ST being null of T, you can also see that range of ST equals range of T. It is almost as if… This property is very important. So when S is restricted to the range of T, it is one-to-one. So it does not do any further damage to the transformation, okay? It does not reduce dimension in any other way, okay? So this special case is very, very important. And this special case applies for us when it comes to adjoint, okay? So you will see why that is interesting, okay?

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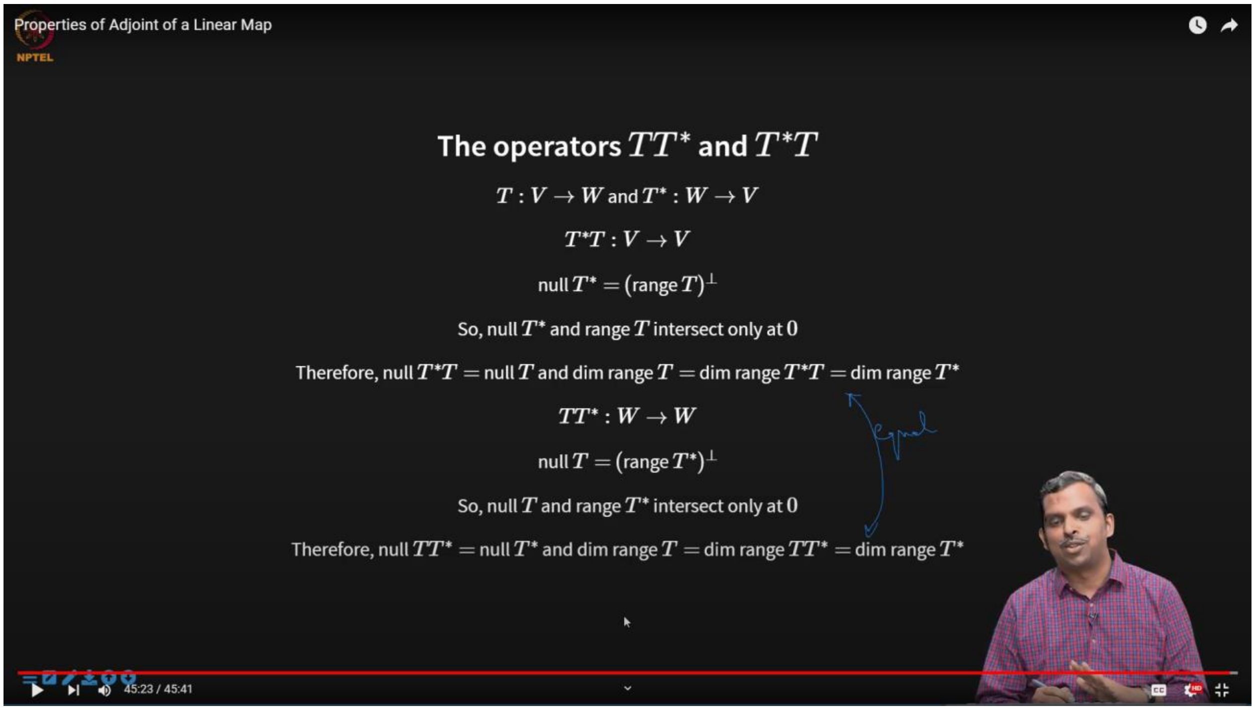

So these two operators especially are very crucial in a lot of linear algebra studies, okay? So you have a linear map T, you have its adjoint T^*, okay? Now I can define an operator and I am right, this is an operator this time. I can define two very, very interesting operators, two compositions TT^* and T^*T, okay? These are two different compositions clearly, right, because V and W are not the same. TT^* is an operator from W to W, okay? And T^*T is an operator from V to W, okay, sorry V to V, okay? TT^* is an operator from W to W and T^*T is an operator from V to V. These two operators are very different even in terms of what they do, but you know, they are very nicely connected and they have some very nice properties. In fact in the previous properties that we saw of how operators compose, right, how one operator does not destroy the other operators thing, these two operators are like that. They really have those nice properties of composing properly to give you what you want, okay? So that plays an important role later on. So let us see each of these things correctly.

So T^*T is an operator from V to V and now I know null of T^* is actually $(\text{range of } T)^{\perp}$, right? So T and T^* are like that. So clearly when this is true, we know that null of T^* and range of T intersect only at zero, okay? So our good composition result is true, right? So it's almost like the invertibility composition. So null of T^*T is null of T and dimension of range of T becomes equal to dimension of range of T^*T. And we also know its equal to dimension of range of T^*. So this TT^* and T^*T, they are all very closely related. So many of their properties. And they will all look very similar in some sense, right? So this is the high level feel for why that is true, okay? So this is the reason. Same thing with TT^*. TT^* takes you from W to W. It's an operator and I mean… So where are we heading here? See when you look at just the linear map $T: V \to W$. One of the problems in studying that was we couldn't repeatedly apply T, right? So when we could repeatedly apply something, with operators we got these wonderful eigenvalues and that shed a lot of light on what that operator was. Now this TT^* trick will give you that repeated application possibility, okay? So you have T, you apply it, compose it with T^*, right? T^*T. You get an operator now and this operator looks sort of similar to T, right? It's close to T in some sense, in a loose way it sort of represents T. It doesn't take anything away from what T is doing. So and now TT^* is an operator, T^*T is an operator. You can start thinking of eigenvalues for it. You can start thinking of invariant subspaces for it and all sorts of interesting things you can study, okay? So this is a nice thing. So again for TT^*, the same thing is true. Null of T is $(\text{range of } T^*)^{\perp}$ perpendicular. So these two intersect only at zero. So you have all these nice relationships that are true. And you can see also even here, right? So these two are also equal. I didn't call it out, these two are all equal, okay? So TT^*, T^*T, they are all equal in dimension, okay? So hopefully this gave you a more clear idea of adjoint and its use when studying linear maps and its connections to linear operators etc., okay? So in the next lecture or so we will start looking at operators and their adjoints. So far we have been looking at linear maps and adjoints and how adjoints play a role in understanding linear maps and all that. Now we will start looking at adjoints of operators in the next lecture and look at eigenvalues and nice properties from that point of view. Thank you very much.