Applied Linear Algebra
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Week 10
Self-Adjoint Operator

Hello and welcome to this lecture. We are going to study in this lecture one of the most important types of operators in all of linear algebra, which is called the self-adjoint operator. It's extremely popular in practice. In fact, you can say without any doubt that if you are not sure what an operator should be, you can sort of assume it's self-adjoint and proceed. So it's not a, it's so prevalent, it's so common that everybody uses this. It's got wonderful properties, these self-adjoint operators and they're very useful and simple to describe as well. So we'll look at why that is so. You'll see from the properties, it'll be sort of clear to you why self-adjoint operators are very, very important, okay? So let's get started looking at self-adjoint operators.

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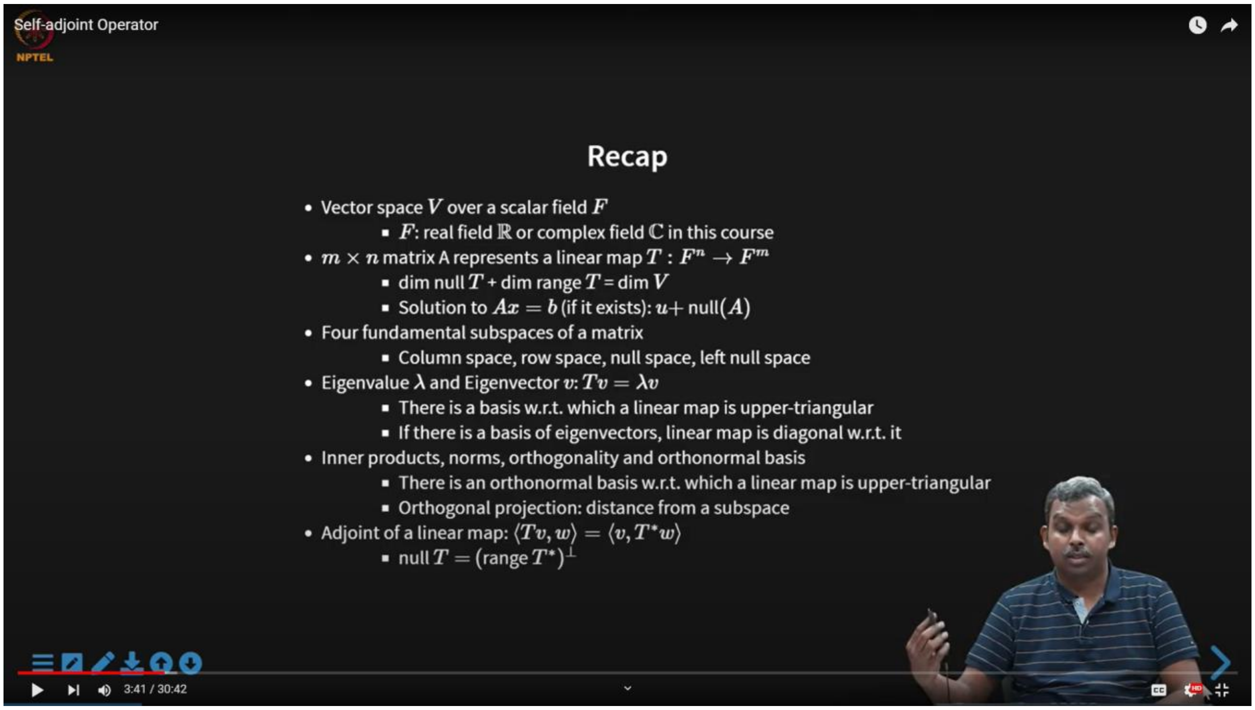

A quick recap. This is a new week. So let's just do a sort of a good recap. We started by defining these abstract vector spaces over a scalar field, restricting to real or complex fields in this class. And then we saw how there is this matrix representation for linear maps from one vector space to another. Notion of basis, dimension, linear independence, all of these enter the picture. And there's this fundamental theorem of linear maps which relates the null space and range space of a linear map, and this helps us solve linear equations $Ax = b$, okay? And these four fundamental subspaces of a matrix or a linear map are important to understand. And the various relationships how row space and null space have intersection which is trivial etc. etc., all of that is something that one needs to be careful about. So think about that carefully. The various intersections between these spaces and properties that they have. Then we looked at invariant subspaces, particularly one dimensional invariant subspaces which leads us to the definition of eigenvectors, eigenvalues. And then we found that there is a basis in which any operator, any linear map becomes upper triangular, okay? So that's a good result to have. And then in particular, if you have a basis of eigenvectors, then the linear map becomes diagonalizable. It's represented by a diagonal matrix which is the simplest form of a linear operator in some sense, right? And then we looked at inner product spaces and we saw inner product spaces bring in this notion of orthogonality, and that helps us solve non-trivial problems like, you know, distance from subspace. And also we saw that there is an orthonormal basis with respect to which any linear map is upper triangular. So these are some nice results we saw. And just last week we looked at adjoint of a linear map, and it is a map which connects the inner products in one from V to W, if you have a linear map $T: V \rightarrow W$, the inner product $< Tv, w >$ and, you know, $< v, T^* w >$ are the same, okay? So this is this linear map which is the adjoint which nicely connects it. And then there was this nice relationship that the null of T and the range of T^* are orthogonal complements of each other, okay? So that picture is important. We will dwell on this a little bit more as we study more in this lecture, okay?

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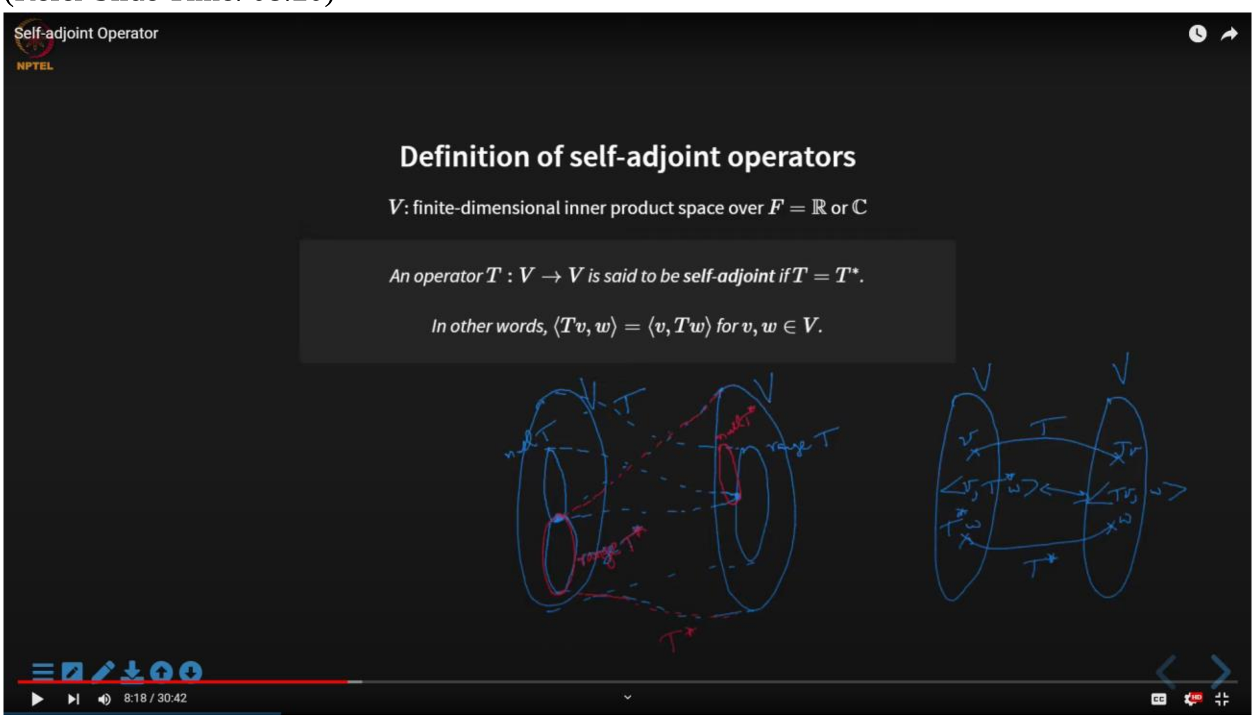

So this lecture is all about self-adjoint operators, okay? So let's start with the definition of what they are. Once again, we'll start with the finite dimensional inner product space. So this is what gives us their adjoint and all that. So we start with the finite dimensional inner product space over a real or complex field. Here's the definition. An operator $T: V \to V$ is said to be self-adjoint if T equals T^*, okay? So the definition and the word are very clear. Self-adjoint. The operator is its own adjoint, okay? So $T = T^*$. In other words, right, what is the definition of the adjoint? $< Tv, w >$ is $< v, T^*w >$. Now if you have a self-adjoint operator, $< Tv, w >$ and $< v, Tw >$ are exactly the same, okay? So this calls for a bit of a picture and some description. So let us do that, okay?

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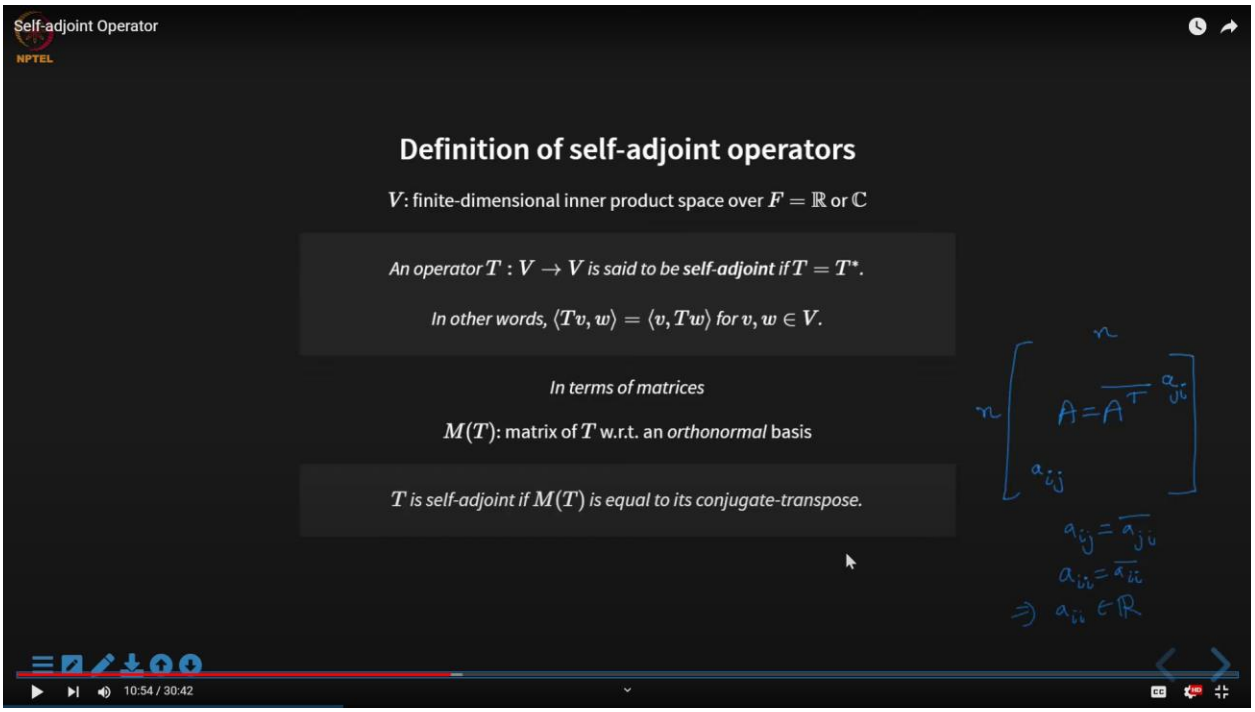

So you have V, okay? I will make another copy of V to sort of drive home the point, but remember these are all the same. Both of these are the same. There is this null of T and there is a range of T. So let us say, we will put it like this, okay? Then null T and then sort of T maps you from V to the range, and then null gets mapped to let us say the zero which I will put at the end, okay? Now what about T^*? T^* we know is going to map V to V again, but it will take you to a range which is, you know, orthogonal complement of null T, right? So range of T^* is an orthogonal complement. So in that case it is not going to intersect much with null T. So you would have something like this, right? So this is probably range of T^*, okay? So that takes, that's, so let me just do this properly… Maybe I should do another color for this. T and T^* are sufficiently distinguished and you would have a null of T^* which is going to be, you know, orthogonal complement of… This is null of T^*. Range of T^*, okay, T^* and then you have T, okay? So this is sort of the picture we have in mind with T and T^*. Now notice what happens when T is T^*, okay? A lot of things here are going to sort of coalesce, right? So if T is T^*, then the range of T and null of T are the same. Null of T^* and null of T are the same. So all sorts of orthogonality is going to happen, and things are going to really collapse, okay? So that's one way of viewing it. And the other way of viewing it is: if you look at a v, if you look at an arbitrary v, so maybe I should do it in a different part here… So let me do that separately. Another way of viewing it is this inner product, okay? So v takes you to Tv, and you have a w, right? And then there is this inner product $< Tv, w >$, right? And then w takes you to T^*w, and there is this inner product $< v, T^*w >$ and both of these are the same, okay? So this is the picture with respect to the adjoint. Now notice what happens when you have self-adjoint. The same T is going to take you back. So $< v, Tw >$ and $< Tv, w >$ are the same, okay? So remember this picture of what an adjoint should do and what happens when the operator is self-adjoint. So this will sort of help you visualize if you think that's useful, what's going on between these kinds of things. So self-adjoint means the operator becomes its own adjoint, okay?

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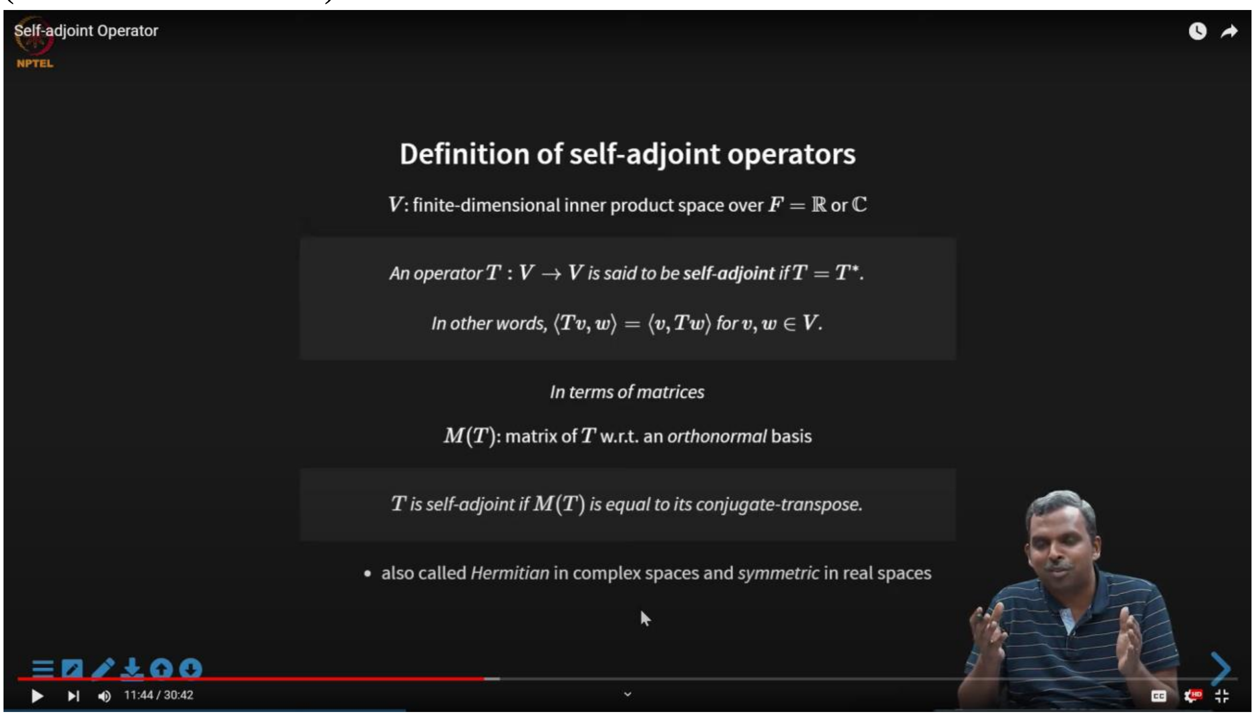

So it's most useful to think of it in terms of matrices, okay? So yeah, it's one thing to visualize the linear map itself and what's happening with the null space and range space. But if you think in terms of matrices, things become much more concrete. So let's say we have an orthonormal basis for V with respect to which you have a matrix for T and that's $M(T)$, okay? So it's going to be a square matrix. Since we are in finite dimensions this will be clearly a square matrix, okay? $n \times n$. And what is the matrix for the adjoint? It is a conjugate transpose of this matrix, right? So when is T self-adjoint? If the matrix is equal to its conjugate transpose. So let me just do this. A^T. And then conjugate, okay? So this is the property. So you do a transpose of the matrix and take conjugate, you should get the same thing. So what does that mean? $A = A^T$. So if you have a_{ij}

somewhere here, okay, and a_{ji} somewhere there, $a_{ij} = \overline{a_{ji}}$, that's it, okay? So this is the definition. So if you are starting to fill out a matrix, okay, you want to construct a self-adjoint operator by constructing its matrix, you start filling out the entries of a matrix. For an arbitrary linear map, you can put whatever entry you want everywhere. For a self-adjoint linear map, you are allowed to freely pick only elements on top of the diagonal matrix. What will happen to the bottom of the diagonal matrix? It will transpose and become conjugate, okay? Not only that, notice what happens to a_{ii}. So that implies a_{ii} is real, right? So on the diagonal, you should put real values and start putting whatever complex values you want on top of the diagonal. And below the diagonal, it will simply transpose and you can construct by this. So it's almost like with you have like half of the degrees of freedom in picking your linear map and you can get a self-adjoint linear map, okay? So that's a good picture to have in mind, it's a very, very good picture to have in mind to describe a self-adjoint operator. In a more abstract sense you can think of what happens. I will tell you what happens in the next slide. You'll see the null, null T^*, range T^*, all of them sort of become the same, okay?

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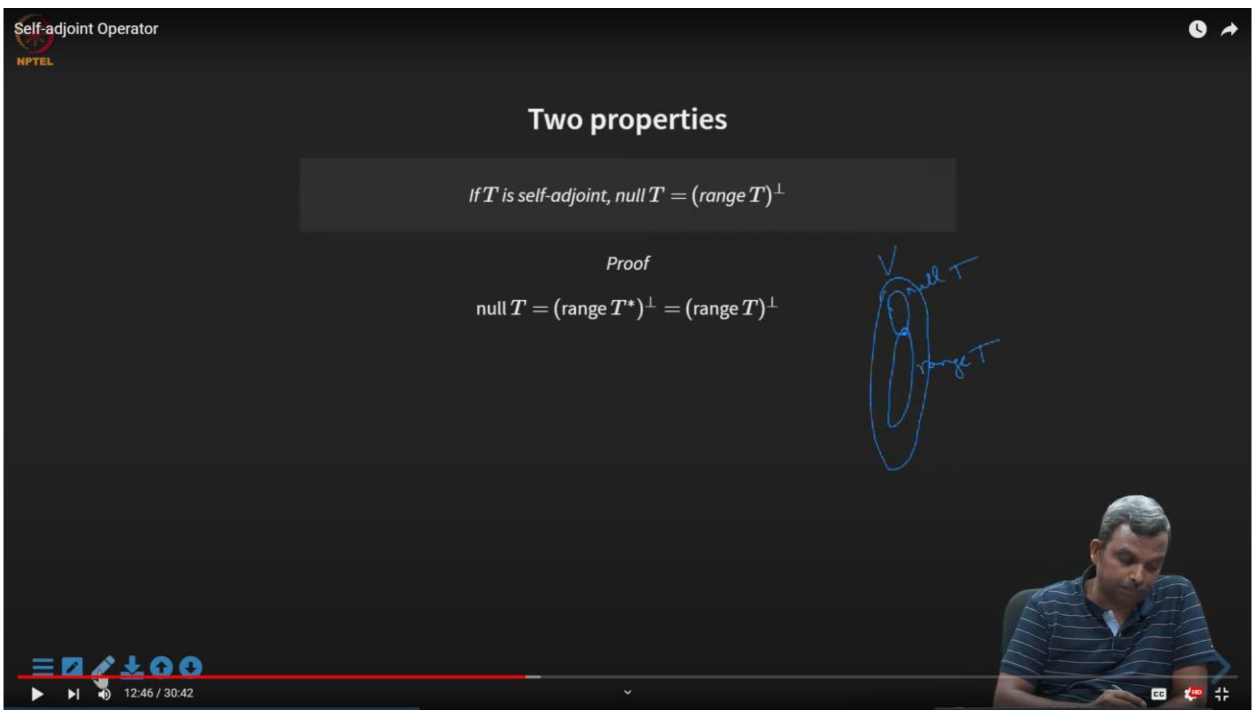

So the self adjoint also has other popular names. In complex spaces it's called hermitian, symmetric matrices, in real spaces it's simply called symmetric matrix. So what happens if it's in real space? All your matrix entries are real, so there's no conjugation, right? Conjugation is the same. So it becomes symmetric, as in above the diagonal is equal to below the diagonal. And diagonal is whatever you want. In complex, I describe to you what happens, what has to happen. Diagonally you have to put real values and above the diagonal you put whatever complex values.

When you conjugate, you will get the transpose location. a_{ij} and a_{ji} are conjugates of each other, okay? So this is the self-adjoint operator definition. Hopefully it is clear. You can see, you can easily construct self-adjoint operators. And there are quite a few of them, right? So you still have like half of the entries of the matrix to fill. So you can, there's quite a few self-adjoint operators.

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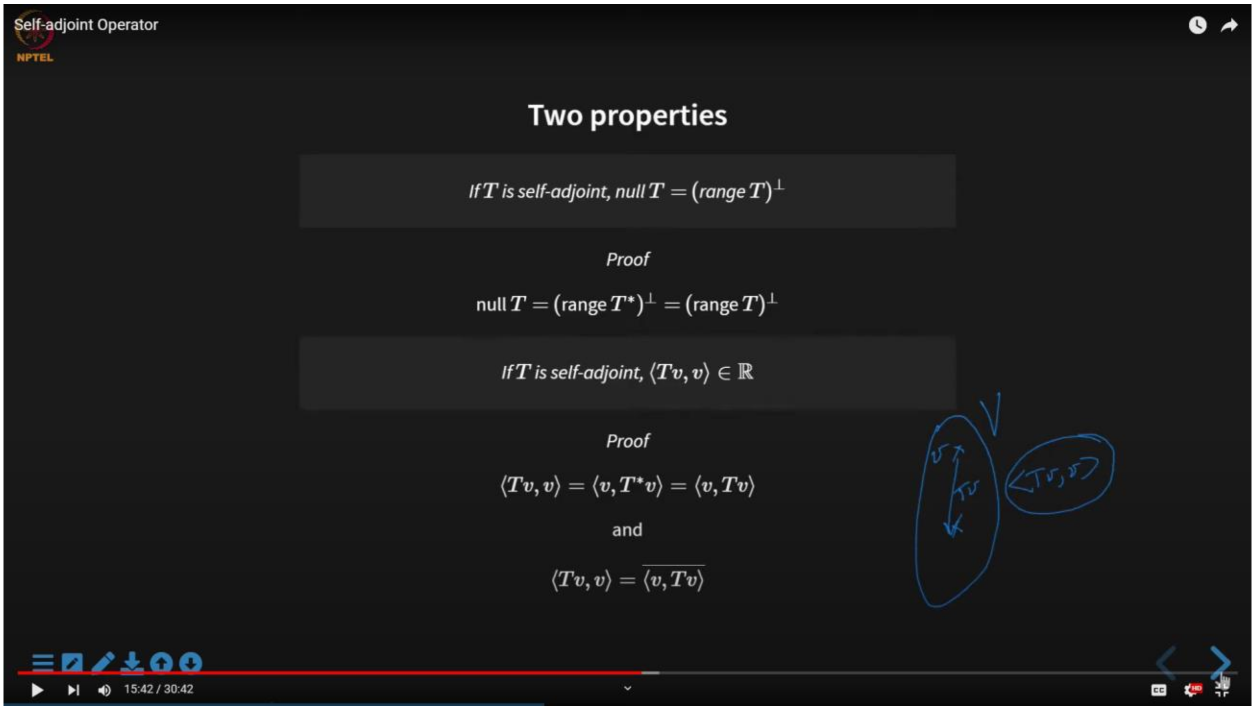

Okay. So a couple of important properties. In the rest of this lecture, we will just keep looking at properties of self-adjoint operators. And you will see eventually, by the end of this week you will see why they are so important, okay? So the first and easy property is: null(T) = (range T)$^\perp$, okay? So if T is self adjoint, so you can see why this is true, right? So null T is always (range T^*)$^\perp$,, and if it's self adjoint $T = T^*$, so your T becomes like this. So now you can see what will happen if you have a self-adjoint operator, right? So you have v and you have range of T and then null of T, and these two are going to be perpendicular to each other, okay? So these are all orthogonal complements of each other. Null T and range T. And that is a good picture to keep in mind for hermitian or, you know, self-adjoint operators. Another very interesting property is $< Tv, v >$, okay? So we saw that, you know, there is some real thing going on. Even in the hermitian symmetric, we saw suddenly the diagonal values have to be real even in a complex space, right? So we saw why that is true. Because the conjugation has to be the same. So this is sort of like an extension of this. So what this says is: if T is self-adjoint, then $< Tv, v >$... So you take v and then look at what happens when you hit it with T. So it goes to Tv. And if you do $< Tv, v >$, the inner product of $< Tv, v >$, that has to be real, okay? You cannot have anything other than real then, okay?

The proof is quite easy. You will see that it works out quite straight forward. See, $< Tv, v >$ has to be $\overline{< v, Tv >}$, right? So the whole thing conjugate, this is the definition of inner product. And here if you look at $< Tv, v >$, it is $< v, T^*v >$, and the case where it's self adjoint is $< v, Tv >$, okay? So $< Tv, v > =< v, Tv >$. And also $\overline{< v, Tv >}$. So $< v, Tv > = \overline{< v, Tv >}$. I am sorry, conjugate. So these two are equal. So these two have to be real as well, right? So this is real and everything is real, so it works out based on this observation. So the fact that, you know, this adjoint takes you from the first argument to the second argument, and when you flip the two things you have conjugacy, they're conjugates of each other, this is just the property that tells you this, okay? So $< Tv, v >$ seems to be something interesting. So in general, this $< Tv, v >$ will play an important role when you look at, when you look at self-adjoint operators. So you have v and it goes to Tv, okay? And this inner product $< Tv, v >$ is something that will play a very very central role in sort of deciding and understanding and characterizing what self-adjoint operators are, okay? So this, it's sort of, you'll see why it sort of determines a lot of things, this inner product $< Tv, v >$, okay? So already we see that it has to be real, okay? The same v you cannot have something which is not real, okay? For a self adjoint operator.

So these are all useful. So for instance, quite often you will have, you will have a, you will have to decide whether an operator is self adjoint or not. These are all easy checks you can do. So if $< Tv, v >$ is not real, then you know it's not a self-adjoint operator, okay? So the next interesting case is $< Tv, v >$ being zero, okay? So forget about real. What about $< Tv, v > = 0$, right? So is that possible? What happens in those cases? What happens when $< Tv, v >$ is real, what about the reverse of this, okay? Supposing $< Tv, v >$ is real for all v, is it true that T is self adjoint? So these are the kind of questions we'll sort of look at. We'll also look at eigenvalues of self-adjoint operators. That is also a very big result, okay? So let us go forward.

Okay. So here is a question on eigenvalues. So we already saw this. Once you say the linear map is, the linear operator is self-adjoint, a lot of things are forced to be real, right? So in the matrix representation, orthonormal basis representation, the diagonal values are real. So we see $< Tv, v >$, the inner product $< Tv, v >$ is real. What about eigenvalues? It turns out for self-adjoint operators, eigenvalues have to be real, okay? So to understand the context, let us just ask a general question. When are eigenvalues real, okay? Now there are two cases here. We've been looking at a case where the field is real and the field is complex. If the field is real and you say v is an eigenvector of T with eigenvalue λ, clearly v has to be real, right? So the eigenvector itself has to be real. So Tv is real, v is real. So λ has to be real, okay? So when you're talking about real spaces and eigenvalues and eigenvectors there, eigenvalues are always real, okay? So there's no problem there. That's not so difficult to look at. But when you look at complex vector spaces in general, eigenvalues can be complex, right? So even if your matrix has only real entries, like, for instance, here is an example. A being $[0, 1; -1, 0]$. The matrix has real entries, okay? But its eigenvalues are complex. Its eigenvectors are complex, okay? Such things can happen, okay? So even with real entries, if you are dealing with complex spaces… In real spaces there will be no eigenvector here, right? So we will look at this soon enough. Why is this example special. This example does some things. So in real, there are no eigenvectors. But when you go to complex, there are eigenvectors. But you know eigenvalue is complex, eigenvector is also complex. So this can happen, okay?

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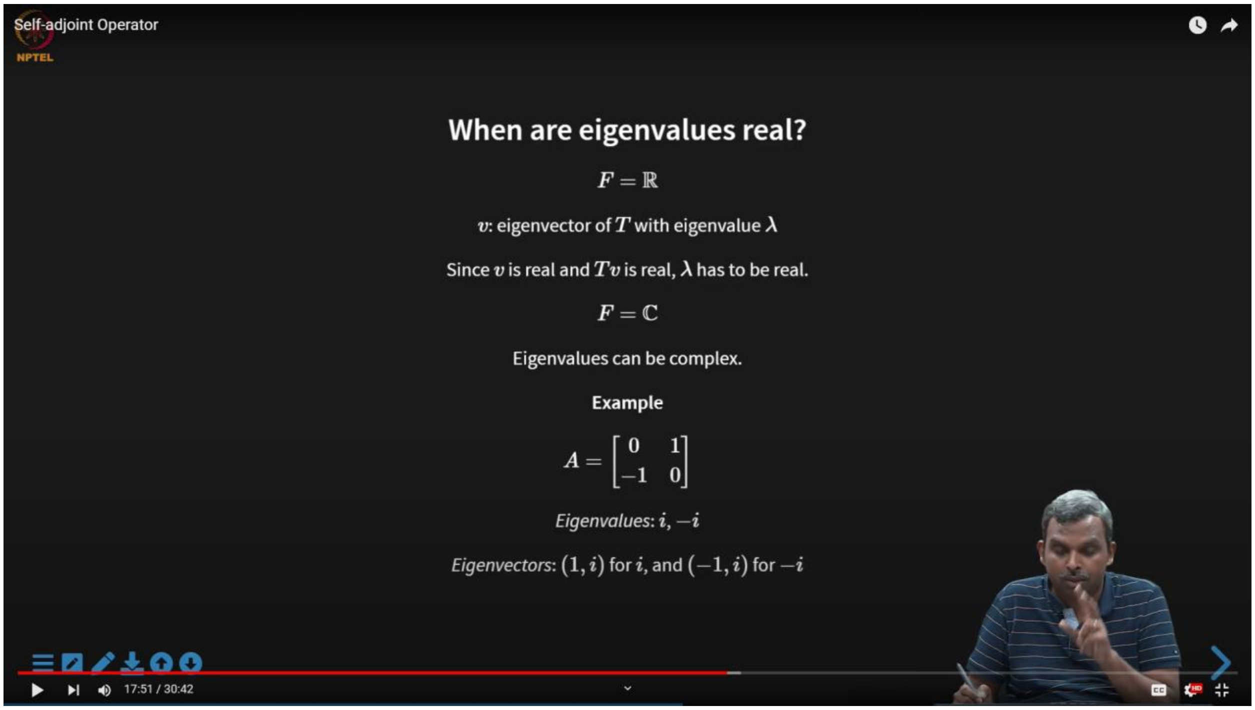

So what about self-adjoint operators? What happens if the operator is self-adjoint? It turns out that cannot happen, okay? If the operator is self-adjoint, then every eigenvalue is real, okay? So if you go back and look at this example, this is not a self-adjoint operator, right? It's not symmetric. 1 and -1, it's sort of like, you know, opposite to symmetry, negative of symmetry. So it's not symmetric. So we saw clearly an example where it was complex. But on the other hand, if the operator is self-adjoint, then every eigenvalue has to be real, okay? So once again it is not very difficult to prove and you will see this $< Tv, v >$, $< v, Tv >$ will play a crucial role. Like we kept saying, this $< Tv, v >$, when you take inner product after being operated on by a self-adjoint operator with itself, that becomes crucial to look at, okay? So that controls a lot of properties.

For instance here, let us say T is self-adjoint and λ is an eigenvalue with eigenvector v. You look at this $< Tv, v >$ being $< v, Tv >$, right? And then you sort of expand, start from the middle and go on both sides. $< v, Tv >$ is $< v, \lambda v >$. And then λ will come out, but λ will come out with the conjugate. On this side, $< Tv, v >$ is again $< \lambda v, v >$ because v is an eigenvector, right? With eigenvalue λ. But this time λ ended up on the first argument. So it will just come out as λ.

$\lambda||v||^2 = \bar{\lambda}||v||^2$. And v is non-zero, right? Any eigenvector is non-zero, okay? By definition. So you can cancel the $||v||^2$. You will get $\lambda = \bar{\lambda}$ okay? So eigenvalues become real. So notice this, the fact that you have self-adjoint makes a lot of things real. Its diagonal entries, $< Tv, v >$, λ and eigenvalues. Eigenvalues become real, okay? So all of these are nice to see. So it looks like this self-adjoint operator is a very interesting class of the general operators, okay?

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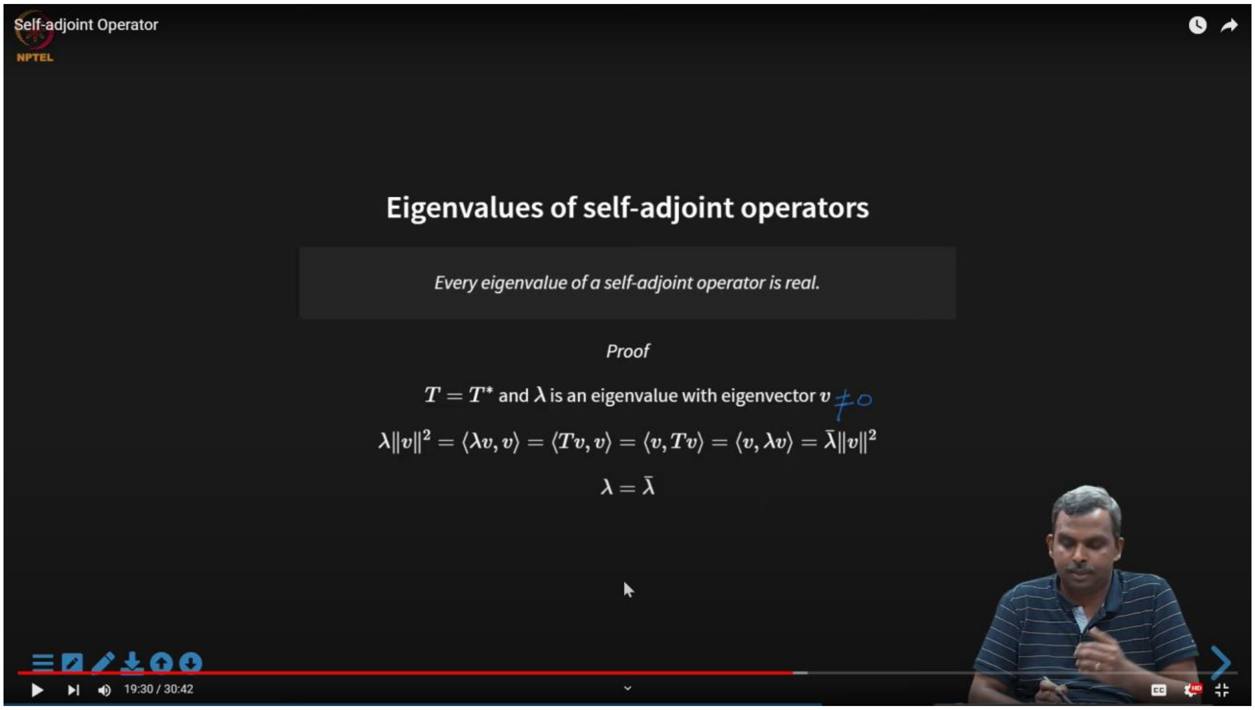

Okay. So the next question to ask is: what about $< Tv, v >$? Can it be orthogonal always? Okay? So it turns out it's possible in the real case. So the example I considered before, $[0, 1; \ -1 \ 0]$ is this interesting case. It's sort of like a 90 degree rotation if you think about it. If you take any real x, you will have $< Ax, x >$ being zero. See what will happen. So if you take $x = (x_1, x_2)$. Ax becomes $(-x_2, x_1)$, okay? And then if you do $< Ax, x >$, you simply get $-x_1 x_2 + x_2 x_1$ and that is zero, okay? So this is what will happen if you have real, okay, x being real is important. Notice when, only when everything is real I can do $x_1 x_2$, $x_2 x_1$. Otherwise I have to do some conjugate and things will go wrong here, okay? So there is no conjugation here, it is all real. So it just works out in this way, okay? So it's possible that you have a linear operator in a real space, so that $< Ax, x >= 0 \ \forall \ x$, okay? $< Tv, v > = 0$ for all v in a real space. It’s possible. It’s sort of like a rotation by 90 degrees, right? So think of, in a real space, you can do rotation by 90 degrees, it’s possible, okay? So we saw this. So what about complex inner product spaces, okay? So if you take the same example… Same example A and you now look at, allow for complex vectors, you can see that if you set x equals $(1, i)$, Ax you see is $(i, -1)$. And this inner product I have shown here will become $i \times 1 + -1 \times \bar{i}$. So this $\bar{1}$ is 1, $\bar{i}$ is $-i$, so you will end up getting

$2i$, okay? And that is non-zero, right? Okay. So the same matrix as before, in a real space gives you $< Ax, x >$ is zero for all $x \in \mathbb{R}$. But if you allow for $x \in \mathbb{C}$, you end up getting something non-zero, okay? So there exists something complex for which $< Ax, x > \neq 0$, okay?

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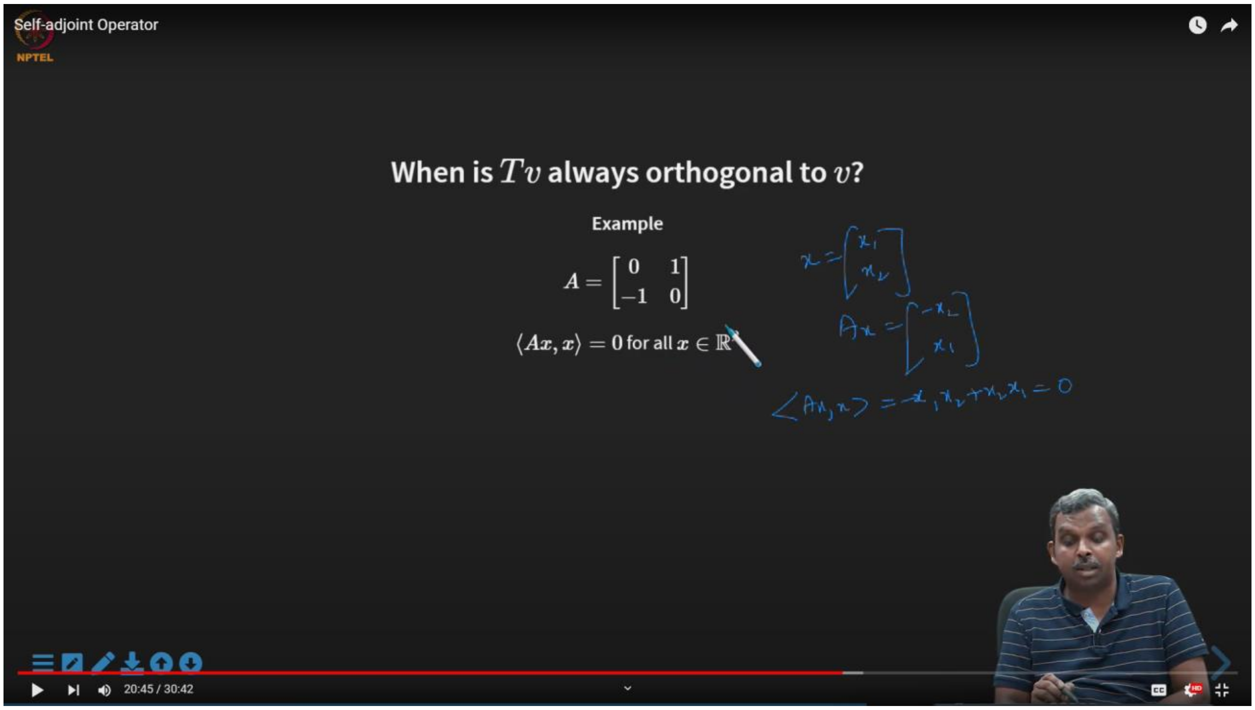

So what's going on in complex inner product spaces? And here is the interesting result, okay? There's lots of interesting things you can say in complex inner product spaces, particularly when you look at $< Tv, v >$. $< Tv, v >$ is sort of enough to fix all the inner products in complex inner product spaces. So let's say we have inner product space over complexes. If $< Tv, v > = 0 \ \forall \ v$, then T is zero, okay? So in a complex inner product space, you cannot have these 90 degree rotations. Seems strange, okay? So you cannot have these 90 degree rotations which make the orthogonal property come true. Right? So $< Tv, v > = \ 0 \ \forall \ v$. Then T is 0. Of course the converse of this is true, right? If T is 0, then $< Tv, v >$ is always zero. So $< Tv, v >$ being zero is enough to check whether a complex linear map is zero, okay? But in the real case it is not true, right? So $< Tv, v >$ is not good enough for you. But, so $< Tv, v >$ represents whether or not the complex, the inner product space, that this linear operator is zero or not, okay? So that is something nice to know. So why is this true? So here is this interesting relationship. So this relationship, it's sort of, I mean, if you, once you look at it, it's obvious. But it's not clear why it should be true. So notice why, I mean, it's sort of clear why it should be true. But you may not be able to come up with it that easily on your own if you haven't seen it before. So it turns out $< Tu, w >$, that inner product you can write as a linear combination of four other inner products. And what is special about all these four inner products? What is special about all these four inner products? They are all of the form $< Tv, v >$ for some v, okay? On the left hand side you have a more general case. $< Tu, w >$, what seemingly is a more general case. But it ends up being a linear combination of just $< Tv, v >$ for different v's, okay? The first one v is $(u + w)$, the next one v is $(u - w)$. $(u + iw)$ and $(u - iw)$, okay? And the linear combination involves some i and all that, okay? So if $< Tv, v > = 0 \; \forall \; v$, then $< Tu, w > = 0 \; \forall \; u, w$, okay? So that's interesting, isn't it? If you, so this is, this happens only in the complex space. This i is very crucial. If you do not have the i, you cannot make it happen in the real space, right? So we already know the counter example is true, right? So it cannot be happening, okay? So $< Tu, w >$ is zero. So just by saying $< Tv, v >$ is zero for all v, you're actually forcing $< Tu, w >$ to be zero for all u's, w's, okay? This is really a property of this complex number field, complex field, I'm sorry. So that is why this is very important, okay?

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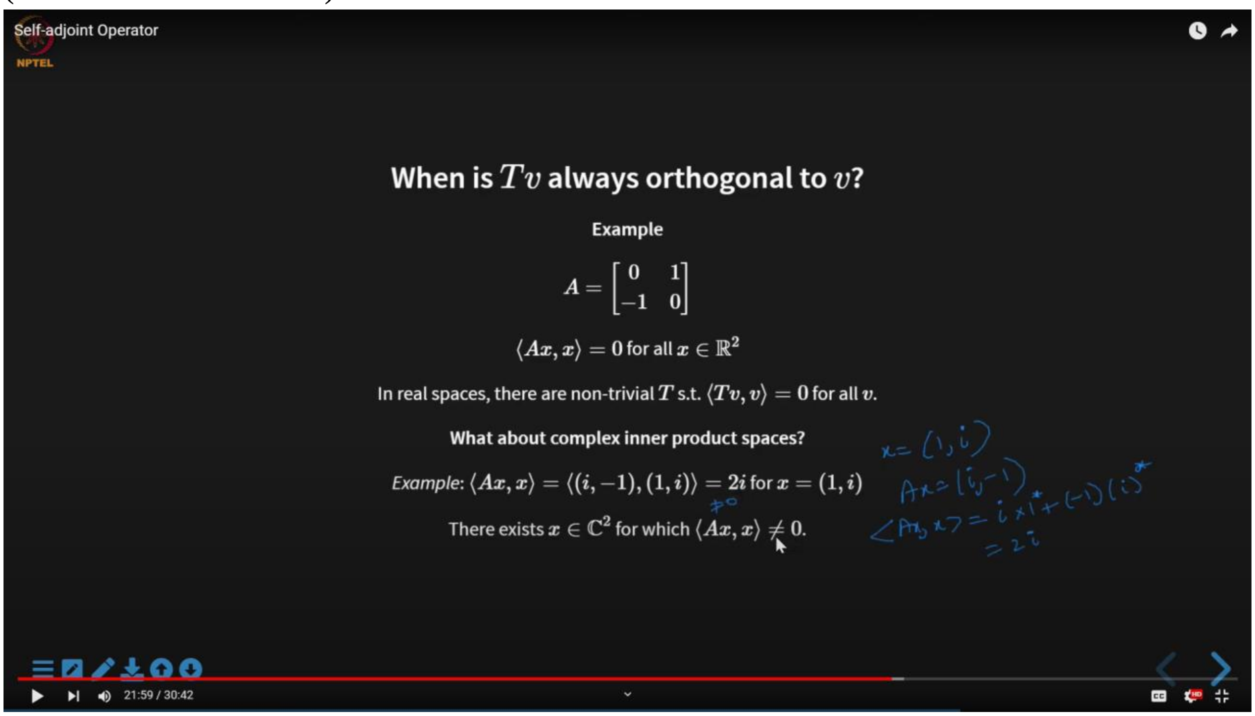

So now you can set w to be Tu, okay? This is true for any w, so you set w to be Tu. So you get $Tu = 0 \; \forall \; u$. $||Tu|| = 0 \; \forall \; u$, which means T itself is 0, okay? So that's the idea, okay? So quickly one can see this result. And notice how interesting it is. So lots of interesting things come together to make this happen. So this, when you have any operator in an inner product space, complex inner product space, there is no self-adjoint going on here, right? So this is just any linear operator in a complex inner product space. And if you look at $< Tv, v >$ and if that is zero for all v, then that's enough to make T zero, okay? But in the real inner product space, this is not true, okay? So some interesting contrast between what happens here. Okay. So now if you force the self-adjoint property and look at $< Tv, v >$, a lot more interesting things happen, okay?

So we saw before that if T is self-adjoint, $< Tv, v >$ is real. So that was an easy thing to prove for us. It turns out the opposite is also true. The converse is also true, not opposite, the converse is also true, okay? What is the converse? If $< Tv, v > \in \mathbb{R}\ \forall v$, then T is self-adjoint also. So that's an interesting result. And the proof is sort of similar to what we did before. You look at $< Tv, v > - \overline{< Tv, v >}$), okay? So notice when you say $\overline{< Tv, v >}$, that becomes $< v, Tv >$ isn't it? So this is just the definition of inner product. $\overline{< Tv, v >}$ is $< v, Tv >$, okay? And now what do I know about $< v, Tv >$, okay? $< v, Tv >$ is the same as $< T^* v, v >$, okay? And then you have $< Tv, v > - < T^* v, v >$, and that is $< (T - T^*)v, v >$, okay? So this is a general result for any T, right? Nothing to do with self-adjoint or something. For any T this is true, okay? So $< Tv, v > - \overline{< Tv, v >}$) is $< (T - T^*)v, v >$, okay? So this is a general result. Now this is enough for us to prove what we want, okay? So if $< Tv, v > \in \mathbb{R}$, then the left hand side is zero. Which means $T = T^*$, right? Easy enough to see. If $T = T^*$, the self-adjoint case which we knew before. If T is T^*, then clearly $< Tv, v >$ is real, okay? So both ways this is true. So that's the first result in this, okay? So you see with this T being self-adjoint and $< Tv, v >$ being real, all of these are very, very true. So these self-adjoint operators have so many simplifying properties and all of this will sort of come together with these kinds of simple proofs. I mean, there's nothing major going on here. But this notion of adjoint and the operator, the linear inner product being the same when you conjugate, then all that is playing the key role here.

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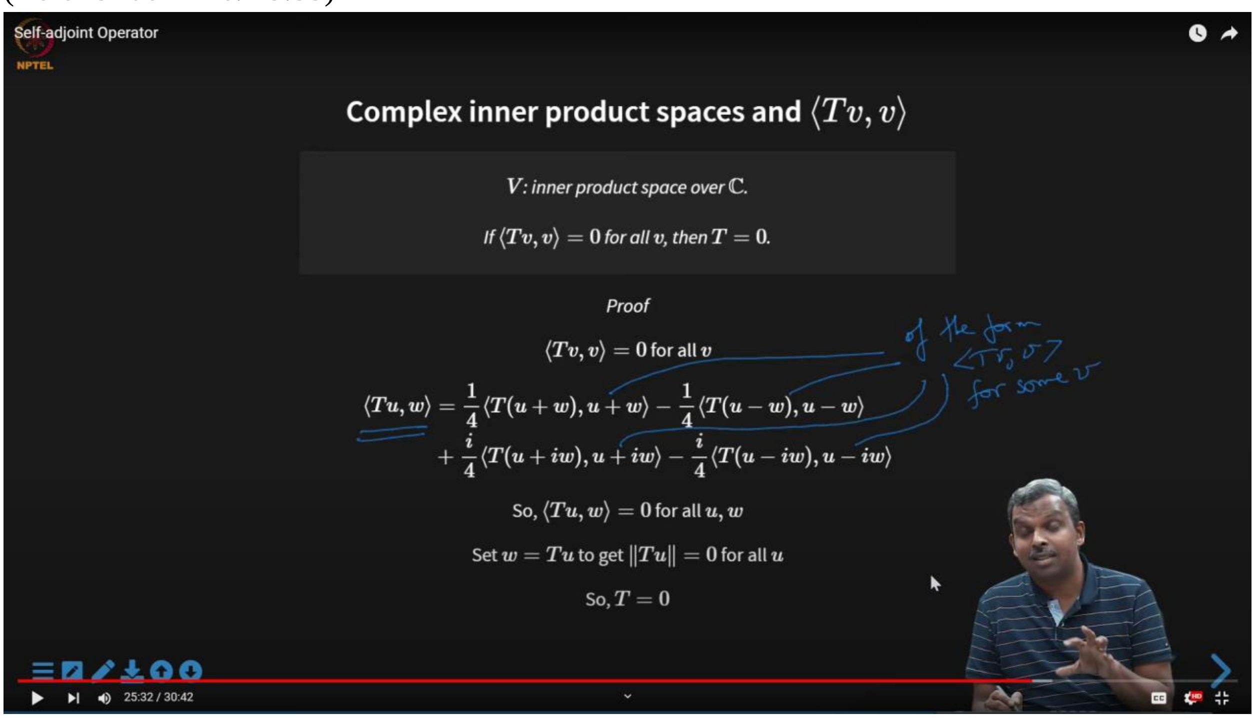

Okay. So there's one more result and this result is also interesting. So we saw that in a complex inner product space, if $< Tv, v >= 0\ \forall\ v$, then T is 0. Now it turns out if T is self-adjoint and $<$

$Tv, v >= 0 \; \forall \; v$, then T has to be zero. Doesn't matter whether you are in real space or complex space, okay? So notice once again, contrast this with the result. If you don't put in the self-adjoint condition and only say $< Tv, v >$ is 0, and the inner product space is real, you cannot say T is 0 because there is the 90 degree rotation that makes $< Tv, v \geq 0$, okay? Even when T is non-zero in real inner product space. In complex inner product spaces we know that can never happen.

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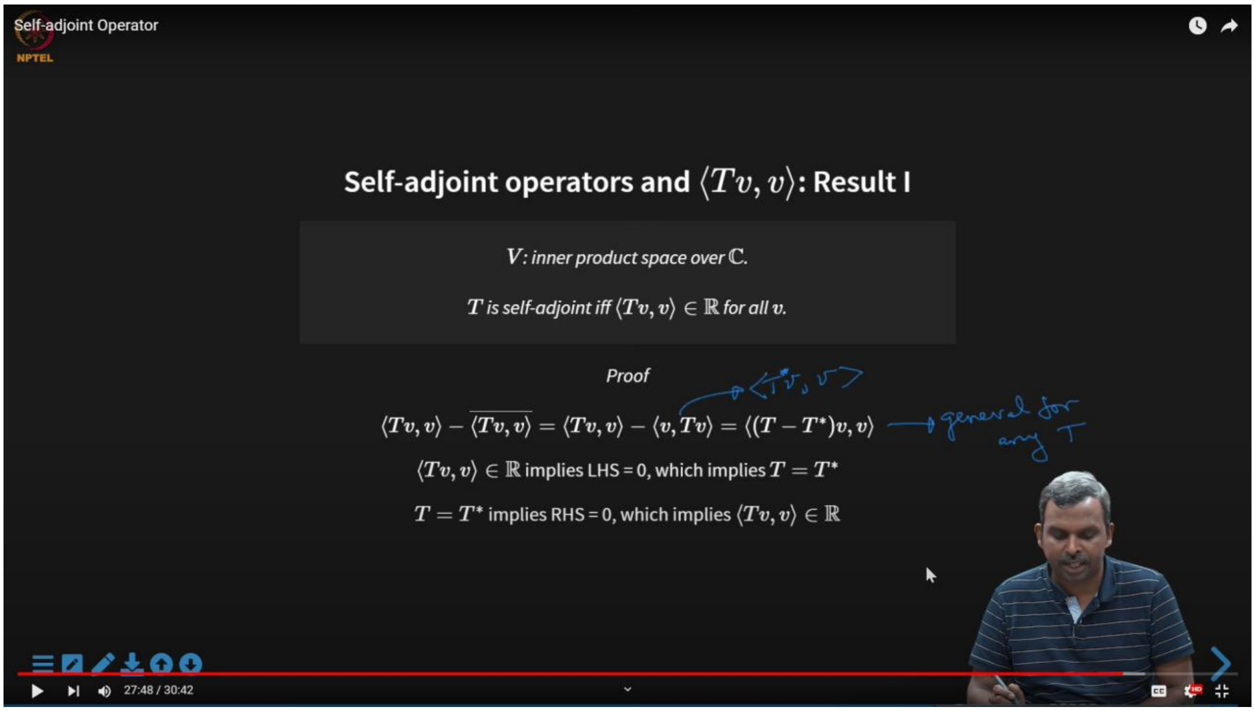

Now if T is self-adjoint, even if you are in a real vector space, if you have $< Tv, v >$ being zero, then T has to be zero, okay? So that's a nice property in real vector spaces, that self-adjoint gives you, okay? Proof is not very hard. Over $\mathbb{C}$, the result is true, right? So there is nothing much to prove. Over $\mathbb{R}$, it turns out when T is self-adjoint, you can have this kind of a linear combination picture, okay? So this is true only when T is self-adjoint. If T is not self-adjoint, this will not be true. You can simplify and check this. Even the previous result I did not prove for you, I just gave that result to you and asked you to check. You can just use the properties and check this. So here if you additionally use T is self-adjoint, you will get this combination, okay? So once again you see here that these two are of the form $< Tv, v >$, okay? And this is true even in real, okay? And that happens because T is self-adjoint okay? So $< Tu, w >$ is zero. And then you said w is Tu, and you will get $||Tu|| = 0$, and then that implies $T = 0$, okay?

So that concludes this lecture. So hopefully the definition, and it is quite a simple definition in terms of matrices, in terms of operator. And you see it has a lot of very interesting sub-results going particularly with respect to what happens with $< Tv, v >$, that inner product. And what happens with range of T, null of T and all that with a self adjoint operator. So there's lots of nice simplifying characterizations. Later on towards the end of this week, we will see something called spectral theorems which really simplify the description of self-adjoint operators. We will see it in more detail at that time. But this is the starting point, okay? Thank you very much.

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