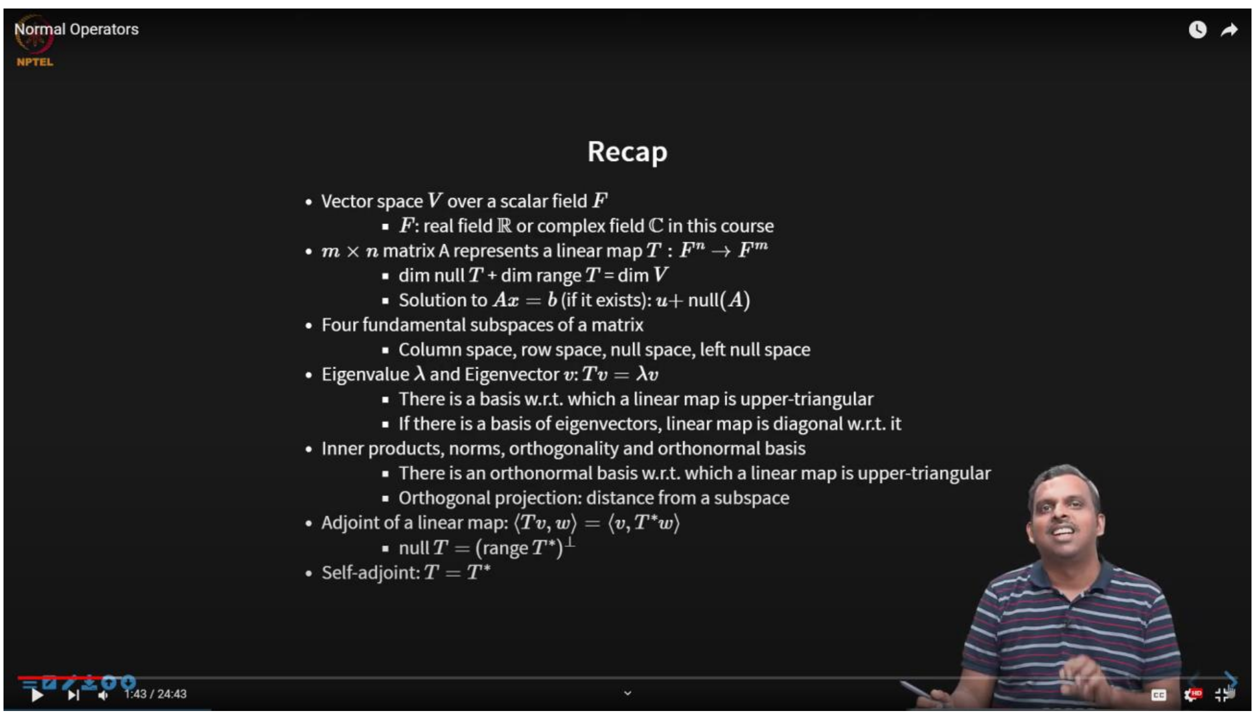
Applied Linear Algebra
Prof. Andrew Thangaraj
Department of Electrical Engineering
Indian Institute of Technology, Madras

Week 10
Normal Operators

Hello and welcome to this lecture. The main topic of this lecture is normal operators. The word normal has several meanings in regular life. You think of normal as something that is routine, nothing extraordinary or different. But these normal operators are actually quite special operators. So in spite of that, they're called normal for some other reason. You'll see why the name comes about. It's related to the norm as opposed to other things. But these operators are very crucial. They have, they sort of generalize self-adjoint operators which we studied in the previous lecture. They generalize it to a very important class and these class of operators have some very special properties. So anything we prove for normal operators also holds for self-adjoint operators. So in that sense, normal operators are a very useful generalization. It's interesting as well, okay? So let us get into it.

(Refer Slide Time: 01:43)

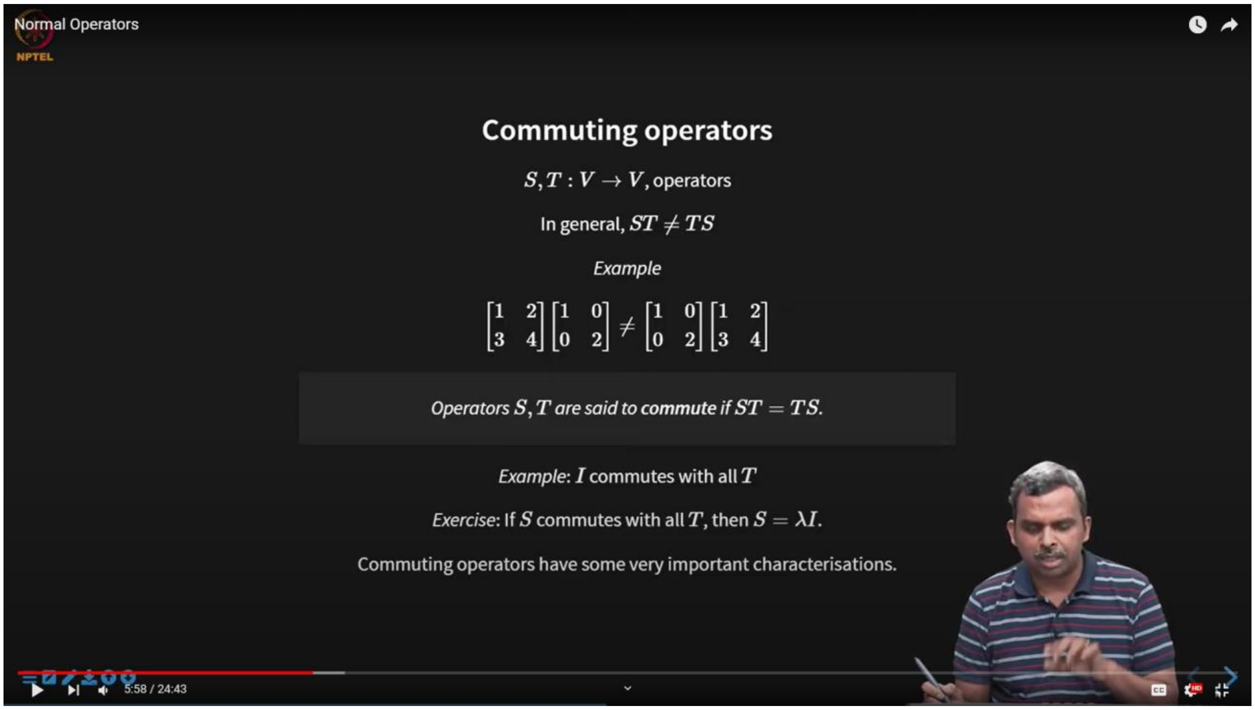


Okay. So a quick recap of the, in the previous lecture we've been looking at the adjoint and properties of adjoint and adjoint-operator product and all those things in the last few lectures. Specifically, in the previous lecture we looked at self-adjoint operators, operators which are equal to their adjoint, okay? So and then that it's… That gives us, gave us a lot of interesting properties. So we're going to sort of push that property a little bit further, generalize it, include more operators than self-adjoint operators. Operator maybe need not be equal to its adjoint but should be close enough in some sense. And is that good enough for us to derive some interesting properties? So that's what's, that's what we're going to do in this lecture. And that useful extension is what's called normal operators, okay?

So before that, let us look at this notion of commuting operators, okay? So if you have two operators S and T from the same vector space V to V, these are two operators, and in general ST need not be equal to TS, right? So this is something to keep in mind. And there are very easy examples that one can come up with where ST will not be equal to TS, right? So this is a property. If, particularly if you keep matrices in mind, you can bring up a lot of examples, okay? So here's a very simple example with diagonal matrices, right? So normally you think of diagonal matrices as being a very simple operation. But here's an example where, you know, the first matrix is not diagonal. $[1\ 2;\ 3\ 4]$. The other matrix is diagonal. $[1\ 0;\ 0\ 2]$. And the 2 in the diagonal matrix makes a lot of difference. And if you put it to the other side, it doesn't commute. You can do the multiplication and check for instance. The $(2,1)$ entry, right, the bottom left entry in the left hand side will end up being 1, right? And on the right hand side, the bottom left entry will end up being… Sorry, the bottom left entry in the left hand side will be 3, the bottom left entry in the right hand side will be 6, okay? So these two are not equal. So the matrices don't commute. And so what happens when two operators actually commute, okay? So you expect something special to happen and that's always something interesting. So this word commute is used in this fashion here. So normally when you think of "commute", in normal usage, it's going from one place to the other and coming back, right? So sort of like that. So this is sort of the notion we use for defining the notion of commuting with operators.

So two operators ST commute if $ST = TS$, right? So, you know, that's the definition. So there are lots of examples of commuting operators, and the simplest and easiest example is when one of the operators is the identity operator. If you have I as being one of the operators, clearly it commutes with any other operator. Doesn't matter whether you do identity before an operator or after an operator, nothing really changes. So it commutes, okay? And you know other non-trivial examples one can come up with, but it's not… I mean, we will look at them as we go along, but this one is a very simple example. So quite often if you want to show some examples with commuting operators, you can use I and T and check if that works out for that case. That's useful to have, a good example, okay? So in fact there is a very interesting sort of converse to this. I commutes with all T. If you have some other operator, somebody comes up and says "hey, I have an operator S, but this commutes with all other operators T", supposing they say that, then it turns out that that S has to be equal to λI. So this exercise is a useful result. You can keep that in mind as well. So there's no other operator which commutes with everything else. So it's not very easy to find these kinds of commuting operators. But they're quite plentiful also. So you'll see later on there are interesting examples we can come up with for commuting operators, but this is a good condition to look at.

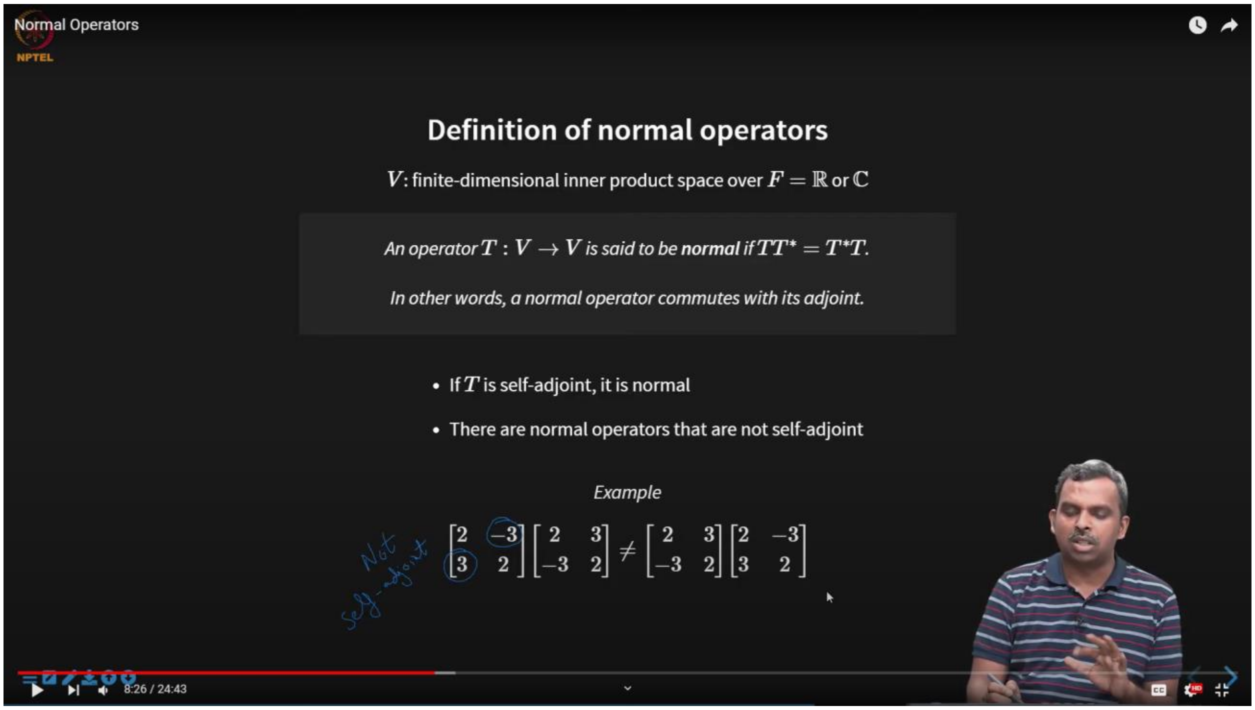
(Refer Slide Time: 05:58)



Now I just want to quickly tell you that, you know, I mean, only for operators this commuting and all sort of makes sense. If you have S and T being, you know, linear maps, general linear maps, $ST = TS$ immediately, so that's only possible when you're talking about operators. You can't do this kind of a property with a linear map even though ST and TS may both be defined. Only when they are operators, you can think of equality and all that. Otherwise it doesn't make sense, okay? So there's lots of interesting properties for commuting operators. When two operators commute, a lot of things happen in common for them and you can sort of maybe expect it, but still some non-trivial things are common to commuting operators and that's something that will come up later on when we study more, okay? So, but this is an important theme and this theme of commuting operators will quite often enter when we study normal operators and various other operators as well.

Okay. So here is the definition of normal operators, okay? So you are in a finite dimensional inner product space over the reals or complexes. And if you have an operator T, it is said to be normal if $TT^* = T^*T$, okay? The operator products, the composition of the operators TT^* should be equal to T^*T. So in other words, we just saw in the previous slide, the normal, a normal operator commutes with its adjoint, okay? So you can very quickly see a couple of very easy properties. One is: if T is self-adjoint, if T becomes equal to T^*, then clearly it is also normal, okay? So because, you know, $TT^* = T^*T$. If $T = T^*$, you're simply saying $T^2 = T^2$, which is always true. So definitely it's normal. There's no problem there. So self-adjoint implies normal, okay? And normal is actually a strict generalization of self-adjoint. What do I mean by strict generalization? There are normal operators which will not be self-adjoint, okay? So if both were the same, then there is really no point in defining, you know, more general normal operators. But it's not the same. So self-adjoint is normal. So self-adjoint is contained inside normal strictly. There are normal operators which are not self-adjoint, okay? So one can come up with a very quick example.

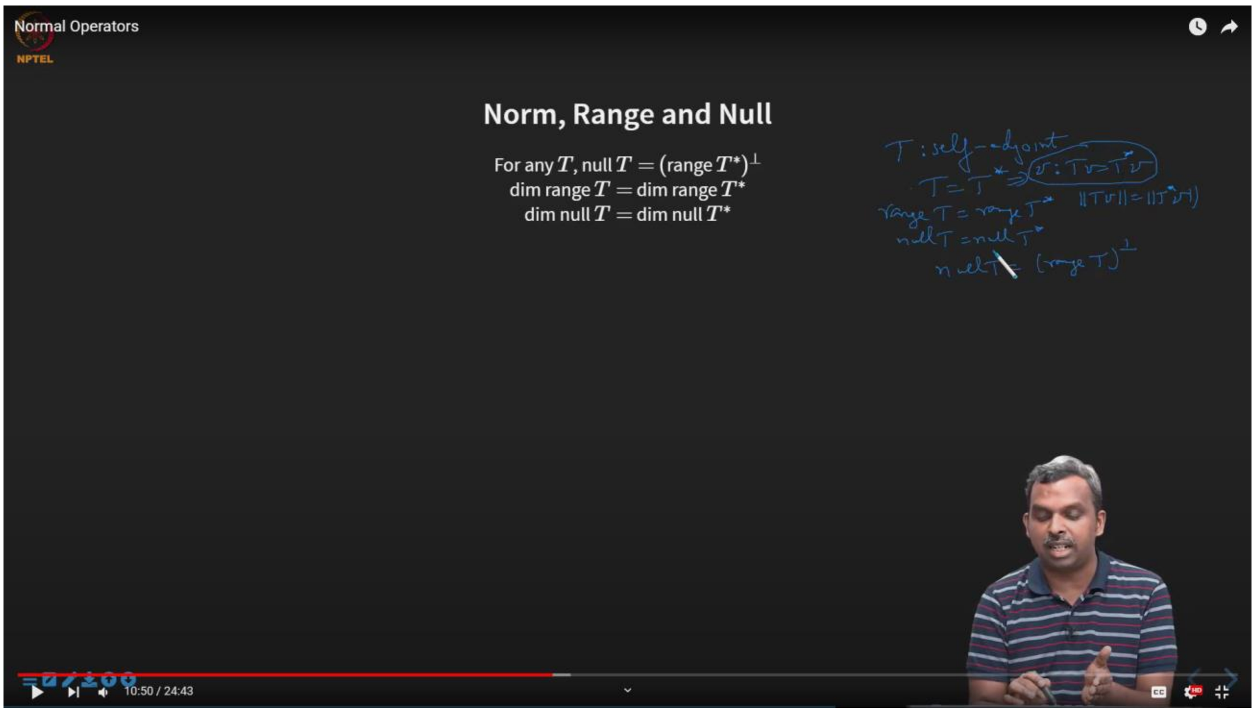
(Refer Slide Time: 08:28)



Here is an example. This is in your book as well. $[2\ -3;\ 3\ 2]$. Clearly it is not self-adjoint, right? So you can see it's not self-adjoint, it's not very difficult to see that these two are not the same. So it's not self-adjoint, okay? But it is clearly normal. You can check with the product, with the transpose either on the right or the left you get the same answer, okay? So this ends up being an example, if you want, of a normal operator which is not self-adjoint. So normal operators are a strict generalization, okay? So in this lecture, we will prove some properties for normal operators. But remember, every single property I prove for normal operators also holds for self-adjoint operators, because self-adjoint operators are normal. But there can be some properties for self-adjoint operators which may not hold for normal operators, okay? So that is something that we will see, okay? So keep that in mind. I will keep pointing out this tension between these two, and what holds, what doesn't hold and all that we'll see as we go along, okay? So these are normal operators.

Okay. So let's first begin with norm, range and null, right? So if you have a general operator T, you have these relationships that null T = (range T^*)$^{\perp}$ and dim range T = dim range T^*. dim null T = dim null T^*, okay? You may want to keep the self adjoint case in mind. Suppose T is self-adjoint, okay? What all do we know is true? Well, self adjoint is $T = T^*$, okay? So not only is the dimension equal, the range itself has to be equal, right? So range of T equals range of T^*, okay? All of these will be true, okay? So not just this. So $T = T^*$ implies what? For v, for any v, $Tv = T^*v$, okay? So it's exactly equal. So whether you hit v with T or T^*, you get the same answer, right? So $Tv = T^*v$ for v, okay? So all these things become true.

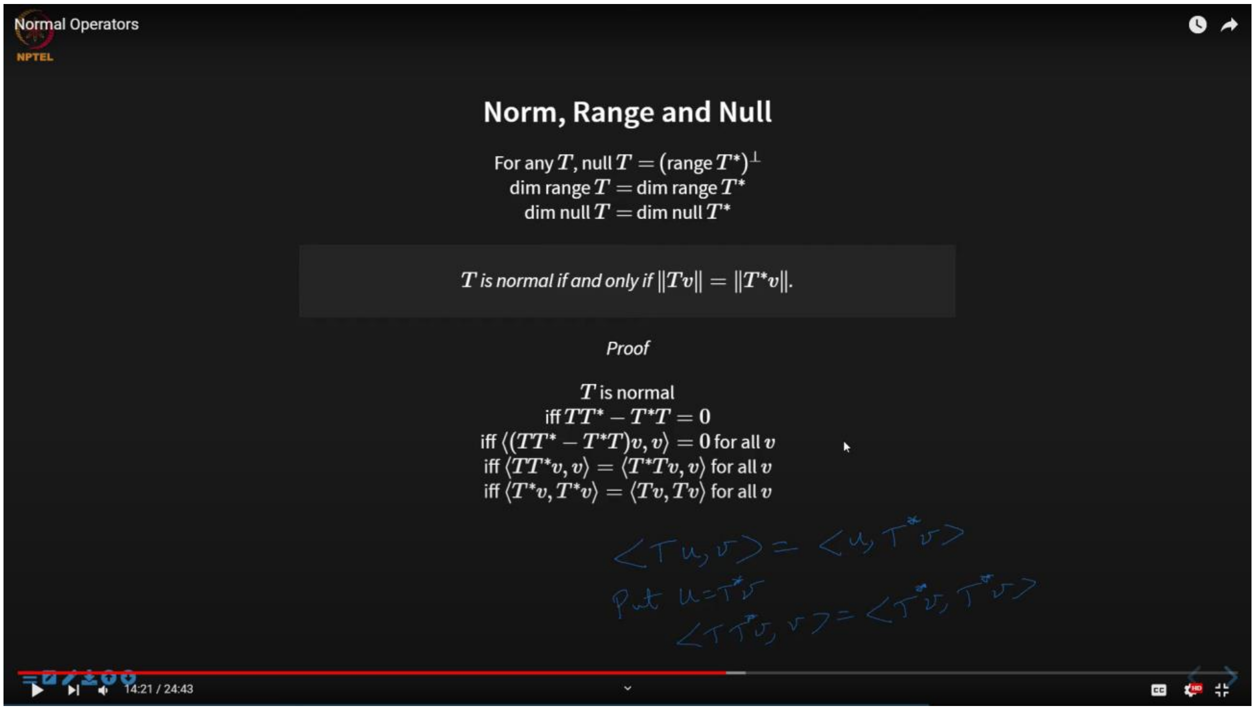
(Refer Slide Time: 10:52)



But this is the most important relationship when you have self-adjoint. Whether you hit it with T or T^*, you get the same answer, right? So T and T^* are identical, okay? For a general operator T, even these are not true, okay? Range of T need not be equal to range of T^*. So only the dimension has to be equal, okay? But this null T = (range T^*)$^{\perp}$ is always true. But for self-adjoint operators, you have this because of this relationship. null T = (range T^*)$^{\perp}$, okay? Null and range are, you know, orthogonal complements. That's true for self adjoint operators. So keep this in mind. So all this is true for self-adjoint operators while for a general operator you can only say these things. So as you specialize more and more, you will get more and more strict conditions, okay?

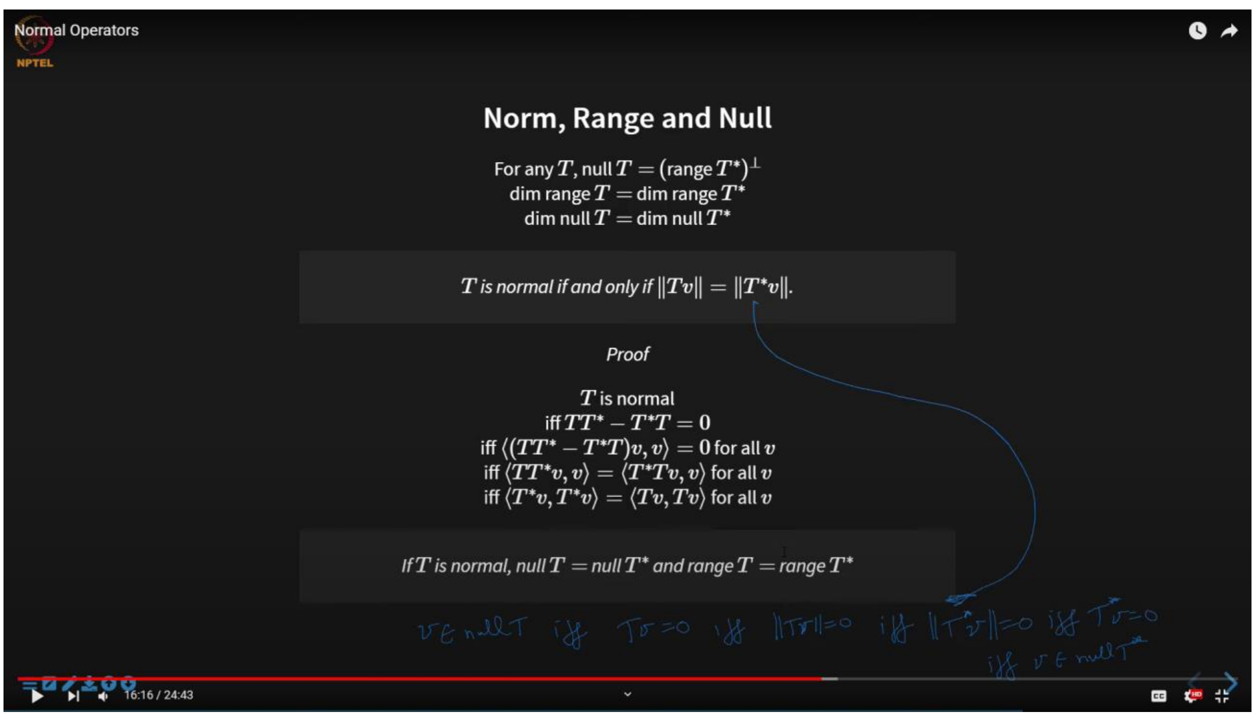
So remember, for any v, $Tv = T^*v$. So, so many more things are true. So for instance, $||Tv|| = ||T^*v||$. All of these will be true for a self-adjoint operator because fundamentally, you know, it's identical, the same. Both operators are the same. T and T^*. So all these other things are also becoming the same, okay? But we will now see, when you generalize to normal operators what are true, what are not true, right? So we will see that. Of course what holds for any operator will hold for normal operators also. What more from the self-adjoint world holds, okay? Do any of these things hold is what we are going to see next, okay? So a lot of interesting things will happen here, okay? So the first thing you can show is in fact a very strong if and only if connection, okay? T is normal if and only if $||Tv|| = ||T^*v||$, okay? So clearly, you know, Tv will not be equal to T^*v. If Tv is equal to T^*v for all v, then that's self adjoint, right? So that is not where we are. But for T being normal, the $||Tv|| = ||T^*v||$. So maybe, perhaps this is the reason why they call it normal. I don't know, I'm just conjecturing here. There's nothing usual or routine about these kinds of operators which are so special, that make $||Tv|| = ||T^*v||$. So this is a very important relationship for normal operators. This is a very crucial relationship. So normal operators, $||Tv|| = ||T^*v||$, okay? So whether you hit it with the operator or its adjoint, you get the same norm, okay? So that's what it means.

(Refer Slide Time: 14:21)



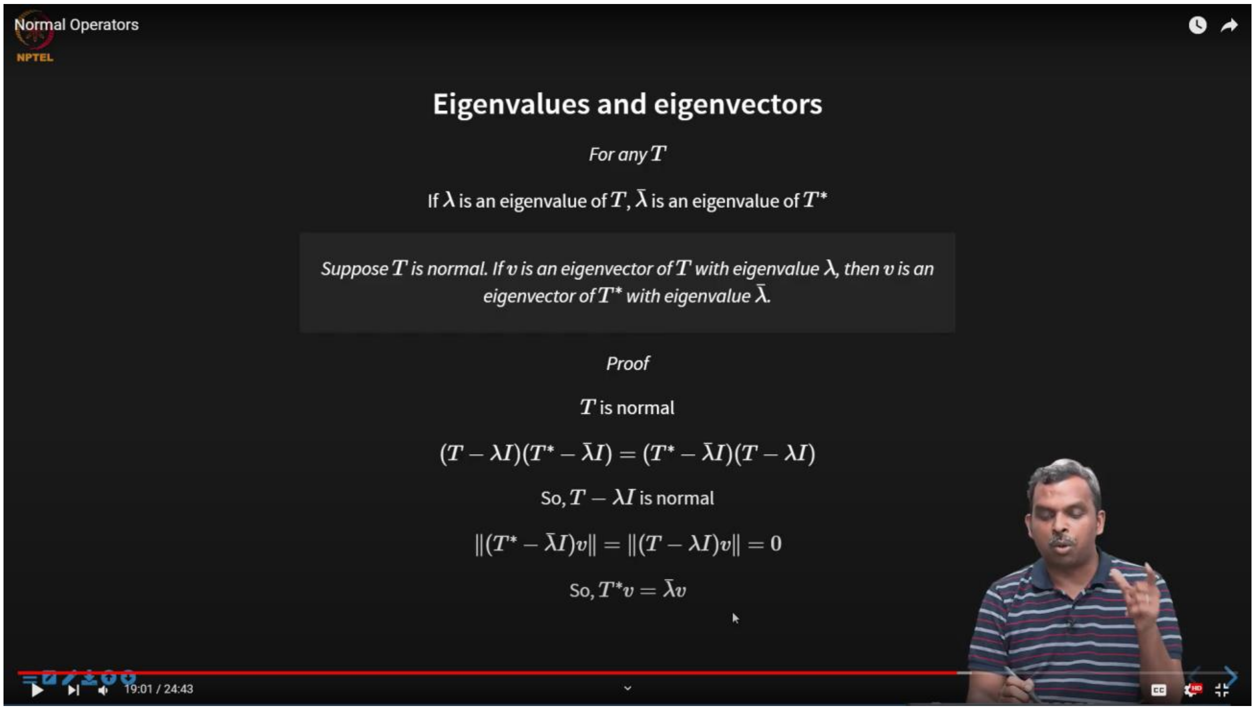
So the proof is actually quite interesting and you can see the inner product and all that will enter the picture here. And I'll just quickly go through the proof. This is the same proof which is in your book as well. So T is normal if and only if this is true, right? So this is just the definition $TT^* = T^*T$. I have brought it to this side. Now an operator being 0 is also equivalent if and only if to that operator operating on v, inner product with v being 0 for all v. This is an if and only if, right? So operators are like this. This is a condition that is true. Only the zero operator satisfies this property for all v, okay? Now you expand this out. Use the linearity on the left hand side. So you get $< TT^*v, v > = < T^*Tv, v >$. And then you use the properties of the adjoint, right? So T operating on $< T^*v, v >$, that you can write the same as T^*v... Bring this T to the other operator, that would go T^*, right? So you can see this relationship. Maybe I should write this down a little bit more clearly. See, $< Tu, v > = < u, T^*v >$, isn't it? You simply put $u = T^*v$, put $u = T^*v$ and you will get this relationship. What do you get? $< TT^*v, v > = < T^*v, T^*v >$, okay? So it's just the same thing. So this, one of these things, when they move to the other side, you get the adjointing automatically, right? So same thing you use here. $< T^*Tv, v >$ is the same as $< Tv, Tv >$. And that is what you want, right? So this is $||Tv||$. Well, $\left||T^*v|\right|^2 = \left||Tv|\right|^2$. But, you know, norm is the positive square root. So $||Tv|| = ||T^*v||$, okay? So a very quick and easy proof for a, quite a non-trivial sounding result for normal operator T. So I had previously written down a bunch of properties for self-adjoint operators. Well, for normal operator Tv is not equal to T^*v. We saw the example before, right? There is an operator which is normal, but not self-adjoint. [2 3; -3 2], right? So for that, clearly $Tv \neq T^*v$. It's an easy example you can come up with. But $||Tv|| = ||T^*v||$, you can't change the norm whether you hit it with T or T^*. So that property is true, okay? Now once this becomes true, it might seem slightly unobvious to you, but once this becomes true, then $null\ T$ will be $null\ T^*$. And because null T is null T^*, range T is also range T^*, okay?

(Refer Slide Time: 16:16)



So why is null T = null T^*, okay? So this is not very hard to prove. So supposing if $v \in$ null T. This is if and only if $Tv = 0$, which is if and only if $||Tv|| = 0$, which is if and only if $||T^*v||$, okay, equals 0. So, which is if and only if $T^*v = 0$ and if and only if $v \in$ null T^*, okay? So this T being normal, this relationship is used here, okay? And you can see why this has to be true, okay? So the proof is quite easy. So once you have $||Tv|| = ||T^*v||$, then the null is the same, range is the same, okay? So you can see that the norm, range and null and all these things for the normal operator are slightly more, you know, strict than for a general operator. For a general operator, clearly the norm need not be equal. The null spaces need not be equal, range spaces need not be equal. Only the dimensions have to be equal. But for a normal operator, the norm has to be the same, the null spaces have to be the same, range spaces have to be the same. But of course Tv need not be equal to T^*v, that's true for self-adjoint operator, okay? So that's the difference. So sort of like normal operators are a mild sort of generalization of self-adjoint. You can think of it that way. I mean, they do generalize in different ways. But not way, way generalized, okay? So this is a useful property to remember.

(Refer Slide Time: 19:01)



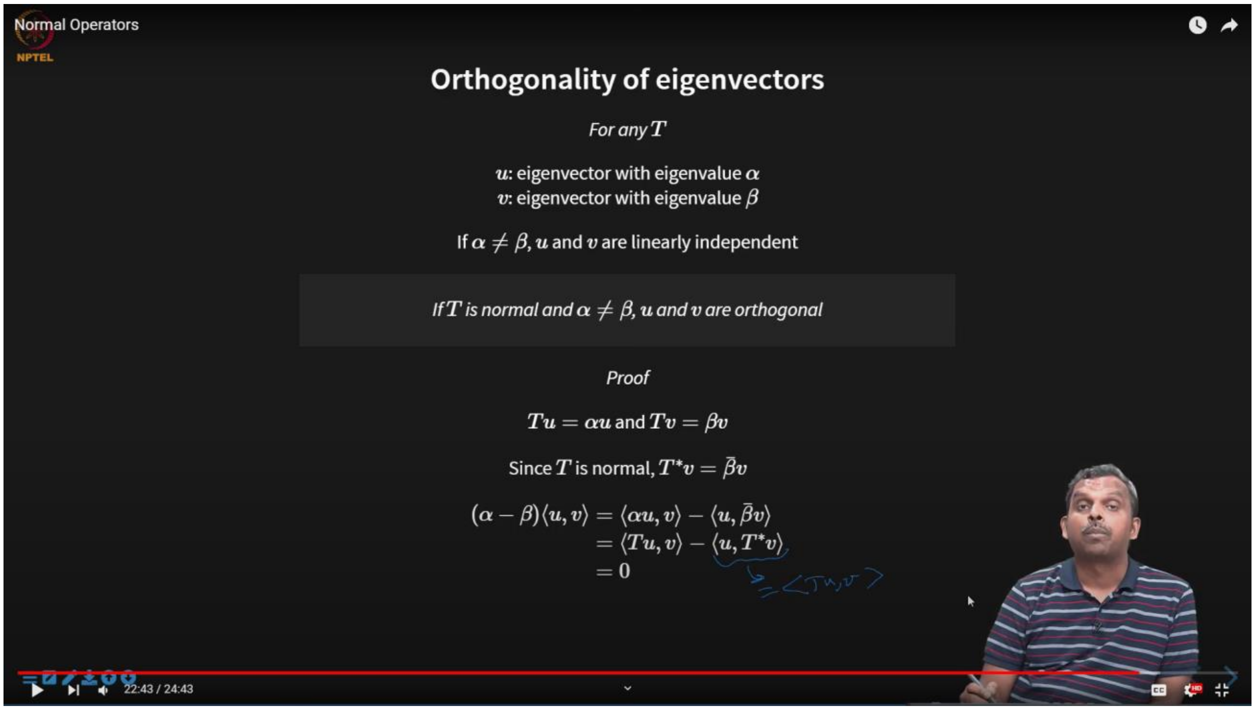
The next is eigenvalues, eigenvectors, right? So what do we know about, for a general operator T, if λ is an eigenvalue of T, then $\bar{\lambda}$ is an eigenvalue of T^*, right? So this is a property we know for general T. Now of course for self adjoint operators, T is equal to T^*, so all eigenvalues, eigenvectors of T are the same, eigenvalues and eigenvectors for T^*, T and T^* are identical if self-adjoint. But in general for an operator, any operator T, we already showed that if T has an eigenvalue λ, T^* will have a corresponding eigenvalue $\bar{\lambda}$. But we don't know anything about the eigenvectors, right? So we know that the eigenvalues will have this property, but we don't know anything about eigenvectors. It turns out for normal operators, you can say something about eigenvector of T^* as well, okay? So this is an interesting result for normal operator. Suppose T is normal. If you have an eigenvector for T with eigenvalue λ, the same v is an eigenvector for T^*, but with the eigenvalue $\bar{\lambda}$, okay? So the eigenvalue gets conjugated, the eigenvector remains the same. So T and T^* share the same eigenspace, the eigenvector space is the same. The eigenvalues become conjugates, but that's okay. But that's also not too bad, like, λ, $\bar{\lambda}$, what's the big difference, it's sort of similar in some sense, okay? So this is a strong connection. So in terms of eigenvectors also, there are strong connections. Of course, if T is self-adjoint, everything is the same, right? Same eigenvector, same eigenvalue. So this is identical. But here it's a mild generalization. So eigenvectors remain the same but λ becomes $\bar{\lambda}$, okay? So you see this commutation really makes things very similar for these operators. It can't go off and be all the way different, okay?

So proof is actually very simple. So if T is normal, you can check that this product is true, right? So because TT^* is the same as T^*T and all the other terms are exactly the same in this product. So clearly $(T - \lambda I)$ is also normal, okay? So this is a crucial property. If T is normal, $(T - \lambda I)$ is also normal. Once $(T - \lambda I)$ becomes normal, you use the normal property of normal, which is the $||(T^* - \lambda I)v|| = ||(T - \lambda I)v||$ and that is both equal to 0, right? So clearly, you know, $(T - \lambda I)v$ is 0 because λ is an eigenvalue of T with eigenvector v. So clearly $T^*v = \bar{\lambda}v$. So $\bar{\lambda}$ is an eigenvalue for T^* with the same eigenvector v, okay? So proofs are all very simple, very elegantly building up. There's no major technical wizardry in the proof, but you can see how important the idea of this normal operator commuting with this adjoint makes a mild generalization of self-adjoint and gives you a lot of interesting connections, okay? So this is a very important property for eigenvalues and eigenvectors.

Okay. Finally, one of the most important and interesting properties for normal operators is orthogonality of eigenvectors, okay? So what have we seen for a general operator T? If you have two different eigenvectors corresponding to two different eigenvalues, that's what's important, if you have two eigenvectors, each corresponding to a different eigenvalue, okay, one is eigenvalue α, another is eigenvalue β, and α and β are not the same, $\alpha \neq \beta$. Supposing that is the situation. We know already that u and v are linearly independent, okay? $\alpha \neq \beta$ means u and v are linearly independent, okay? That we know for an arbitrary operator T, okay? Notice what happens for a normal operator, okay? If T is normal and $\alpha \neq \beta$, okay? Then it turns out u and v, the eigenvectors corresponding to the eigenvalues α and β have to be orthogonal, okay? So that's a strong and very simplifying, interesting property, right? The two eigenvectors corresponding to two different eigenvalues are going to be orthogonal, okay? So this ends up putting a lot of simple structure on normal operators, self-adjoint operators. Remember this says T is normal. Of course if T is self-adjoint also this holds because, you know, self-adjoint is a subset of normal. So if T is self-adjoint also this holds, okay?

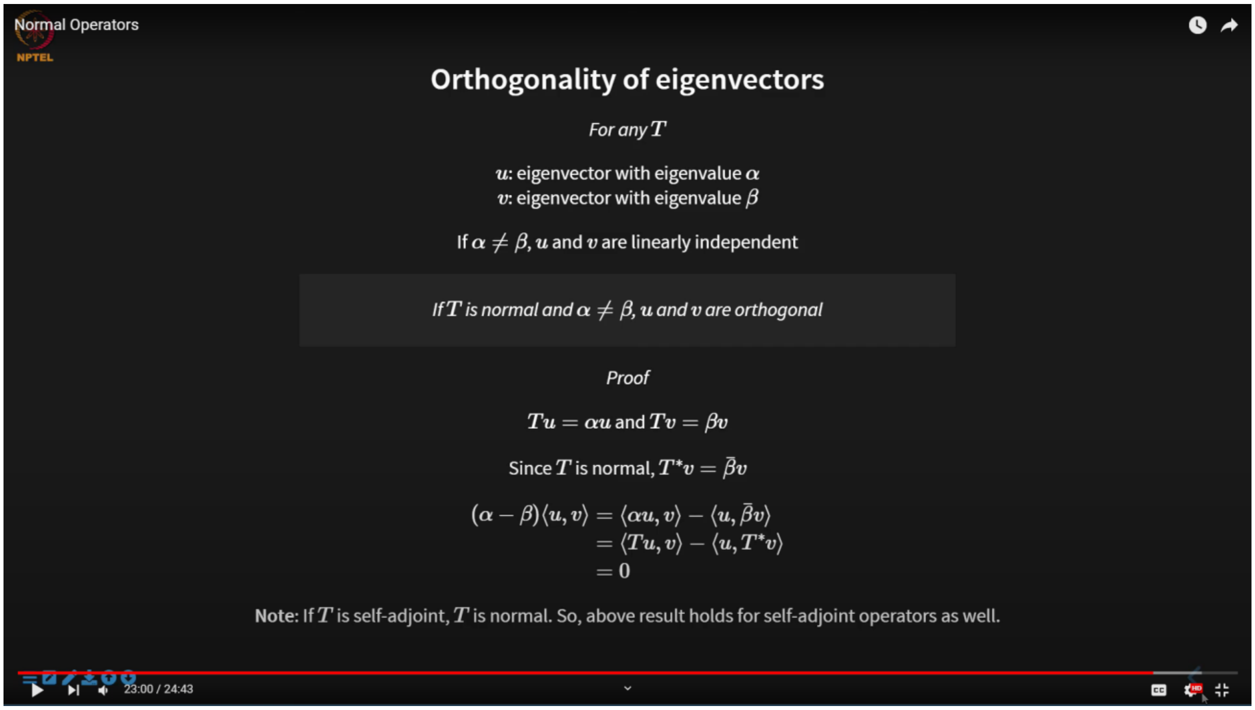
So this is a very, very, very interesting example. You can come up with operators which are, I think there are no operators which are not normal for which this need not be true. So clearly this is sort of like the nice example here, okay? So proof is not very, again, once again, the proofs in all these things are very simple, very elegant and easy to write down. You do not have to, I mean, even the number of lines is very, very limited, okay?

(Refer Slide Time: 22:43)



So $Tu = \alpha u$, $Tv = \beta v$, right? So that's what it means to say u is an eigenvalue v is an… u is an eigenvector with eigenvalue λ. v is an eigenvector with eigenvalue β. Since T is normal, T^*v is $\bar{\beta}v$, okay? So this is important. Of course, T^*u is also $\bar{\alpha}u$, but in this case we're going to use the $\bar{\beta}$ so that's why we put it here. So T^*v is $\bar{\beta}v$. If T is not normal, you can't say this. Of course if T is self-adjoint, you can say more, but T normal is enough, this $\bar{\beta}$ is enough, okay? So notice what happens here. So we will consider this $(\alpha - \beta) < u, v >$, inner product $< u, v >$, okay? So since α is not equal to β, the left hand side is not zero, isn't it? In general, I mean, I'm sorry, left hand side, not left hand side, $(\alpha - \beta)$ is not zero, $\alpha \neq \beta$. So $\alpha - \beta \neq 0$. Remember that. $< u, v >$ could be 0, right? So we want to show $< u, v >$ is 0, but that's where it starts. So you can write this as 2 terms. $\alpha < u, v > -\beta < u, v >$. I'll push the α into the first term, first vector, I'll push the β into the second vector. So when I push it into the first one, I get $< \alpha u, v >$. When I push the β into the second one, I get a $< u, \bar{\beta}v >$, right? So this is what I get. So now what is αu? Tu. What is $\bar{\beta}v$? T^*v. Right? Now what is $< u, T^*v >$? So at this point we are almost done. So this one is, what is this? This is equal to $< Tu, v >$. And it cancels, you get zero, okay? So that's a very quick and easy proof for why eigenvectors corresponding to different eigenvalues for a normal operator are orthogonal, okay? So of course, if T is self-adjoint, this also holds, okay? So this is the last and final property I am going to show for normal operators, okay?

(Refer Slide Time: 23:00)



So we saw that this normal operator is a very useful and simple generalization of self-adjoint. And most of the properties that are interesting for us hold. In particular for normal operators, Tv and T^*v have the same norm, and that gives you the same null space, same range space for T and T^*, right? The spaces coincide even though Tv equals, not equal to T^*v, right? So it's not identical, not self-adjoint. But a lot of other properties are the same. So for instance, the eigenvectors remain the same for T and T^* for normal operators. The most crucial property which is very important is two eigenvectors corresponding to two different eigenvalues, just because the eigenvalues are different, they have to be orthogonal. So just imagine, You have a normal operator. If you have a normal operator, if it has two different invariant subspaces, right, one dimensional invariant subspaces, they have to be orthogonal, right, if the two, if they correspond to two different eigenvalues, okay. So that's a very interesting and powerful result. And we saw how orthogonality simplifies a lot of calculations, right? So that's why, our normal operators, self-adjoint operators are very easy to understand and express in linear algebra, okay? So I'll stop here. And the next two lectures are very crucial. We will use all that we have studied about self-adjoint, adjoint and normal operators so far and put together two results which are called spectral theorems. So these are very important theorems. They really nail down what a self-adjoint operator can be, what a normal operator can be in very clear terms, in very simple terms using all these properties. So we will close this week with those two spectral theorems: the complex spectral theorem and the real spectral theorem as your book calls it, okay? Thank you very much.