Applied Linear Algebra
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Week 01
Subspaces, Linear Dependence and Independence

So we've seen so far linear combinations and notion of span. The next very, very important notion in a vector space is that of a subspace, okay? What is a subspace? Suppose somebody gives you a very big vector space and you're a little worried about whether you'll be able to understand something so big or not. You can look at portions of it, smaller pieces of it which are already vector spaces of their own, okay? So if you have such subspaces, then studying vector spaces becomes really, really easy, okay? So that is the idea behind looking at subspaces. Smaller subsets of the vector space which are vector spaces by themselves, okay? So that is what this definition means.

A subset of V, U, okay, is called a subspace if U itself is closed under vector addition and scalar multiplication. So if U itself is like a vector space or a vector space actually, right? So U itself is a vector space over $\mathbb{F}$ and then you say it is a vector subspace, okay? So you might wonder why you study subspaces. You'll see later on there are really powerful theorems that are very useful and all of them rely on these subspaces, and a lot of ideas surrounding them okay? So subspaces are very, very important to understand the vector space, okay?

So I have given here examples for subspaces, okay? Remember - a subspace needs to be a vector space by itself. So let's look at $\mathbb{R}_2$ and $\mathbb{R}_3$ for instance, right? So in $\mathbb{R}^2$ and $\mathbb{R}^3$, which is, you know, the plane and the, just the real plane and the real vector space $\mathbb{R}^3$, right? Lines and planes through the origin are all subspaces, okay? So let us take a look at how that would be, okay? So if you look at $\mathbb{R}^2$ which I'll have to once again draw... Let me draw the plane axis. I'll draw the next one here, okay? Okay, so this is $\mathbb{R}^2$. Any line through the origin, okay? So let us make the origin a little big here, okay? So any line through origin is a subspace, okay?

So this naturally begs the question - what about lines that don't pass through the origin, okay? So let us say you take a line like this - this is also a straight line, okay? So very interesting question is whether this is a subspace, okay? So this is also a subset right? A line in 2D plane is also a subset of the vector space. A question you can ask is - is this a subspace, okay? So it turns out this is not a subspace, okay? A very, very easy check for a subspace is - see, a subspace should be a vector space by itself, which means what? It should definitely have the zero vector, right? A vector space should have the identity element. So if the subspace does not have the origin, there is no chance that it can be a subspace, right? If any subset does not have the origin in it, there is no chance that it will be a subspace, okay? So that is one very easy way to eliminate lines that don't pass through the origin. They will not be a subspace. You can also check. You can take any line that is not passing through the origin, take two points on that line and then add them together. You will get a point that's not on the line, it will be outside, okay?

So it's most easy to illustrate with a very simple example. Supposing I take this very simple example of a line, you know, that's parallel to the y-axis, right? Okay, so if I take a point here this would be, let’s say 1, 2, 3, 4... (4, 0) and then I take another point here. This would be maybe 6... I am sorry, this is (4, 1), okay? And this would be (6, 1). If I add these two what happens? (4, 1) plus (6, 1) I get (10, 2) okay? Where is that? That is somewhere here. Okay, so this is (10, 2) and clearly this is not on the line, right? Okay, so you can clearly see that lines that do not pass through the origin will not be subspaces, okay?

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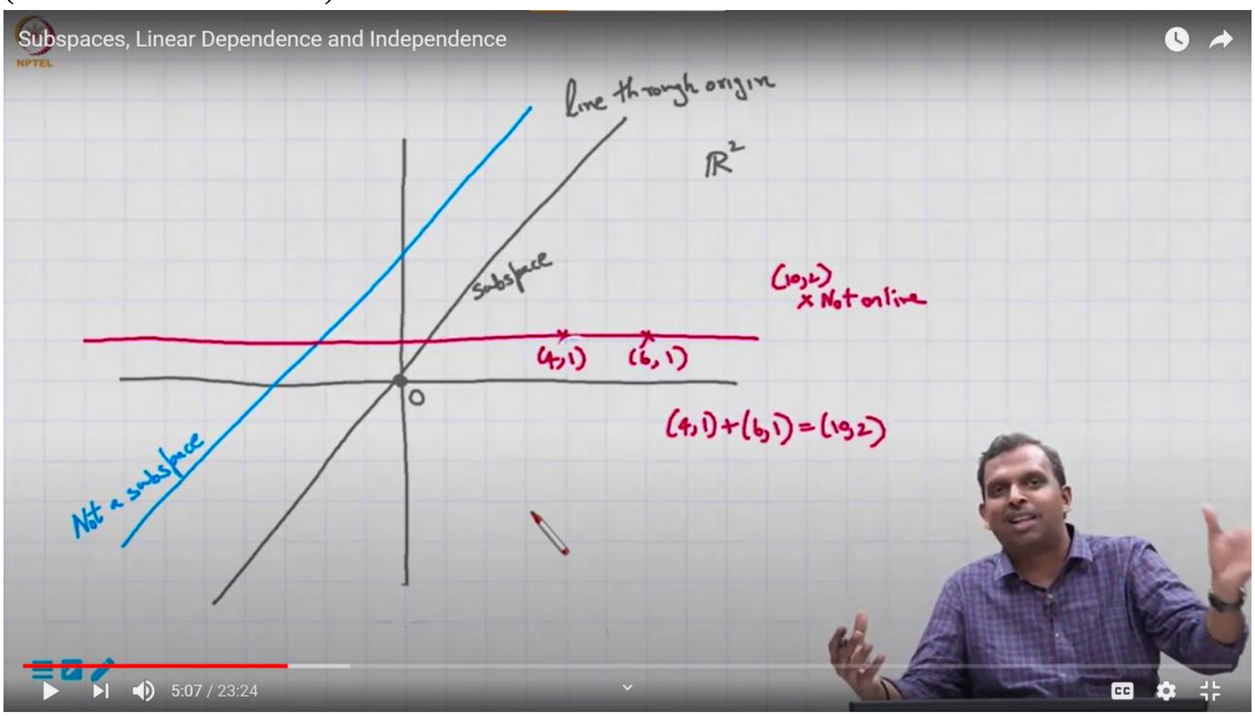

Same thing is true with $\mathbb{R}^3$. You can think of $\mathbb{R}^3$ as the three-dimensional space here and then if you have planes that go through the origin you have subspaces. But... And lines that go through the origin you have subspaces. But if a line does not go through the origin, plane does not go through the origin, or any other surface, you know, sphere or cylinder or something else, cone etc. they are all not subspaces okay? So you can quickly see why that would be not true, okay?

So that gave you a picture of, you know, visualizing how a subspace would be. And here are a couple of more definitions which I like. This is also a way in which people typically describe subspaces. Look at this definition, okay? So you can check that this is a subspace. You can take two points of the form, you know, (x_1, x_1, y_1) and then you add it to, you know... You do linear combinations. a times this, b times x_2, y_2... I'm sorry it's (x_2, x_2, y_2), you will get, you can check that this would be $(ax_1 + bx_2, ax_1 + bx_2, y_1 + y_2)$ right? This also belongs to the same space, it has that form. So the first x... (x, x, y), what does that mean? The first two coordinates are identical, the third coordinate can be anything else. That is a subset, that's the way I am defining my subset of $\mathbb{F}^3$ and that ends up being a subspace because the linear combination is closed, right? It's within that, so it becomes a subspace. You can check that, that's very interesting.

So you can think of what that will be in $\mathbb{F}^3$ right? So if you think of the $\mathbb{F}^3$, three dimensional space, what will be (x, x, y)? If you think of the 2D plane, right? x-y plane, right? (x, x) is the line, 45-degree line, through the origin and then y or the z coordinate can be anything, right? So it's a plane like that, okay? It's a vertical plane and it cuts the x-y plane at the 45-degree line, okay? So that's how you can visualize if you like. Of course, if you have a much larger dimension vector space it's difficult to visualize it like this but here you can do that, okay?

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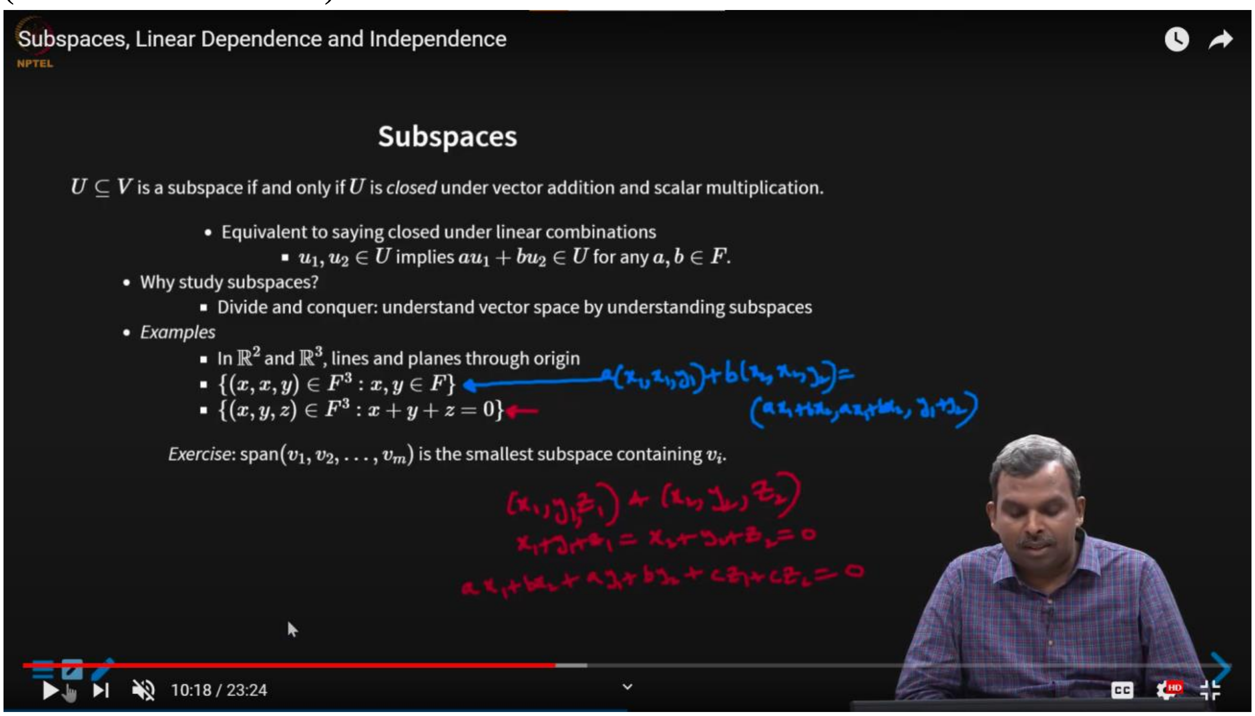

So look at the next one. The next one is also very interesting. So you have this x, y, z such that x + y + z equals 0 okay? Again if you take two points (x_1, y_1, z_1)... So let me write right here. So for this one, (x_1, y_1, z_1) and (x_2, y_2, z_2). Remember $x_1 + y_1 + z_1$ would have been equal to 0 and $x_2 + y_2 + z_2$ is equal to 0, right? So $x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = 0$, I know that that's true, okay? So then what will happen if I do $ax_1 + bx_2$, right? So you see clearly that $ax_1 + bx_2 + ay_1 + by_2 + cz_1 + cz_2$, this is also zero, okay… So if you have, if you take linear combinations (ax_1, ay_1, az_1) plus (bx_2, by_2, bz_2) so that vector also belongs to the same subspace, so it becomes a subspace. So you can create subspace using these kind of linear relationships between coordinates, right? So that's also possible, something to keep in mind so that you can get subspaces in different forms.

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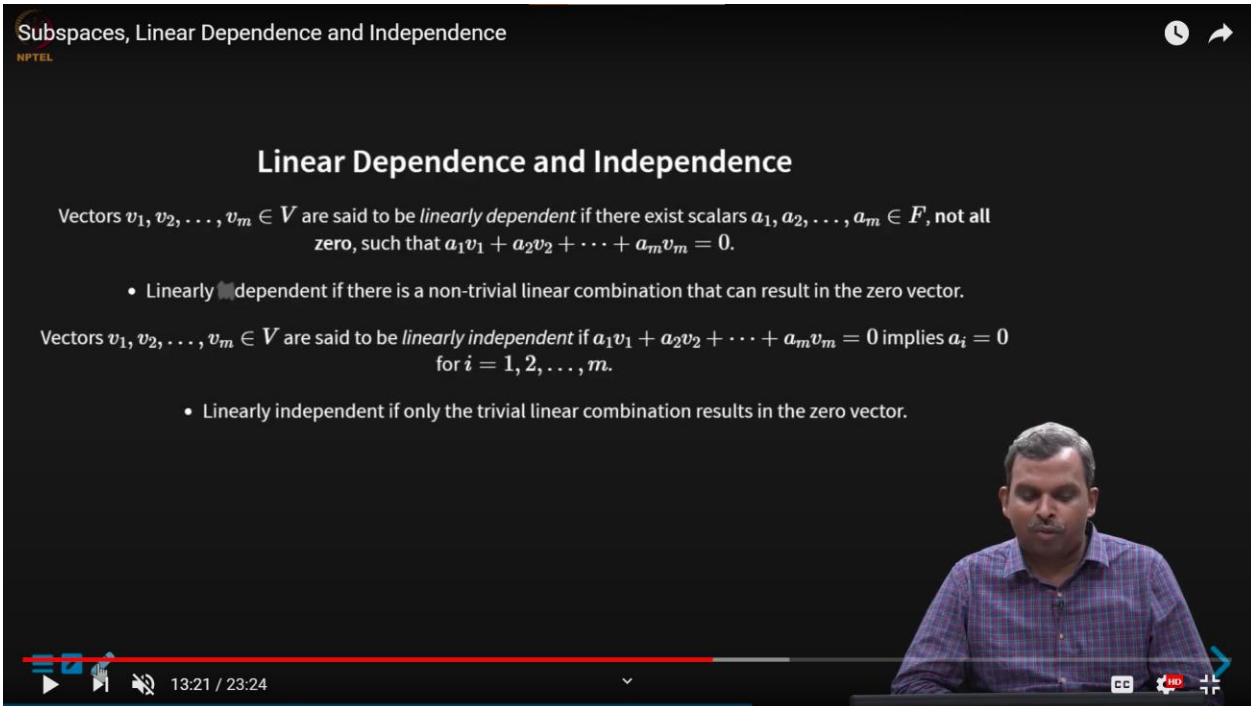

Now how will (x, y, z) such that $x + y + z = 0$ look on the, you know, the three dimensional plane? It should be a plane through the origin, right? So if you think of z equals zero on the x-y plane, you have to have x + y equals zero, so that will be the line. So y is minus x, okay? So you will have this line like, you know, minus 45 degree line right? So that's the line that will be. And then how will you push it to larger and larger z? Think about it, so once you have x and y fixed, so you can think of that line, and then as you vary z you will simply get, for a particular z on that plane, it has to be x + y equal to that z, right? So it will be a line which will keep changing. x plus y equal to minus z. So you will get a different solution sort of a line, and from there you can get a plane. That will picture it for you okay? So think about how that would look, it's a bit of an interesting thing. So it's this... various ways to do these things it will be, it will not be vertical or anything. It will be sort of at one angle... What angle it will be, you can think of all that and imagine how the subspace would look. But it will definitely be a plane that passes through the origin, that's for sure okay? So think about why that's true, okay?

So in general here's an exercise that I've given you. You can try to prove it, you can show that the span is basically the smallest subspace that contains the entire, all the vectors, okay? So if somebody gives you a set of vectors and asks you the question - what is the smallest subspace, what do I mean by smallest? How do you define size of a subspace? So subspaces, if you say smallest, it should be smallest in size in some sense right? So it should be contained… Any other subspace that contains this should contain the smallest subspace also, okay? So this is a nice exercise to think about. Why should span be the smallest subspace containing the v_i okay? So anyway... So hopefully these examples gave you some idea of what subspaces are. If you have a big vector space, subspaces are smaller subsets which have all the properties of the vector space itself and you can use it in interesting ways, okay? So that's subspaces for you.

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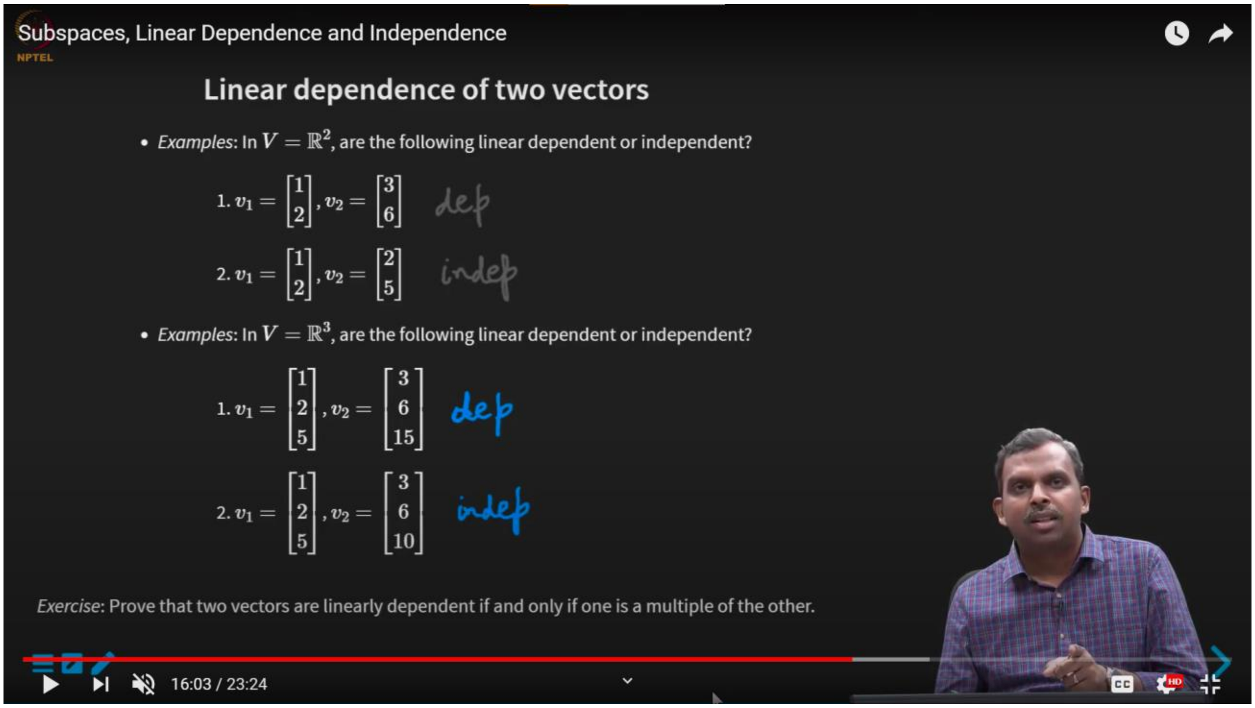

So let's move on to the next important idea which is of linear dependence and independence, okay? So in any vector space over some scalar field, one can define the notion of a linear dependency and linear independency, okay? So all of this starts with the set of vectors, in this case we are going to start with m vectors $v_1, v_2, \ldots, v_m$ in this vector space V. I will say that this is a linearly dependent set, $\{v_1, \ldots, v_m\}$ is linearly dependent if there are scalars which can make a linear combination of v_1 to v_m and give you 0, okay? So now I have to be careful here because if all my scalars are chosen to be 0, right? If a_i is 0, every a_i is 0, then the linear combination is automatically 0, okay? So there's nothing great about it. So of course if given any set of vectors I can make a linear combination which will give me 0, which is the trivial linear combination where all my scalars are 0. Now that is not the notion of linear dependence. In linear dependence at least one of these coefficients should be non-zero, okay? You cannot make all of them zero. At least one has to be non-zero and still the linear combination should end up being zero. If you have a situation like that, then these vectors are supposed to be linearly dependent okay? So that's what I meant here in the bullet point right below that linearly...

Okay, so something is wrong here, so let me correct it… It is not linearly independent it is linearly dependent, okay? So that should not be there... So vectors are linearly dependent if there is a non-trivial linear combination that can result in the zero vector, okay? So that way we say vectors are linearly dependent.

The opposite is linearly independent. So what is the opposite? There should be no non-trivial linear combination. Or another way to put it is if somebody tells you there are scalars a_i such that $a_1 v_1 + a_2 v_2 + ... + a_m b_m$ is 0 then what should be true? All the a_i should be 0, okay? So only the trivial linear combination results in the zero vector, okay? So this is something that's sort of important because, you know, if there is a set of vectors and they are linearly dependent, there is something redundant about them, right? So it's almost as if one can be formed through the others, right? So if you have a linear combination with a coefficient non-zero you can keep that term alone on one side, move everything else to the other side, right? And then you can divide and you get one vector in terms of the other vectors, okay? So a linearly dependent set has some redundant vectors in some sense, linearly redundant sort of vectors. So maybe you can use that and that's something useful. On the other hand linearly independent sets, every vector adds some new information in some sense, okay? So this is something to keep in mind when you think of linear dependence and linear independence, okay?

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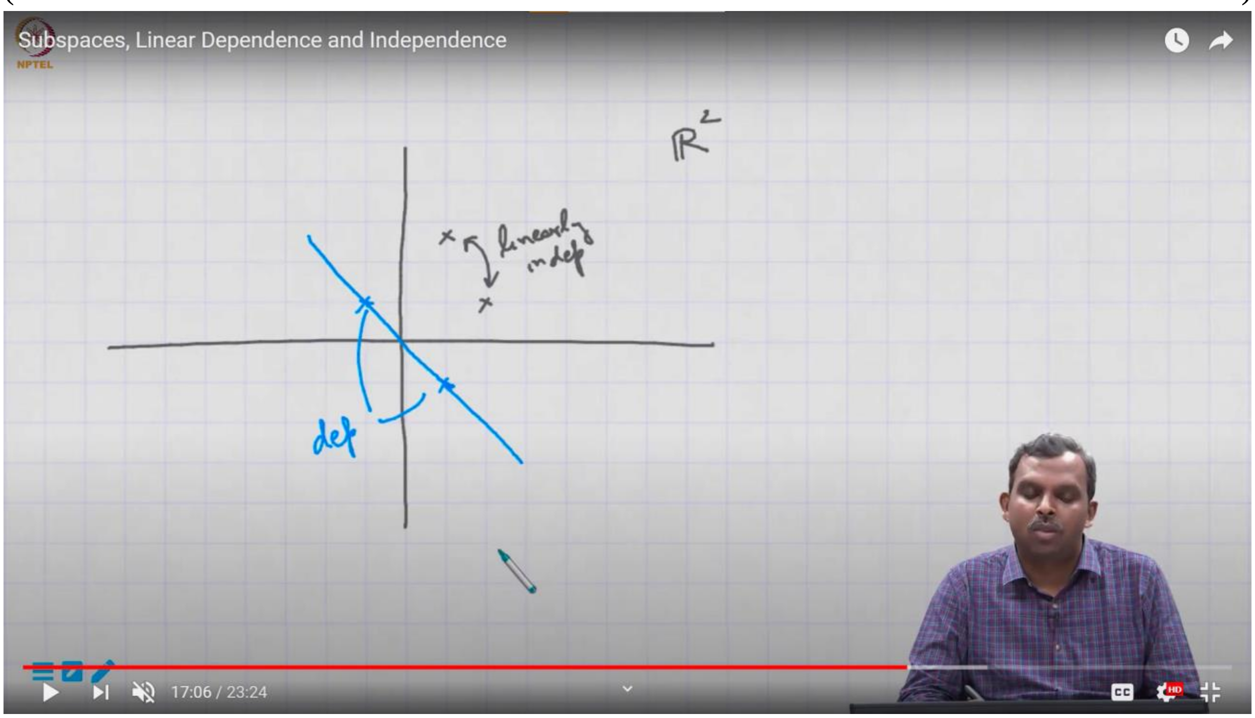

So here are some examples, okay? So the examples are in $\mathbb{R}^2$ and $\mathbb{R}^3$ and you can see, you, if... So given two vectors how do you find whether they are linearly dependent or not, okay? So that is the next question naturally, you know, I mean we start with very simple examples. Before that, what about one vector? If somebody gives you one vector how do you find out if they are linearly dependent or not? What's the meaning of saying one vector linearly dependent? You know sometimes definitions you have to pay attention to these kind of trivial corner cases to make sure everything works out. So if you have only one vector, that vector has to be the zero vector. If it is the zero vector then it is sort of linearly dependent because you can have non-trivial combinations giving you zero. But if it is a non-zero vector then it is linearly independent.

So one vector is not so interesting, but if you have two vectors you can ask this question - are they linearly dependent, are they linearly independent, okay? So that is a very valid question, so here are two examples I have given here. In $\mathbb{R}^2$ you can see very easily that, without too much thinking you can quickly say these are linearly dependent, these are linearly independent, okay? It’s not too difficult to guess. Likewise, here you stare at it for a while, you see that this is linearly dependent. You stare at this for a while, you know that this is linearly independent, okay? Okay so this is easy to see and here is an interesting result, okay which is, I will leave as an exercise. It is not very difficult to prove. So two vectors are linearly dependent if and only if one is a multiple of the other, okay? So this is an interesting thing to remember - one vector has to be a multiple of the other for the two vectors to be linearly dependent, otherwise they are linearly independent, okay?

So in fact think about other trivial cases. Suppose I give you two vectors and one of them is the zero vector, okay? Then they are trivially linearly dependent again, okay? So the zero vector makes everything trivial in linear dependence, but assuming non-zero... All these kinds of cases are interesting, they have to be multiples of each other, okay? So something to remember. So let me, this question, to also remember. So supposing you think of $\mathbb{R}^2$, in $\mathbb{R}^2$ okay? Supposing I have a vector here and a vector here, right? These two are linearly independent, okay? Why? Because one is not a multiple of the other. If one were a multiple of the other, what should happen? They should lie on the line through the origin, okay?

So here is an example of linearly dependent vectors. So if I have a vector here and a vector here, these are linearly dependent, okay? Why is that? Because there is this line through the origin which connects both of them, okay? So that is the idea, so one becomes a multiple of the other, all right? So keep that in mind. So in $\mathbb{R}^2$ linear dependence can be very easily visualized, okay? If they lie on a line through the origin, then they are linearly dependent, otherwise they are not. So in general in n-dimensional space also, if you can visualize, if you can draw a line through the origin… Linearly dependent vectors, if there are only two of them, then they have to lie on that same line, okay? So that’s how it goes.

All right. So two vectors, quite easy to think of linear dependence. Let us go to the next non-trivial case, which is three vectors, right? So from two, proceed to three, okay? So let us go to $\mathbb{R}^2$ and then ask this question, okay? So the first example, look at the first example. (1, 0), (0, 1) and (3, 17) okay? Are these linearly dependent? Okay, so you think about them for a while. It's not very difficult to see that $3v_1 + 17v_2 = v_3$ okay? So these are linearly dependent, okay? What about the next example (1, 2), (2, 5), (3, 17)? So in this case it's not so obvious, right? I mean how do you find a linear combination for (3, 17) from (1, 2) and (2, 5)? You have to write down some things and think about it for a while. But it turns out they are linearly dependent, okay?

So in fact look at this exercise. Any three vectors in $\mathbb{R}^2$ are linearly dependent, okay? Seems like an interesting result, I am leaving it out as an exercise for you. Think about it, see how you can imagine, look at different situations and prove it. Any three vectors in $\mathbb{R}^2$ will be linearly dependent, okay? So we'll prove it in a different way later on in our class, but for now try and prove it for the $\mathbb{R}^2$ case. It's an interesting exercise if you can show that, okay?

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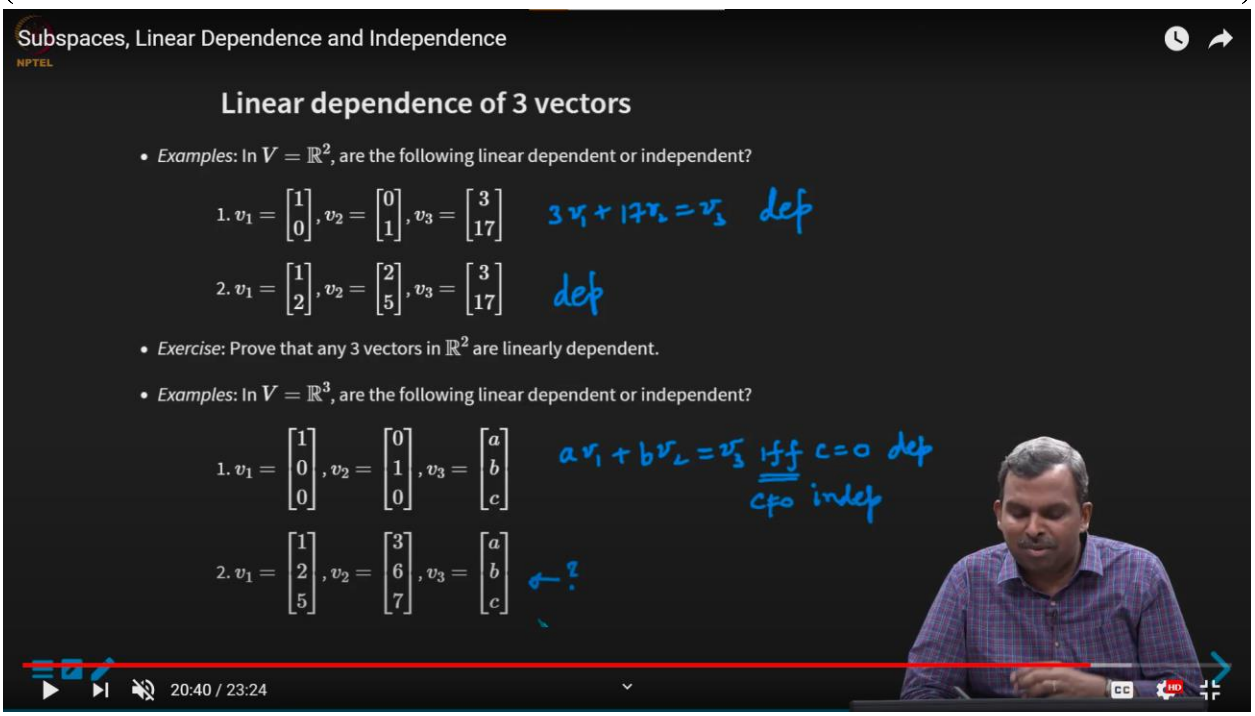

Let's go to $\mathbb{R}^3$, the next more interesting case and ask the same question. So v_1, v_2, v_3, (a, b, c) Suppose I give you an (a, b, c) vector, okay? Notice here what will happen. If you look at v_1, v_2 right? So I can do a times... I have to do av_1 if at all I have to match v_3 as a linear combination, right? And then I have to do bv_2 okay? And if this has to be equal to v_3, this is if and only if, okay? I will write iff. Iff means if and only if, both ways, c equals zero, right? So this variable c is crucial to the story here. If in v_3 the third variable is 0, third coordinate so to speak is 0, then v_3 becomes a linear combination of v_1 and v_2 so they are linearly dependent, okay? So they are dependent if c is zero. If c is not equal to zero then they are linearly independent, okay? There is no way you can make a linear combination of v_1 and v_2 to get v_3 when c is not zero. If c is not zero... Notice the third coordinate in any linear combination of v_1 and v_2 will always be zero, okay? It's like you're stuck in the x-y plane, you don't have any z direction, right? So only x-y plane, whatever combination you do, you can never go there, okay? That's the idea.

Look at the next example here, okay? So what do you do here? It seems much more tricky, right? $(1, 2, 5), (3, 6, 7), (a, b, c)$. Can you make a linear combination? Who knows, right? So you have to write it down and think about it. Later on we will see some general methods for how to approach these problems, how to do them very easily. For now I will let you think about it. If you can, come up with some clever ideas for how to find such answers - are these linearly independent or dependent. I am leaving it once again as an exercise for you, okay? Think about it.

So in general we saw quite a few examples in the previous two slides for simple cases, small cases. How to establish linear dependence, independence. It seemed like you have to do some work, you have to think about it a little bit. Now push yourself to this large data regime, right? Where you have thousand length vectors, hundreds of them, given to you. If somebody asks you - are these linearly independent? How would you go about doing it, okay? So we want a very efficient, simple solution. It turns out there are solutions based on ideas called Gaussian Elimination, okay?

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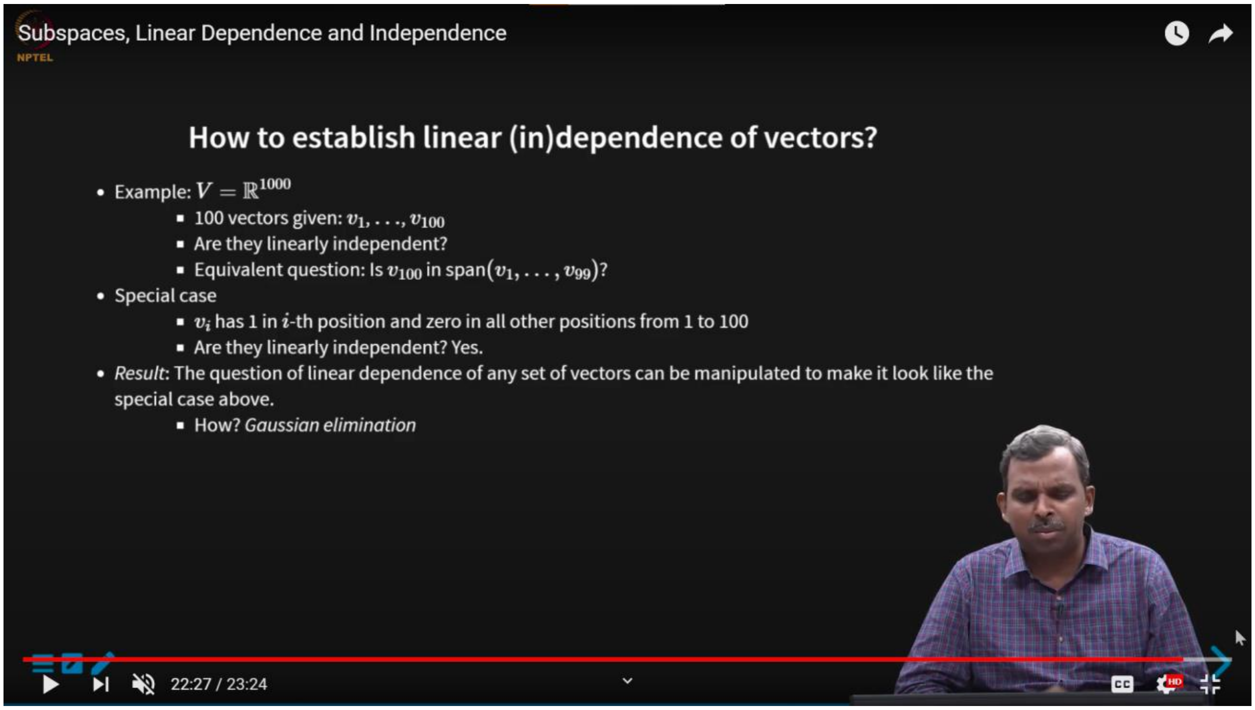

But before that, let me just give you a special case. So here is a special case, right? If each of these hundred vectors v_i, it was 1 in the i^{th} position, okay? v_i has 1 in the i^{th} position and 0 elsewhere, okay? In the first hundred. After hundred I don't care, okay? So this thousand length vector, the i^{th} vector, say the 50^{th} vector, has one in the 50^{th} position, 0 from 1 to 49 and 51 to 100. And then after that I don't care. Supposing you had 100 vectors like that, then it's clearly easy to see that they have to be linearly independent. Why? So if you make a linear combination and you have to get 0, right? Each one of those coefficients has to be 0 right? Otherwise you will not get this. It is easy to see in that case, okay?

So special cases like that for v_i are easy to solve. But what about the general case? It turns out you can use this notion of Gaussian Elimination to reduce any general case to look like the special case, okay? So that was the nice idea. You can manipulate the vectors to make it look like a, like this special case, okay? So we will see this later on in some other context. I will give you some simple introduction to Gaussian Elimination in another lecture, okay? So it is possible to establish linear dependence and independence in a systematic methodical manner. As of now I am leaving you, leaving that as a puzzle to you. You can think about it and then when we look at Gaussian Elimination you will see how the special case that I am talking about here is all that is important, okay?

So the final piece here is once again a quiz for you, okay? I'll urge you to answer these questions. There are some points etc., they are not for actual grading so please go ahead and submit these things. These exact slides will be shared with you, you can take a look at this quiz, answer them. It will give me feedback on how well you've done. Try to do it on your own, it will improve your learning and other skills, okay? Thank you very much, this is the end of this lecture. We'll meet again in the next lecture.