Applied Linear Algebra

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Week 10
Complex Spectral Theorem

Hello and welcome to this lecture. We're going to look at one of the cornerstone results of linear algebra which is this complex spectral theorem. It's a simple result. Now that we've built up all the necessary pieces, the final result will come in in a very easy way. The proof is not too hard, but the impact of this result is really huge. In applications all over, people use this spectral theorem inside out. Self-adjoint operators, normal operators satisfy the spectral theorem, it's very, very powerful. Particularly in physics, there's a lot of applications for this result, okay? So let's go in and see what it is.

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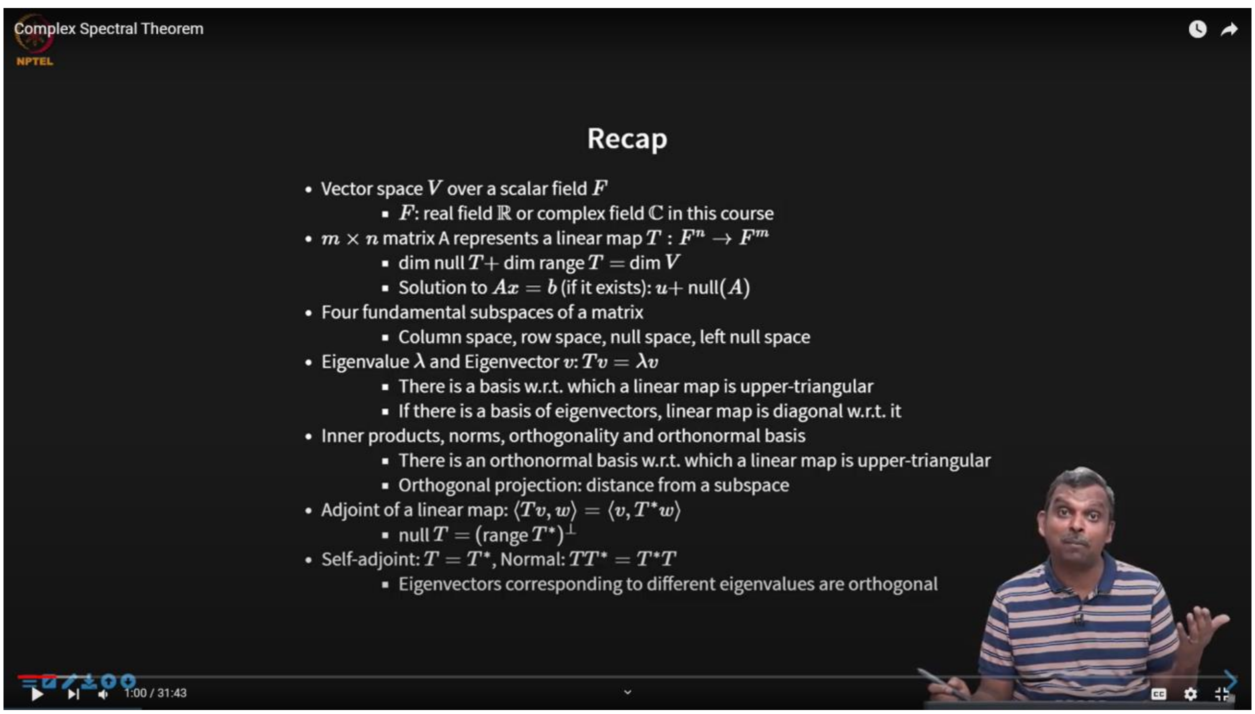

A quick recap. I think most of it is sort of familiar. I've been repeating it so many times. There's this adage that, you know, if you repeat three or seven times it's good for learning. So it's good to repeat. So we've been looking at adjoint closely in the last few classes, last few lectures. Adjoint is related to sort of the connection between inner products and linear maps, right? So when you look at a linear map and inner product, they're both linear in some ways. And what is the connection? Adjoint gives you a very nice connection that way. And that equation down there, <

$Tv, w > = < v, T^*w >$ defines the adjoint for you. So T^* is another linear operator from W to V which does this. And there are these nice relationships between null space and range of T^*. They are the orthogonal complements of each other, and in particular you can define special operators using the adjoint in terms of what properties the operator shares with the adjoint. If it is equal to adjoint, it's called self-adjoint. These are very special operators, sort of like, you know, the book, your book talks about the analogy between, in the world of complex numbers, the real numbers are special, right? So they are equal to their conjugate. So self-adjoint operators are like that, okay? You can think of them like that. They are equal to their adjoint. And then we have these special other operators called normal operators. Self-adjoint operators are normal operators. But there can be normal operators which are not self-adjoint. The definition is that they commute with their adjoint. TT^* is T^*T and it's a larger class, and this class of operators is crucial in this complex spectral theorem. So you will see that the spectral theorem nicely holds for these normal operators. We saw towards the end of the last lecture, this nice result, so, that if you have two eigenvectors which correspond to two different eigenvalues of a normal operator, then they are in fact orthogonal, okay? So previously we saw results of how, you know, if you have an operator and there are two eigenvectors from two different eigenvalues, they are linearly independent. That is always true for any operator. But for, in an inner product space, when you have normal operators, two eigenvectors that correspond to two different eigenvalues are in fact orthogonal, okay? So they are not even related to each other, sort of, right? So that is the modern interpretation for what these vectors are, okay? So this is a quick recap. With this, let's jump into the complex spectral theorem, okay?

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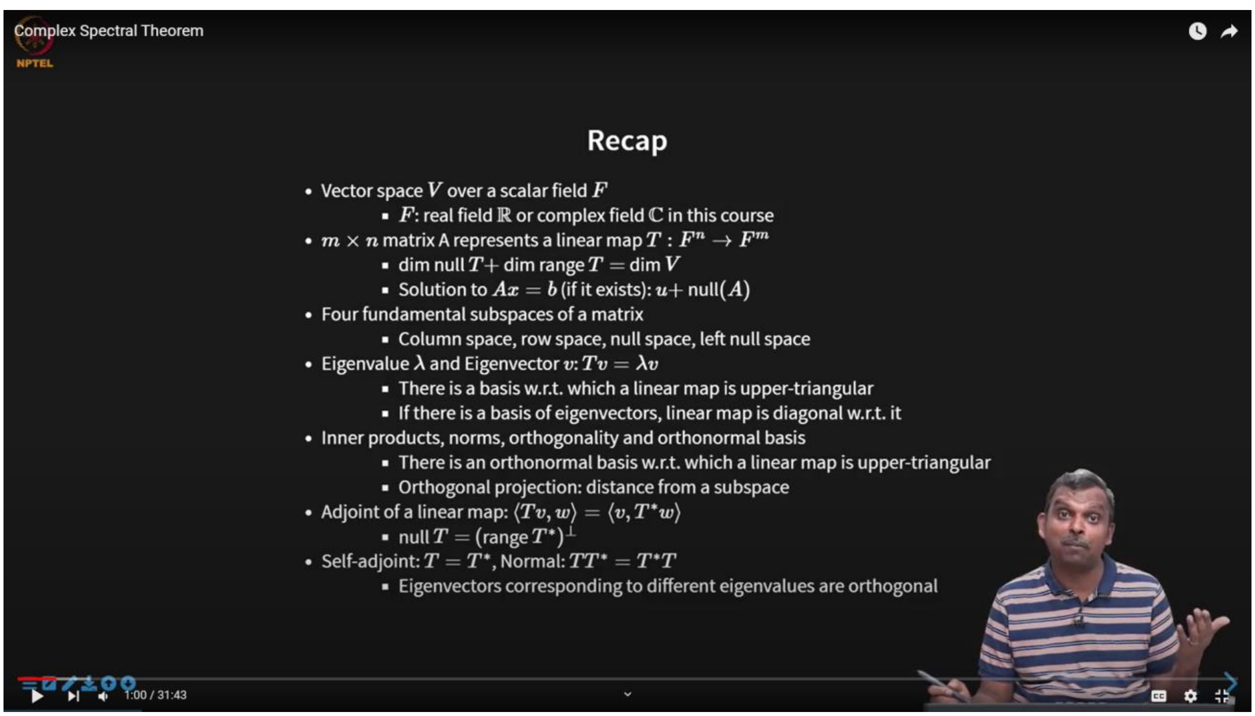

So before we jump into the complex spectral theorem, it's sort of like a diagonalization result, okay? At some level, it's a diagonalization result. So let us first look at diagonalizable operators. And I will remind you of what we have studied about diagonalizable operators and how to think of them, how to express them and how to work with them, okay? So first look at, we'll look at this and then we will look at the corresponding picture for normal operators, where you have orthogonality and all that, okay? So let us start looking at it. So if you have an operator $T: V \to V$, and if I say it is diagonalizable, then it means that there is a basis of eigenvectors of T for the vector space V, okay? The entire vector space V has a basis which is made of all eigenvectors of T. Every vector in the basis is an eigenvector of T. Of course, the basis vectors have to be linearly independent, they have to span the space. So this is all going to work with just eigenvectors of T. So when you have that, you know, that Tv_i is going to be $\lambda_i v_i$, and if you look at the matrix of this operator T with respect to eigenvector basis, it simply becomes a diagonal matrix, okay?

So a couple of things I want to point out. One is these λs can be equal, okay? I have not said λs have to be all distinct, I just put λ_1, λ_2 simply to distinguish it, but they could be equal, there could be multiplicity. That's one thing. Another thing which is very interesting is: see, once you have an operator like this, it's sort of, you know, because there is a basis for the whole space made of eigenvectors, there's something interesting going on here. So supposing you take any vector v, right? So what does that mean? So once you have an operator like this which has an eigenvector basis, any vector v can be written as $a_1 v_1 + \cdots + a_n v_n$ and each v_i is an, v_i is an eigenvector of T, okay? So this is something very interesting. So, so any vector v can be written as a linear combination of eigenvectors of an operator, of a diagonalizable operator, okay? So that is a nice property to have. So it is as if… And, you know, once you write it as a linear combination, you know what T does to v, right? So Tv is going to be simply $a_1\lambda_1 + \cdots + a_n\lambda_n$, okay? Sorry, I forgot to put the $v_1, \ldots, v_n$, okay? So it is just diagonal. Each v_i gets scaled by the λs, okay? So the operation of T is easy to describe. But more importantly, the eigenvectors sort of span the space, right? So operator T sort of controls the whole space in some sense. So this is important and this is useful. So supposing, so you want to define the operator T in some close way with v. This sort of helps you do that, okay? So I will comment more on this when we look at self-adjoint and normal operators later, okay? So this is the basic definition for diagonalizable operators.

Now in terms of matrices, supposing you want to start looking at matrices for diagonalizable operators. Let us say you have A, which is an $n \times n$ matrix and let us say it represents a linear operator T in the standard basis, okay? And now this T is going to be diagonalizable, okay? I am going to assume that T is diagonalizable. I think maybe I should write that down somewhere here. So it is diagonalizable. So T is diagonalizable, okay? Something to keep in mind. So, and then v_i's are the eigenvector basis, right? So just like before, I mean v_i's are the eigenvector basis, T is a diagonalizable operator. Now I am saying A is the $n \times n$ matrix in the standard basis, right? So we know that we can make this matrix S, right? We have used this before. This is the change of basis from the eigenvector basis to the standard basis. How do you do that? So if every column is v_1 to v_n… What do I mean by column is v_1? The coordinates of v_1, right? So the first column is coordinates of v_1. The first eigenvector basis, the vector, the coordinates of v_1 in the standard basis, you put that in the first column. Likewise in the i^{th} column you put the coordinates of v_i in the standard basis. You make a matrix like this, this matrix will change basis for you from the eigenvector basis to the standard basis, right? Coordinates from the eigenvector basis get converted to coordinates in the standard basis if you do multiplication with S. So this is your change of basis vector, okay? Now this will play a good role in, you know, diagonalizing A. We know how that works, right? So we have seen that before.

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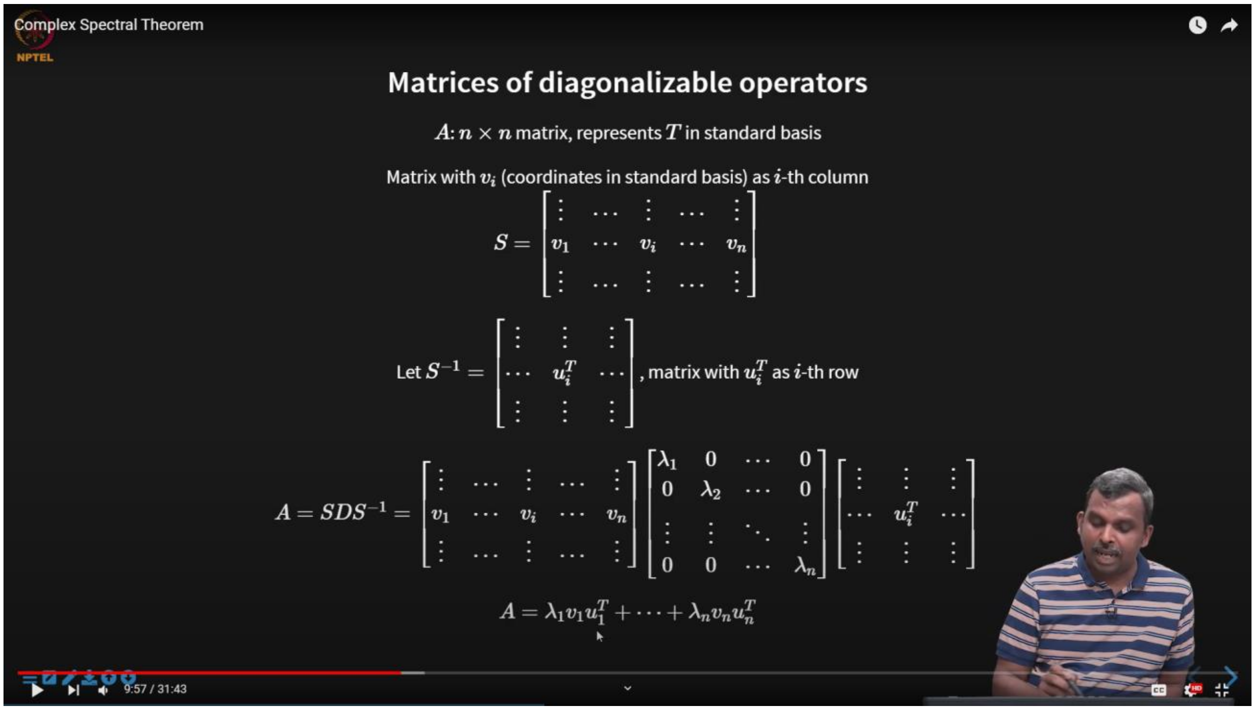

So we can make S^{-1}, you can, I mean this will be an invertible operator. It is a change of basis operator, so it is invertible. So you can, you can compute S^{-1}. And for S^{-1} I like to call the rows of S^{-1} as something. You'll see why this plays an important role. But anyway, S^{-1} can be computed. After you compute S^{-1}, the i^{th} row of S^{-1} I will call as u_i^T. I am putting transpose because we represent column vectors as vectors and then row vectors become transpose. Just a notational convenience. So we write u_i^T here for the i^{th} row, okay? So now what is true, I know, is that A can be written as SDS^{-1}, right? So the way I sort of interpret this is: S^{-1} converts from the standard basis to the eigenvector basis, and D is the operator in, diagonal operator in eigenvector basis, and then from eigenvector basis you go back to standard basis. That should give you A, okay? So SDS^{-1}, I think that's correct, so that's the matrix in standard basis that you write, okay? So now what is S? S is this v_1, I've written it column wise like this, D is the diagonal matrix λ_1 to λ_n, okay? So I didn't quite tell you what D is. D is the diagonal matrix. And then I have S^{-1} whose rows I have denoted as u_i^T. Now this product you can rewrite in one very simple and interesting way and that is this, that the matrix of a diagonalizable operator in any basis is going to look like this, right? It is going to be $\lambda_1 v_1 u_1^T + \cdots + \lambda_n v_n u_n^T$ where this v_1 is basically the coordinates of the eigenvector basis in that basis that you have chosen for A, okay? So supposing, let's say we choose standard basis for A. Just to be clear, the matrix of T in standard basis is given simply by this nice simple formula where this v_1 to v_n are simply the eigenvectors in that basis. And u_1 comes from the inverse, okay? So this is very, very useful if you think about it. You can express any diagonalizable operator in this fashion very cleanly, okay? So this is quite nice. We will see an analogous result for normal operators and it will be even nicer. But still this is a nice enough result to start with for diagonalizable operators, okay? So quite a few things I want to point out here. I will point that out in the next slide. So one can think of, now, a general form for matrices of diagonalizable operators. They will be of this form, that any matrix which is diagonalizable is going to have this form. $\lambda_1 v_1 u_1^T$... v_i is the eigenvectors in the basis and u_i is rows of the inverse matrix, okay? So that's very clear. That is nice.

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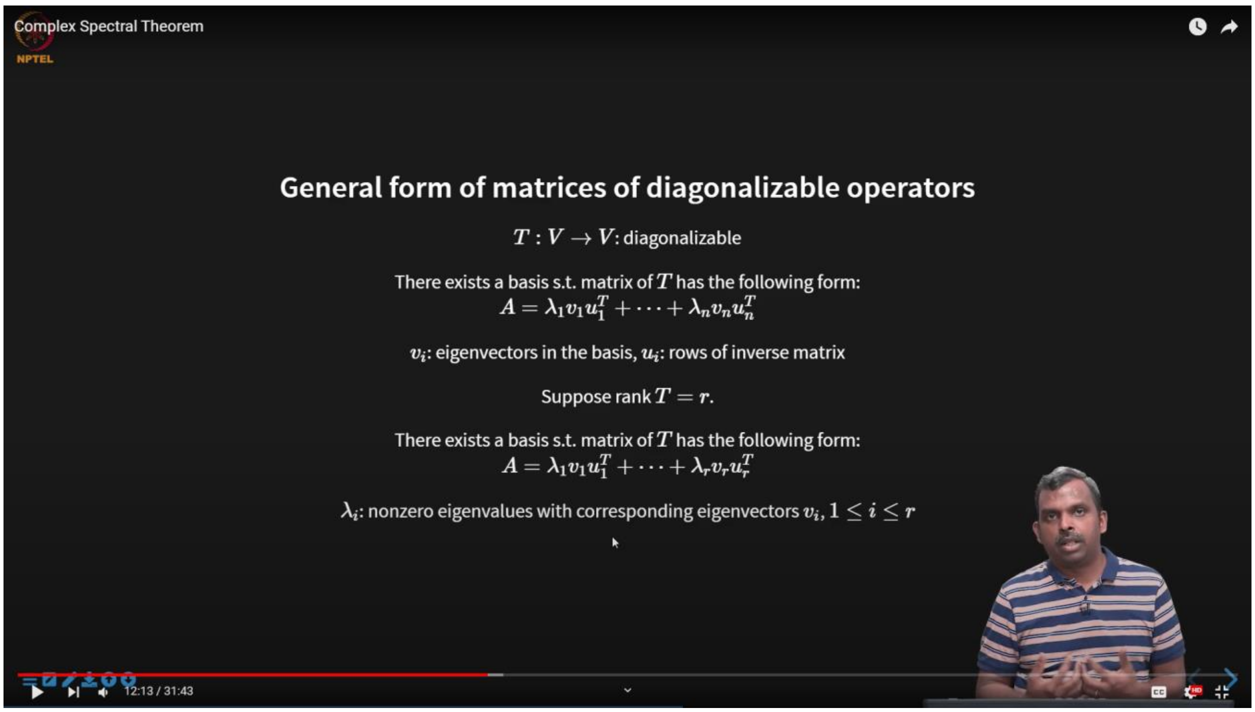

Now there is a connection between this expression and rank. When you want to fix the rank of a diagonalizable matrix as, say, r, it is enough if you take r such terms in the expression, okay? So it is almost as if… So you can see why this is true, right? So if you have rank r, then there are

$(n-r)$ eigenvectors with eigenvalues 0, right? So those will go away and then the remaining r only will have the, you know, non-zero eigenvalues and eigenvectors associated with them. And so A becomes simply that and the remaining eigenvalues are 0 and they go off to 0, okay? So this is the… And the matrix is diagonalizable. Now that is very important. The matrix is not diagonalizable, you cannot do this. But it is diagonalizable and diagonalizable matrices are plentiful in some sense, okay? So you can take, you can use diagonalizable matrices, if you're in doubt, you can use diagonalizable matrices. It's not too bad, okay? Now notice some nice things about this. Each of these terms, right, $v_1 u_1^T$ is a rank 1 matrix, isn't it? $v_r u_r^T$ is a rank 1 matrix. So rank 1 matrices are very simple to understand. So we know we know what rank 1 matrices do. And when you want a rank r diagonalizable matrix, you simply take a linear combination of r rank 1 diagonalizable matrices in this fashion, and you get a rank r diagonalizable matrix. And that's the general form. That's how it looks, okay? That's how, that's how diagonalizable matrices of rank r look always, okay? So this form is very nice and you can use it quite often when you want to derive some things with diagonalizable matrices of a particular rank, okay?

So, so far we have seen general operators, what happens to general operators when they are diagonalizable, okay? So now what about normal operators, okay? So that's what we've been studying so far. Normal. Of course, self-adjoint is a subset of normal. So when I talk about normal, I'm also talking about self-adjoint operators. What about normal operators? What can we say about normal operators from a diagonalizable point of view? It turns out there's a very, very surprising and simple and elegant result that normal operators satisfy, and that's why they are very, very important. And that's this complex spectral theorem that will come up soon. But before that, let us look at this. A few examples of how this works, okay? So I will show you a few examples. You will get an idea. I mean and these expressions are fine but it is good to see examples. We will see some examples and then we will jump into this complex spectral theorem, okay? So let's start with the simplest possible example. So A is diagonal. $[1\ 0;\ 0\ 2]$. It's self-adjoint, diagonal and in this case it is very, very easy to write A as the sum of two rank one matrices, right? $[1;\ 0][1\ 0]$, $[0;\ 1][0\ 1]$ and then you multiply by 2, you can see the eigenvalues. It's very easy, it's very trivial, there is nothing going on here.

Let's take a slightly more general example. $[1\ 2;\ 3\ 2]$. This is not self-adjoint, you can see clearly it's not self-adjoint, so definitely, I mean it could be normal, I haven't checked that, maybe, I don't think it's normal, okay? So anyway, you can check it. It's very unlikely to be normal, but you can check it. I think it's not normal, but anyway, it's diagonalizable. It's got two distinct eigenvalues, okay? -1 and 4, I have picked it so that the eigenvalues are very nice. Once you see two distinct eigenvalues and a 2×2 matrix, you know it's diagonalizable, right? So that is for sure, and you can make this S. you can find the eigenvector corresponding to -1, that's $[1;\ -1]$, it's quite easy to see I think. Eigenvector corresponding to 4 is $[2;\ 3]$, so that's $[1\ 2;\ -1\ 3]$. And then S^{-1} you can easily compute. It's $[\frac{3}{5}\ -\frac{2}{5};\frac{1}{5}\ \frac{1}{5}]$. So all that I have done. And now I know I can use my general form, right? A is λ_1 which is -1… Notice this minus here, do not forget it, times the first column, outer product with first row, this is the outer product, right? The rank one outer product plus the second eigenvalue, right? Second eigenvalue times the second column of S and the second row of S^{-1}, right? So that's what I've written. You can check that this is actually equal to $[1\ 2;\ 3\ 2]$, okay? It's quite easy to check, okay? So in small cases it looks like a bit of an overdoing of what is going on, but I just wanted to illustrate how it works in example so that you clearly see what is going on. In larger matrices, this kind of decomposition is very, very interesting. So notice, this 4 is much bigger than -1. So you can make some judgment calls based on what will dominate etc. And particularly for larger matrices these kind of things make a big difference, okay?

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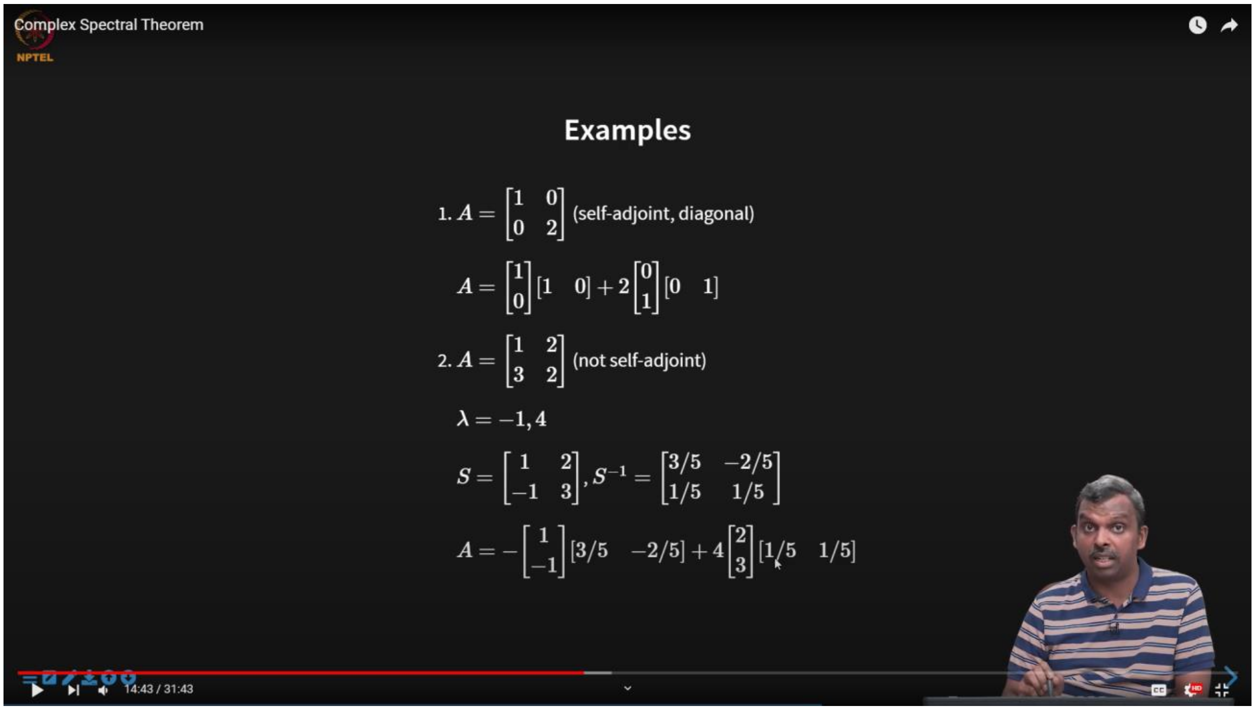

So that was two examples. I am going to take next a normal example. This is an example we have seen before. $[2\ -3;\ 3\ 2]$. We saw that it was normal and it has two eigenvalues: $2 + 3i$ and $2 - 3i$. Clearly, again distinct eigenvalues, so it will be diagonalizable. And then you can find S, S^{-1}. I have done all that, and then you can write A as this, okay? So some of you may argue $[2\ -3;\ 3\ 2]$ is simple enough, why write it with all these i's and all that? But, you know, this is rank one plus rank one. It has its own merits when we look at it. Plus, plus, notice this major, major result. This eigenvector basis is orthonormal, okay? So this $[i;\ 1]$, $[-i, 1]$, these two are orthogonal, okay? The first column of S dot product with this, not dot product, the complex conjugate product with the second column goes to zero, okay? So these eigenvectors of this normal matrix, normal operator are actually orthogonal, okay? So we know that that's true, right? For normal operators, two distinct eigenvalues, the eigenvectors will be orthogonal, okay?

So this orthonormal basis makes it, makes life very, very easy and simple when dealing with A in this fashion, okay? So I will comment more, a little bit, but from this example you see that this is working out to be true, okay? So, are normal operators diagonalizable? Can we say some results like that? It turns out all that is true and all that is captured in the spectral theorem, okay? The spectral theorem is a very nice result which completely characterizes the normal operator in terms of its eigenvalues, eigenvectors and its diagonalizability, okay?

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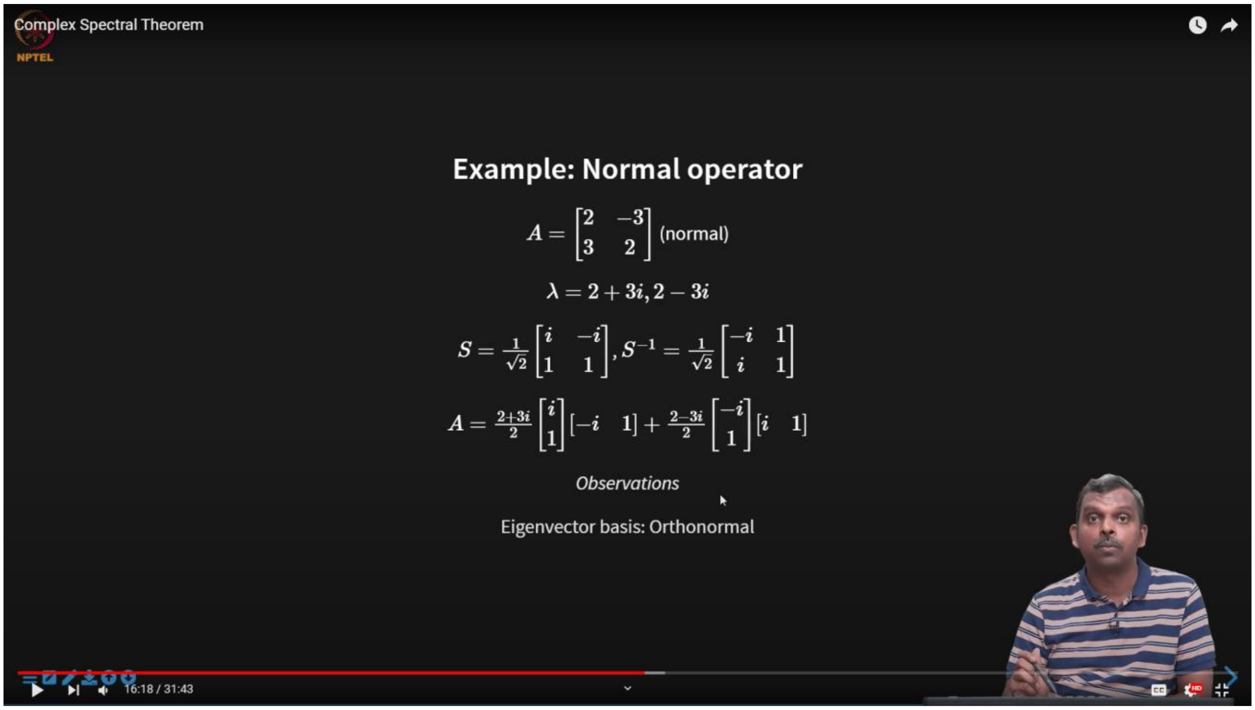

So here is the result. Very interesting and simple and deep result about normal operators, okay? You have an inner product space over the complex field, okay? I am going to use the complex field here because I am doing normal and all that. Then T is an operator from V to V. The following three things are equivalent. What does equivalent mean? They all imply each other, okay? Any one is true, the other two are true, okay? So that's how it works, okay? Equivalent, remember, it's not that one implies the other alone, the other also implies this, everything is true, okay? Not everything is true, every order, everything… 1 implies 2, 2 implies 1, 1 implies 3, 3 implies 1, 3 implies 2, 2 implies 3, everything is implied by any one state, okay? That's what it means.

So what are the three things that are equivalent? T being normal. Look at the next one. v has an orthonormal basis of eigenvectors of T, okay? So you have eigenvectors of the operator T. Not only are they linearly independent and span the space which gives you, makes them a basis, the eigenvectors are in fact orthogonal to each other and they span the bases, so they form an orthonormal basis for V. Supposing you have an operator T which has an orthonormal eigenvector basis, right, which is… An operator T in an inner product space and that inner product space has an eigenvector orthonormal basis, eigenvectors of T orthonormal basis. Then that operator has to be normal, okay? So these two are implied. And if T is normal, then for any operator you can form eigenvectors, and those eigenvectors in fact will be orthonormal, and they will span the basis, span the whole inner product space, okay? So it's very interesting to look at this result. Just that one operator results in a basis for the whole space. And they have to be orthonormal and all that, okay? So it's very nice to see, have that.

So of course 2 and 3 are very obvious, right? So once you have an orthonormal basis of eigenvectors, clearly T is diagonal or diagonalizable with respect to an orthonormal basis, okay? So that is the new word here. We have brought in this orthonormal everywhere, okay? When T is normal, not only is T diagonalizable, right, it is diagonalizable with respect to an orthonormal basis, okay? So it is a really, really strong characteristic for a normal operator, okay? If it is diagonalizable, okay, you might have as well said it is diagonalizable, but it’s not only diagonalizable, it's diagonalizable with respect to an orthonormal basis, okay? So that makes T really, really simple to deal with in practice, and no wonder it's very popular, okay? So people are able to easily use it in various situations, okay? So we're going to prove this result and that will more or less be the end of this lecture.

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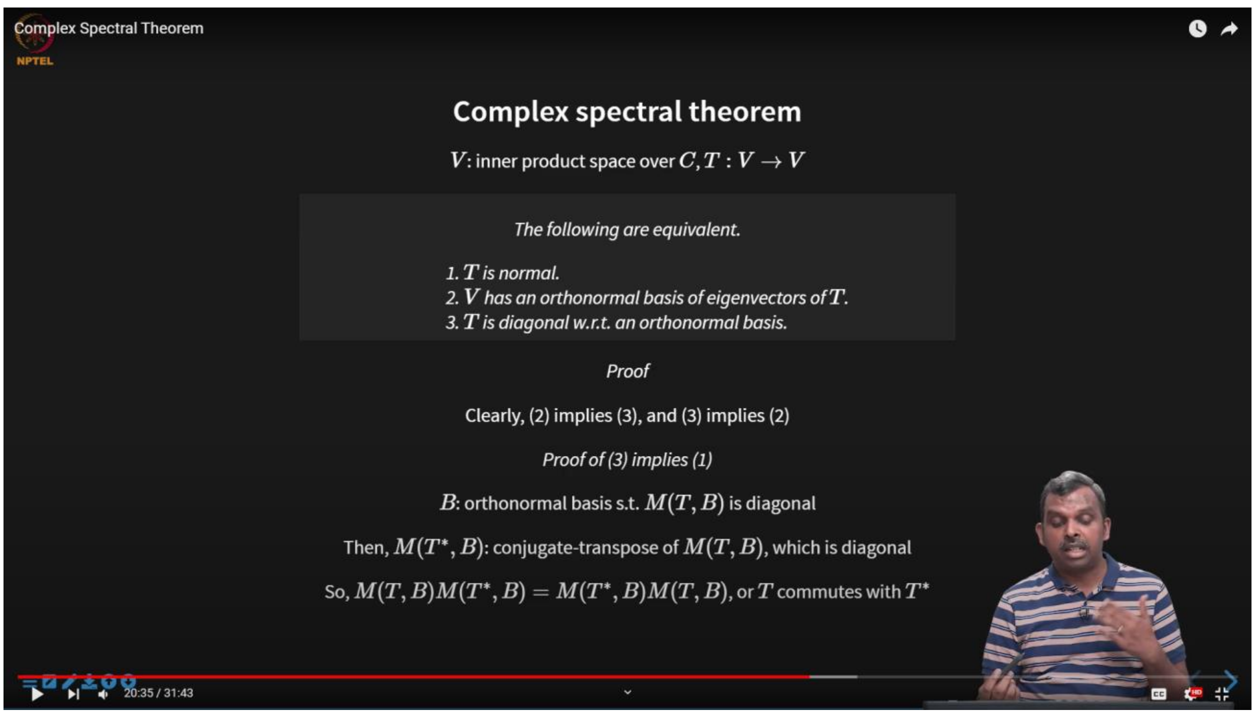

The proof is not very hard. You've done most of the hard work, it's just easily putting some things together. But there is essentially one slightly interesting idea in the proof. Let's see that, okay? So the first thing I'm going to do away with is 2 implying 3 and 3 implying 2, okay? So those two are sort of standard, and we already know this, okay? We'll do 3 implies 1 first, that's actually quite easy. So what is, what does 3 say? T is an operator and there is an orthonormal basis with respect to which T is diagonal, okay? So let us say that the basis is B, okay? B is an orthonormal basis such that the matrix of T, $M(T, B)$ is the matrix of T with respect to B is diagonal. Remember this is a diagonal matrix, okay? Now B is orthonormal. So what is the matrix of T^* in the same basis B, okay? Because B is orthonormal, you have a matrix for T. What is the matrix for T^*? You take conjugate transpose, okay? When you take conjugate transpose of a diagonal matrix, what will you get? You will end up getting another diagonal matrix, okay? So you see, once you have this property that T being diagonal with respect to the orthonormal basis, not only T, T^* is also diagonal with respect to that same orthonormal basis. So once you have two diagonal matrices, they commute, right? Any two diagonal matrices commute. It doesn't matter in which order you multiply. So T commutes with T^*. So T becomes normal, okay? So we have shown 3 implies 1. If an operator is diagonal with respect to an orthonormal basis, it has to commute with its adjoint, okay? It has to become normal, okay? So it is quite easy to see this, all right? So that is nice.

And the only thing left is 1 implies 3, right? 1 implies 3 is the only thing left and there we will invoke Schur’s theorem, okay? What does Schur’s theorem tell you? A normal operator, see, 1 implies 3 means T is already normal, okay? I have to show that T has a, T is diagonal with respect to an orthonormal basis, okay? We already know T is upper triangular with respect to an orthonormal basis, this is true for any operator, leave alone normal operator. So without even using normal operator, we know T is upper triangular with respect to an orthonormal basis. So let's say we pick up that orthonormal basis $\{e_1, e_2, \dots, e_n\}$, okay? And then we find the matrix of T with respect to this orthonormal basis you will get an upper triangular matrix, right? So everything in the bottom will be 0, on the top half will be something, a_{11}, a_{12} like that. I do not know what it is, it will be like that. Now what is the… See, this is an orthonormal basis. So what is the matrix for T^* with the same basis B? It will be the conjugate, right? Conjugate transpose, okay? So you transpose it and then you take conjugate. Is that okay? So that is easy to see. This is, these are the two matrices for T and T^* with respect to the same basis B. This is the basis B by the way, $\{e_1, \dots, e_n\}$, okay? So what we can show using normal… So far we have not used the property that T is normal, right? So this is true for any operator T. When you use the property that T is normal, you can show that this matrix itself is diagonal, okay? With respect to this orthonormal basis itself, the orthonormal basis which supposedly makes any operator upper triangular makes a normal operator diagonal, okay? Not just upper triangular, we will simply show the off-diagonal elements on the upper side of this matrix are 0, okay? Using the normal property, okay? So it is quite easy actually, if you look at the proof. But that is the idea of the proof, okay?

So let us take the first basis vector e_1. What is Te_1? That is the first column, isn't it? Maybe I should write something here. This column is T times e_1, right? That is the definition of a matrix for an operator with respect to a basis, okay? So $||Te_1||^2$ is $|a_{11}|^2$, okay? So that is easy to see. What is T^*e_1? T^*e_1 is this guy, right? Okay. Now I know because T is normal, $||Te_1||$ and $||T^*e_1||$ are the same, okay? So this, now this $||Te_1||^2$ which is $|a_{11}|^2$ equals $||T^*e_1||^2$. Now what is $||T^*e_1||^2$? $|a_{11}|^2 + |a_{12}|^2 + \cdots + |a_{1n}|^2$. Now if these two have to be equal, this a_{11} will cancel out and of course everything else has to be zero, right? These are all sum of positive numbers being equal to zero, each positive number has to be, I mean sum of non-negative numbers being equal to 0 and each non-negative number has to be equal to 0 itself, okay? So that is how you get a_{11} alone will survive, everything else will go to 0, okay? So once you show that all these elements are 0, you can now proceed to the next one, okay? After this, you can proceed to the next one and again you use the same idea, okay?

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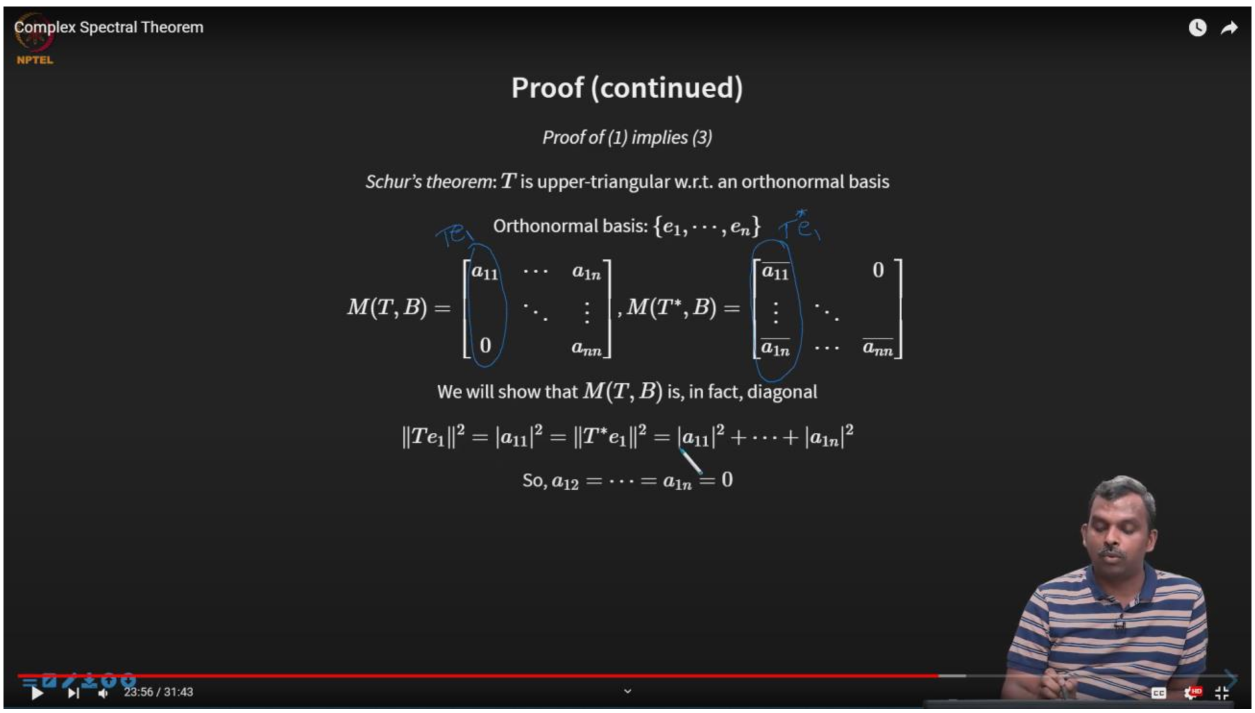

So once you have shown that the top thing is 0, what is Te_2? Te_2 is the second column and the only thing that is non-zero in the second column now is a_{22}, right? $|a_{22}|^2$. Now you look at T^*e_2, that is the second column and that is just $|a_{22}|^2 + \cdots + |a_{2n}|^2$. And then a_{22} will cancel, a_{23} to a_{2n} is 0. So everything else below the diagonal goes to 0 here. Everything else to the right of the diagonal here goes to 0, okay? The same way you proceed, you can show that this matrix itself is diagonal, okay. Quite a simple proof, but it sort of uses the notion of Schur's theorem to get an upper triangular matrix and this notion of normality which makes the magnitude same in a very clean, neat way to get to diagonal, okay? So that concludes the proof of the complex spectral theorem, okay?

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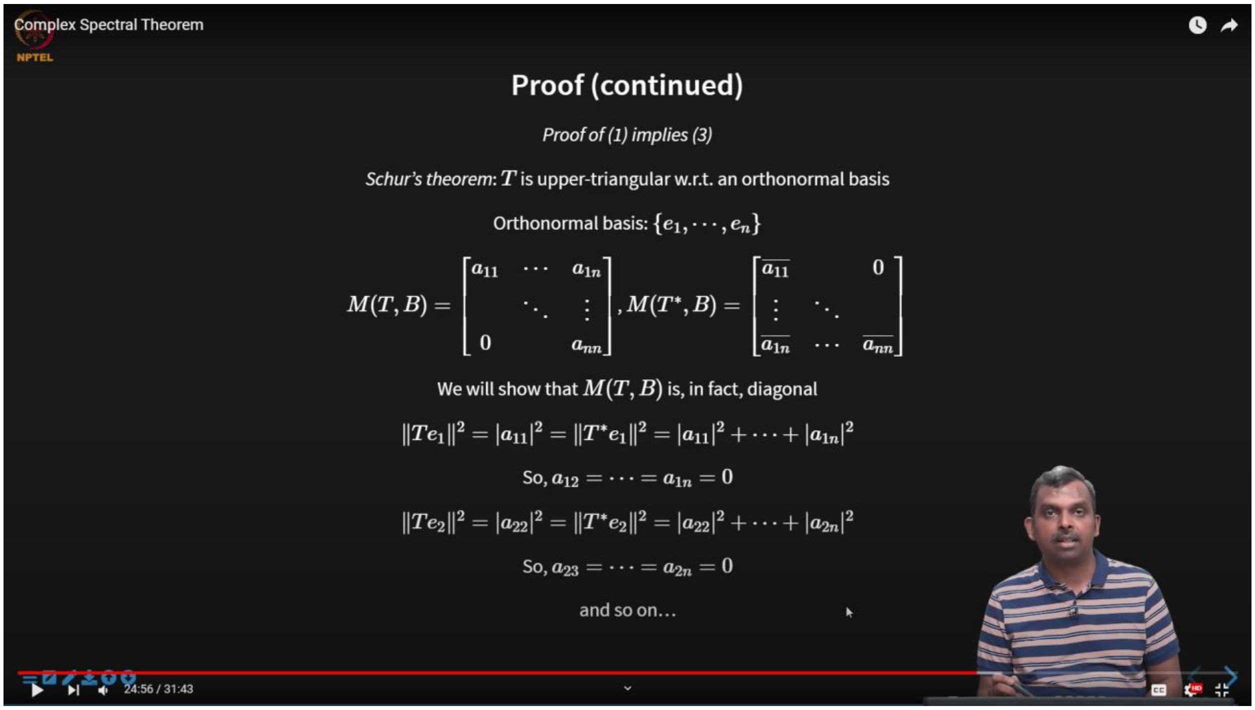

So let us look at a matrix picture of normal operators similar to the matrices of diagonalizable operators that we saw before. We saw that any diagonalizable operator can be written as the sum of a linear combination of rank one matrices with eigenvectors coming in each of the matrices, right? So it was a very simple thing to see. Something very similar you can write for normal operators, but now with orthonormal basis, okay? Very, very powerful, okay? So let us say A is an $n \times n$ matrix which represents a normal operator T, maybe in the standard basis, it could be any other basis, I am just taking standard basis. We know that T is diagonal with respect to an orthonormal eigenvector basis $\{e_1, \dots, e_n\}$, let's say the corresponding eigenvalues are λ_1 to λ_n. Once again, these λs could be equal, okay, I am just taking them to be generically λ_1 through λ_n. Now matrix, I am going to make this S matrix with e_i as the, you know, i^th column, right? Coordinates in the standard basis, same as before. And then of course I can do S^{-1}. But now there is something interesting, right? Because these are orthonormal, I know what the i^th row of S^{-1} will be, right? The i^th row of S^{-1} will simply be $\overline{e_i^T}$, okay? You take e_i and then make a row vector out of it by doing conjugate transpose, that will be the i^th row of S^{-1}, okay? You can easily check that $S^{-1}S = I$. So it's an exercise, it's quite easy to check $S^{-1}S = I$, okay? So you can quite easily check. So, right, because this shows up in the i^th column, so when the i^th column multiplies the columns of S, only the i^th entry will survive, everything else will go to zero, right? The orthonormality ensures that, okay? So you can ensure that. So this $\overline{e_i^T}$ becomes the i^th row. In the general case, when you had a diagonalizable matrix, when the basis was not orthonormal, you couldn't say this, right? The i^th row was some general thing u_i, but here the i^th row becomes the conjugate transpose. So that is very nice.

So once you have that, the familiar form that we had, SDS^{-1} becomes even nicer, okay? So not only do you get, you know, sum of n rank 1 matrices, each rank 1 matrix is given by $e_1 e_1^T$, the same $e_1 e_1^T$, not, you know, $v_1 u_1^T$, something else, but $e_1 \overline{e_1^T}$, okay? So this is very, very clean and you can work with this so easily, it's like, for instance, so this is the comfort of that, right? So look at, look at what we could do before, right? If you had $v \in V$, you can always write v as linear combinations of the e. Notice what happens when you do Av, right? Or I mean you can even do Tv if you like. Look at what happens when you do Tv and when you multiply this T with v, and I know T is of this form, when e_1 multiplies, I know that, you know, this term is the only term, the first term is the only term that will survive, everything else will go off to 0 and Tv can be written in a very, very simple form. So you get $\lambda_1 a_1 e_1 + \cdots + \lambda_n a_n e_n$, okay? As simple as that, right? So it just goes very cleanly. Any v can be written as a linear combination of the orthonormal basis, eigenvector basis vector for T, and then Tv simply becomes λ_1 , …, λ_n, okay? So this kind of property is used, like I said, quite extensively in many applications. One of the applications is in quantum mechanics, when people define states as complex vectors. Then the operators representing actual physical quantities are actually usually taken to be self-adjoint operators, right? Hermitian operators. And the property that is very crucial here is that any operator will have a basis, orthonormal basis which is an eigenvector basis, right, eigenvector orthonormal basis. So now what happens is: any state that you sort of think of for the system can be expressed as a linear combination of those eigenvectors. And one defines probabilities of, you know, measuring and measurements as you know reducing to eigenvectors. And the probabilities depend on the coefficients that are there for the eigenvector. So this is a very popular application and you can see that the property of hermitian being, you know, having eigenvector basis is very very crucial.

And also I think on top of hermitian, you might say, why not pick normal operators, why pick hermitian operators? You do not lose much actually but hermitian operators also have real eigenvalues and in physics you tend to associate real values with physical quantities. So you can say, you know, measurement of hermitian operators results in real physical quantities and so that's why hermitian operators are very popular in quantum mechanics. So that is just one, one application. There are so many other applications. And this form, right, so this form is particularly powerful. So see, particularly for hermitian operators. λ_1 is going to be real, so you can order these $\lambda_1 > \cdots > \lambda_n$. Usually λ_1 is taken as the largest eigenvalue and then λ_2, then λ_3 etc. And when n is very, very large the first few eigenvalues will dominate, okay? So usually people tend to make some assumptions on how the first few eigenvalues are, and there are so many more applications based on this decomposition, okay? So this decomposition writes A as the sum of n rank 1 matrices, each matrix is an orthonormal, I mean they are all orthonormal, and then there is that outer product, okay? So it's really very interesting. So how that, you know, the other rank one matrices do not affect the first rank one matrix in any way, right? So this is sort of orthonormal in that sense, not just independent, okay? So it's very, very, very strong and a simple decomposition.

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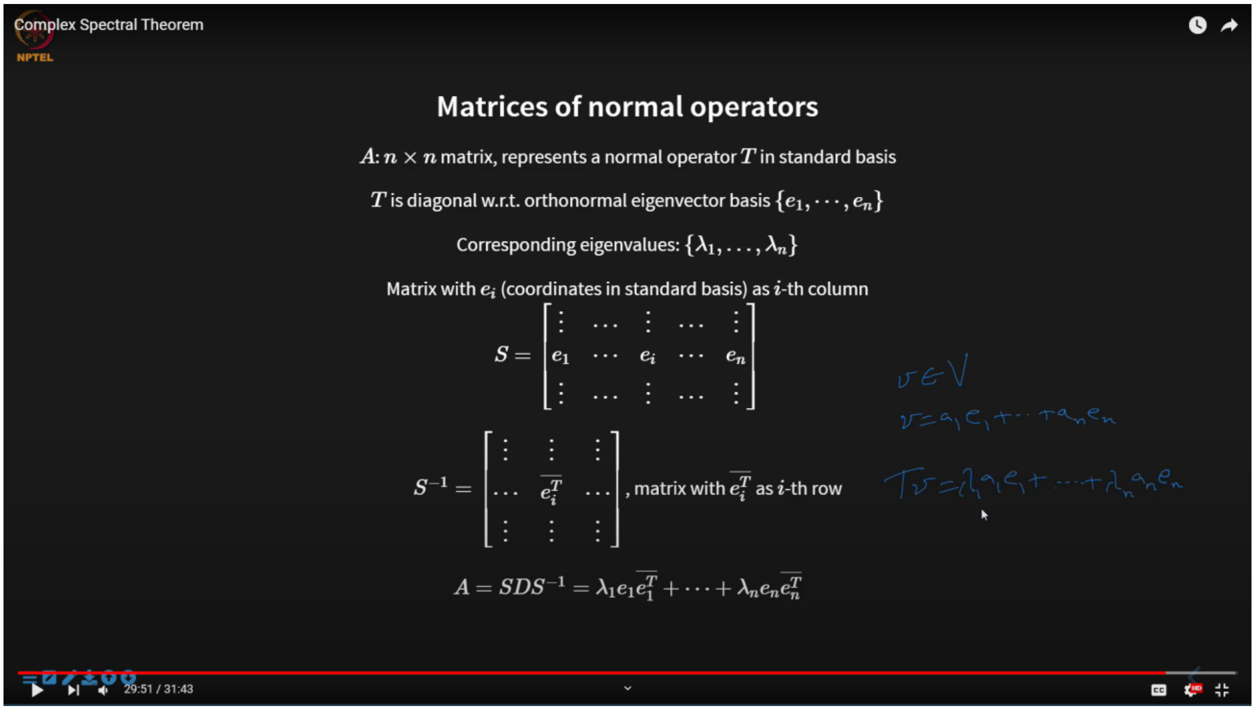

Of course you can also do a rank r assumption here. So if you say rank r, only the first r terms will survive, others will go off to zero, okay? So a self-adjoint operator, normal operator has this form. That's it. No other form can be taken by it, okay? So a very useful form in which you can now prove things. So if you want to prove properties for normal operator, self adjoint operators, you can assume they have this form, okay? That is the end of this lecture. The next lecture we will see what is called the real spectral theorem, which specializes this to the self adjoint case, okay? This is true for any normal operator. What happens for a real, in the real space? What if your vector space is over reals, and you do not want to allow for complex eigenvectors, right? So in that case what do you do? So that is done in the real spectral theorem. We will see that in the next lecture. Thank you very much.