Applied Linear Algebra
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Week 11
Isometries

Hello and welcome to this lecture. In this lecture, we are going to study specific types of operators which are called isometries. Now we have already been studying different types of operators. Maybe all of it is getting a bit clouded in your head. But towards the end we will also, you know, sort of summarize and… Maybe not in this lecture, maybe in some other lecture we will summarize all the various different types of operators and what to keep in mind etc., you know? I mean, so it will eventually, hopefully will be easier for you later on. But for now let's just go through the various types of operators and what they mean.

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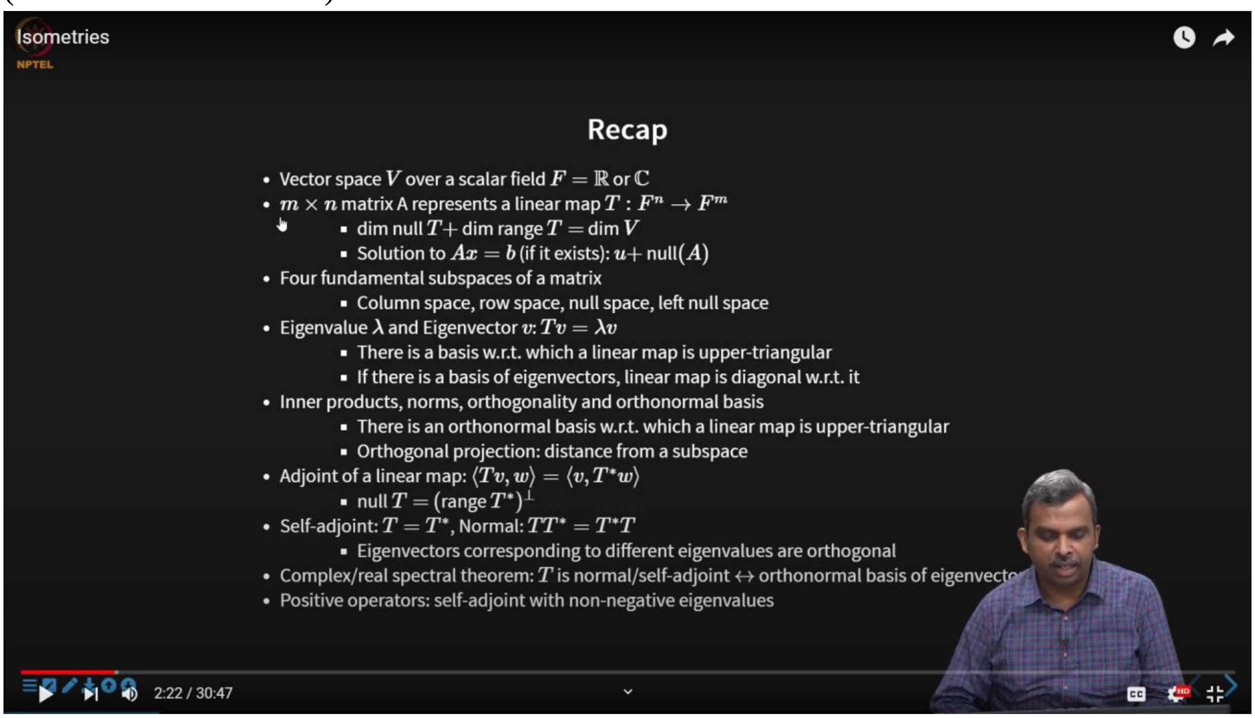

So already we have seen, you know, self-adjoint, normal operators. The basic thing to remember, so always it's confusing as to, I mean, you may remember the definitions, right? Self-adjoint is $A = A^*$. Normal is $AA^* = A^*A$. But what is the crucial defining property? So for self-adjoint and normal, the crucial defining property is that they have an orthonormal eigenvector basis, right? So that's a very strong property and that gives a lot of properties, other properties that you may expect from these kinds of operators. And then there were these positive operators, which, on top of being self-adjoint, they also have these positive eigenvalues. So that gives them a square root. And they behave in some very nice way. The quadratic form becomes non-negative. And then we saw how these operators and the spectral theorem help you solve optimization problems in a very nice way. And then now we are looking at another type of operators which are called isometries, okay?

So isometries are very special and they are also very simple. You'll see all these results are simple in some ways, but they just sort of add up together in a very interesting non-trivial way. For instance, the projection operator is a great example, right? So projection operator has a basic definition and then you see it's self-adjoint, you see that it's positive, all these nice properties come about in a very interesting way for operators, okay? So let's go ahead and study isometries and complete the picture as far as type of operators are concerned, okay? So here's a brief recap of what we've been studying. Let me just skip it for this time. Let's proceed, okay?

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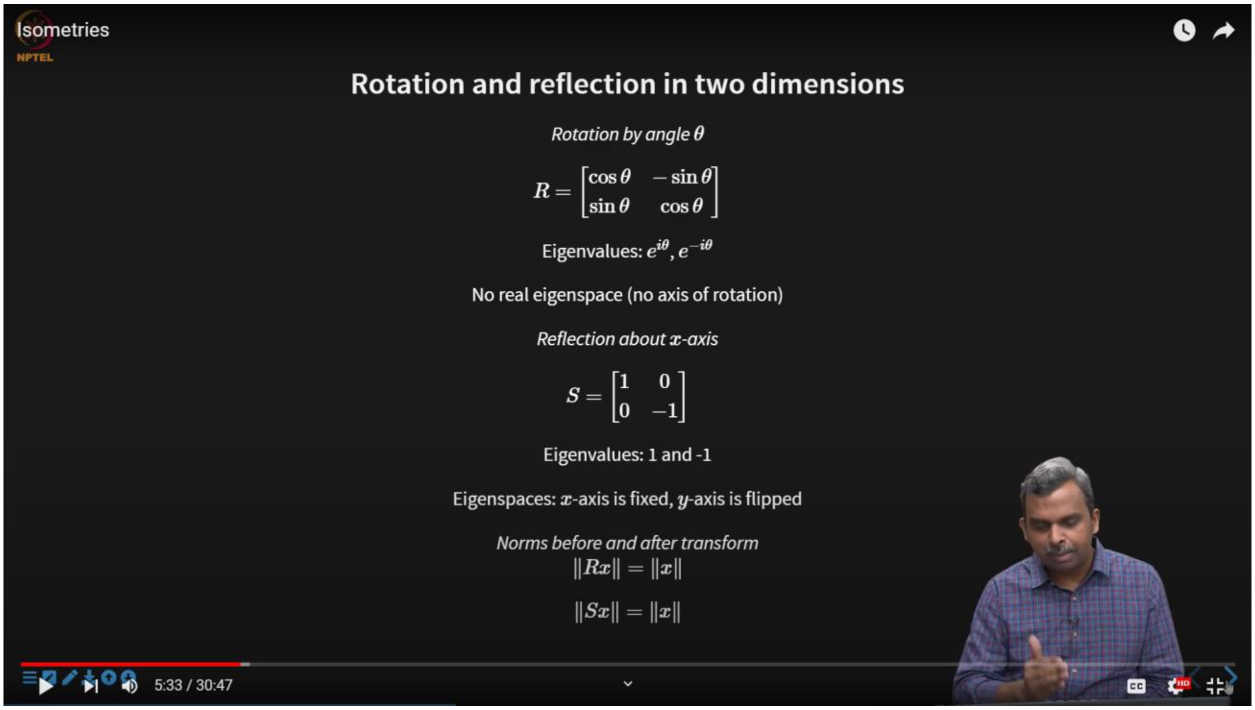

So when we want to look at isometries… Isometries basically mean you can sort of split it as Iso-metry. So iso means something that remains the same metric usually is the norm, right? So isometry, when you think of an operator as an isometry, it is an operator that does not change the norm, right? So that's sort of like a meaning from the word itself. We'll define it formally soon enough, but that's what we're looking for. We're looking for operators which do something to the vector but do not change the norm, okay? So if you take, once again, inspiration from what we know from two dimensions, we have always been doing it, right? So we've been starting with two dimensions, looking at some property and then we are seeing how to generalize that, right?

So if you're looking at two dimensions, we already know a very popular operator which does this, right? So that is rotation by an angle theta. This is a famous matrix. $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ If you put an (x, y), if you operate with this R, you get that (x, y) rotated by that angle θ, right? So it's easy to show. Now if you look at it a bit more closely, it's sort of an interesting operator, as in it does not have any real eigenspace, right? So, because it rotates by an angle θ. So unless θ is zero, right? So it does not really fix any line through the origin. Every line through the origin gets rotated. So there is no real eigenspace. So when you view it in the real, as a real space, so it's a bit of a challenge to look at this. But if you go to complex, once again it doesn't change the norm when x gets multiplied by this. But there are eigenvalues and eigenvectors when you view this as a complex operator, operator on a complex space, right? So this is something interesting to remember.

There is one more type of operator which is basically reflection about a line, and that also fixes the norm, right? So here's an example. $[1\ 0;\ 0\ -1]$ which is reflection about the x axis, right? x goes to x, y goes to $-y$, okay? And this one you can see has a proper eigenspace representation, the x axis is an eigenspace, y axis is an eigenspace. x axis is fixed, right? So whenever you have an eigenvalue 1, it means there is a line which is fixed, right? And that acts like a, you know, axis of the reflection, or axis of the rotation or something like that. So that is a eigenvector with eigenvalue 1, okay? Nothing gets changed on that, okay? y axis sort of gets flipped, okay? Because of the eigenvalue -1. So a couple of interesting examples of operators in two dimensions which are isometries, they do not change the norm of the input, right? The output norm equals input norm, okay? So this is a property that we can see. So now of course our question is how does this generalize to an arbitrary vector space. Now if you're thinking of an arbitrary vector space, thinking of an arbitrary operator, when is it an isometry? What are the various characterizations for isometries in arbitrary vector spaces? Maybe larger dimension vector spaces or even arbitrary abstract vector spaces. So that is where we are going to go forward. But we'll keep this as sort of our inspiration and see if something interesting or something similar to this happens in arbitrary vector spaces also, okay? So let us go ahead.

Here is the definition of an isometry. You have an inner product space over $\mathbb{R}$ or $\mathbb{C}$ and operator $T: V \to V$ is called an isometry if $\left||Tv|\right| = \left||v|\right|\ \forall\ v$, okay? So this is what I meant by saying isometry. It does not change the norm, right? So the metric or the norm of v is not affected by T, okay? So what kind of operators are isometries? That's our main goal in this lecture, we'll try to characterize all the properties that it might have, okay? But let's slowly work our way through, look at a few examples, get some ideas and see where we go, okay? So first example is a diagonal example, right? So always when you are in doubt about operators, you always look for a diagonal example, right? So diagonal is very simple, it will give you a lot of good ideas on how operators will work. So if you were to pick a diagonal operator, okay? Here is an example. If you have λ_1 to λ_n on the diagonal, everything else is 0, then the only condition you need is: absolute value of λ_i should be equal to 1. Anytime this is true, it is an isometry, right? So you can see that if you operate it with x, you simply have, you know, $[\lambda_1 x; \lambda_2 x]$ etc. And then you take $||Ax||$, you only get, you know, $|\lambda_1|^2|x_1|^2$ + etcetera. And if $|\lambda_i|^2 = 1$, then it goes back to the original norm, okay? So this is a very powerful example to keep in mind. A diagonal with absolute value 1 is an isometry, okay? So, I mean, when you say absolute value 1, you are thinking of the unit circle, isn't it? $e^{\iota\theta}$, in complex numbers, you are thinking of $e^{\iota\theta}$. If you are in real space, it's only -1 and $+1$, okay? So that is a nice example to have, okay? So another interesting example. And this is an example in three dimensions and you can see this will also be some sort of a rotation, right? It fixes the x, but then y and z gets rotated by θ, okay? In the yz plane, you rotate by θ, okay? So this is an interesting idea as well, and this has a, you know, you can say the x axis is fixed by this rotation. So x axis is the axis of rotation and then you rotate about the, rotate in the yz plane, you rotate by θ, right? So θ anticlockwise, okay? So that is clearly a rotation as well. And you can prove it, you can write down x and then $||Ax||$, assume complex also, assume x is complex and then write it down, you'll see $||Ax||$ will be equal to $||x||$, okay? So this is also an example. So these are typical examples. A, for instance, the second one is also a, you know, you can think of it as an isometry in real space, right? It's interesting that way. And A is actually real, complex, everything. And so there is no problem here, okay? Hopefully these examples give you an idea of how isometries are going to look. They're going to look sort of similar to what we had in the 2D example, right? Not very different.

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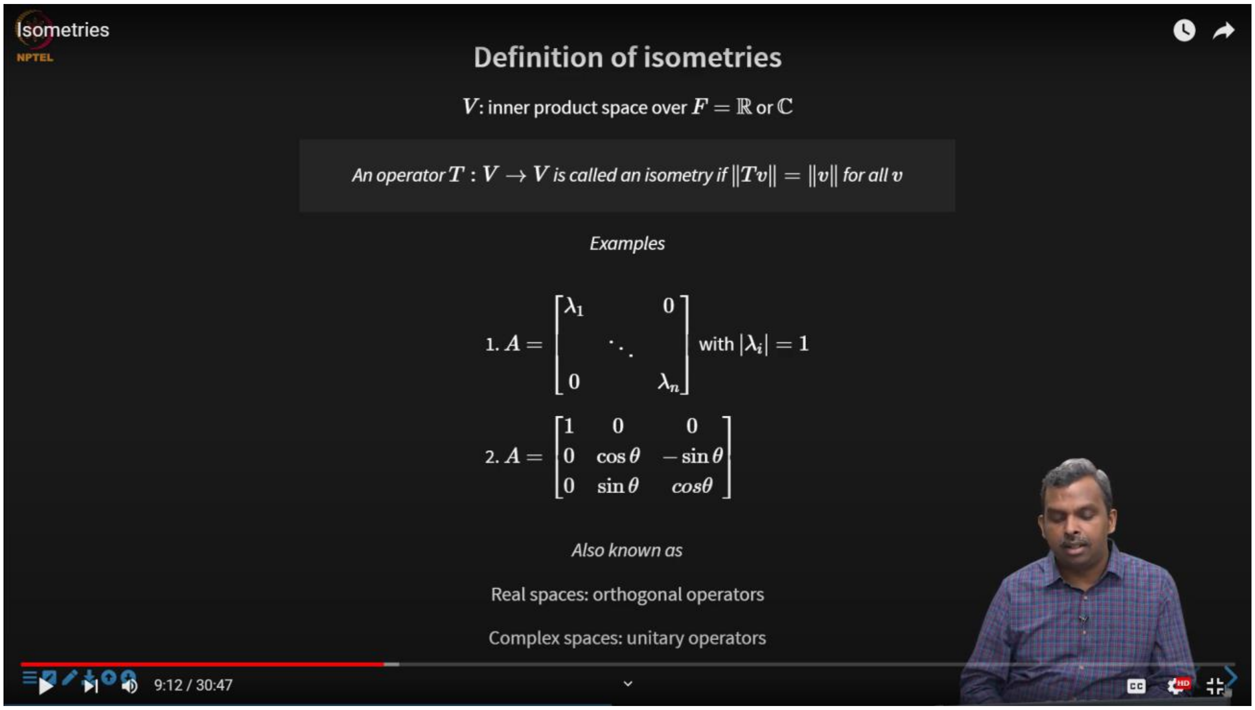

So a couple of other terminologies that you should know. In real spaces, isometries are also sometimes called orthogonal operators. These are called orthogonal matrices, orthogonal operators, you will hear this. And also in complex spaces, isometries are called unitary operators also, you know, unitary matrices, okay? So just like, you know, we kept saying self-adjoint. But self-adjoint in complex is called hermitian, self-adjoint in real is called symmetric, like that there are multiple names. And here again isometry is what is a name that we are using. In real spaces it's called orthogonal, complex basis it's called unitary, okay? So this is the definition. Hopefully the definition is clear. Let's see what the definition means. You'll see in the characterization, we will do a lot of simplification, okay? One interesting question we can ask is, see again, we asked in the examples, we looked at when are diagonal matrices, when do diagonal matrices become isometries? We saw that the absolute value of the diagonal value, diagonal entries should be equal to one, okay? Now normal operators is another question you can ask. Normal operators are almost diagonal, right? There is a basis, orthonormal basis in which it is diagonal. So you can sort of extend that diagonal idea to normal also.

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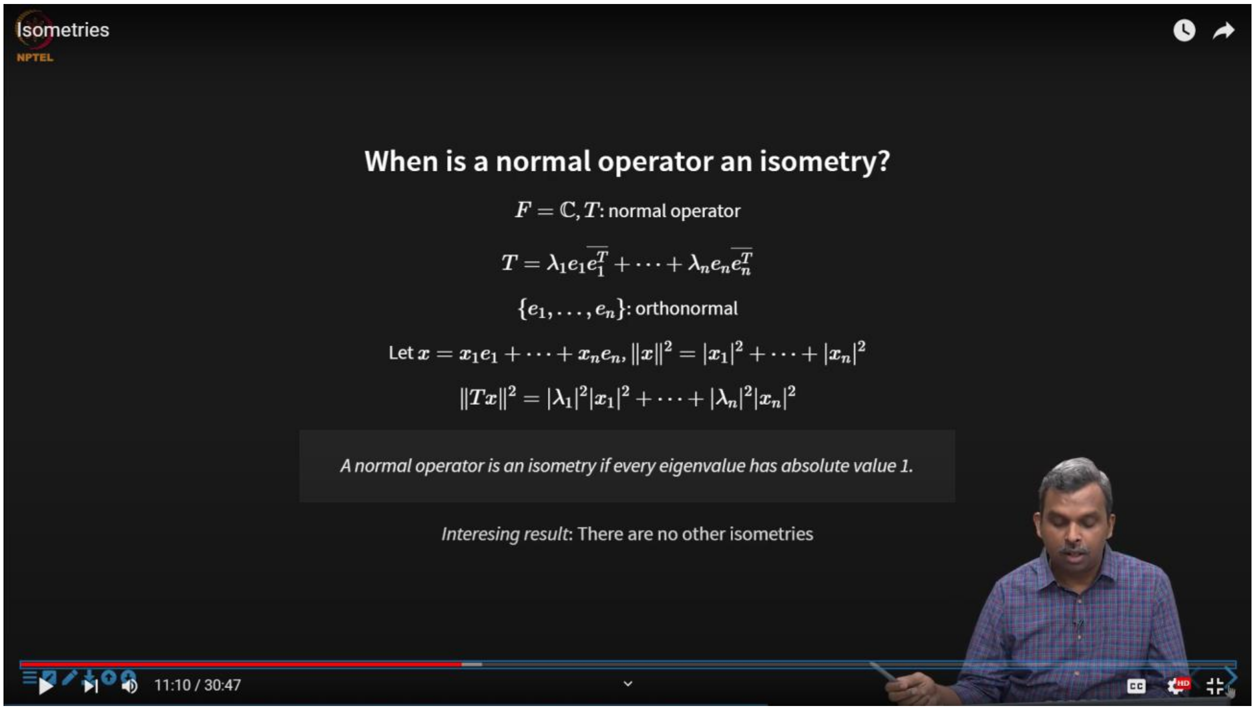

Supposing T is a normal operator. I know T can be written in this form where $\{e_1, \ldots, e_n\}$ is orthonormal. You have an orthonormal basis of eigenvectors. λ_1 to λ_n is eigenvalues for T, okay? We know all this. This comes from the spectral theorem, okay? I am looking at complex space here, okay? So if you start with an x, you can write x as a linear combination of e_1 to e_m, okay? $||x||^2$ of course is $|x_1|^2 + \cdots + |x_n|^2$, okay? If you look at Tx, okay, so what is Tx? If you have x equals this, Tx is simply $\lambda_1 x_1 e_1 + \cdots + \lambda_n x_n e_n$, okay? It's almost like a diagonal, right? So x_1 gets multiplied by λ_1, x_n gets multiplied by λ_n, that is what happens. This matrix, this operator is diagonal in this basis. So this is what you get. So $||Tx||^2$ is simply this. It's almost like it's diagonal. Same thing. I mean whatever we did with diagonal will hold good here also. So we see that a normal operator is an isometry if every eigenvalue has absolute value one, okay? So that is the simple result here. It is sort of similar to the diagonal result, but except that, you know, it is true for the normal operator also, okay? So that is interesting.

So what is interesting is, when we finally characterize, we will see that there are no other isometries. So I mean all isometries have to be normal, okay? Of course, there is this complex problem here, but anyway it's okay, I mean real numbers after all are inside complex numbers so it's not a big deal. So that you, one can sort of say that there are no other isometries. All isometries will have to look like this, okay? So this will be a powerful characterization that we will do to characterize isometries, okay? So that is the next slide, okay? So this slide gives you the complete sort of characterization for an isometry, okay? We are going to start with an inner product space over real or complex and $S: V \rightarrow V$, okay? And once again we are doing this all of the following are equivalent, okay? So we've been doing this quite often. When we say, list a bunch of things and say they're all equivalent, any one implies all the others, right? So that's what it means. And how do you prove it? You sort of prove it in sequence. 1 implies 2 implies 3 implies 4 implies 5 implies 6, and 6 implies one. So there is a cyclical thing. So anywhere you start, you will go back and imply everything, okay? So that's the idea, okay? So what it means, let's just quickly go through them. First is: if S is an isometry, we know that, you know, $||Su|| = ||u||$, $||Sv|| = ||v||$. Not only that, isometries will have to preserve inner products, okay? If you have two vectors u, v and they had a certain inner product before you hit it with S, after you hit with S also $< Su, Sv >$ have to have the same inner product. Remember, in the definition of isometry, we mentioned, we did not expect all possible inner products to be fixed, right? So we just said norm has to be fixed. But just because norm is fixed, inner product also has to be fixed, okay? So it's a very interesting little result. We'll prove it, you'll see maybe some of, you can think of the proof itself. The proof will work out quite nicely. So isometries preserve inner products also, okay? For two different vectors, okay?

So once you preserve inner product, the other results are easy. So if you have an orthonormal set of vectors, after you hit it with S, you will still be orthonormal. Of course, all inner products are preserved, so it will still be orthogonal, okay? Now the next result, right? There exists one orthonormal basis such that this is true. Of course if, you know for every orthonormal basis this is true, so of course you pick one orthonormal basis that you like and this will be true. The 4 is not too bad, but you know, it's sort of like, you know, 4 implies 1, okay? So that is why you should see why that 4 makes sense. If there is at least one orthonormal basis so that after S you are, you continue to be orthonormal, then S is an isometry, okay? S will fix every, I mean, will preserve every inner product. S will preserve every norm, okay? So that's what, that's how you read it. That's why this 4 is interesting, okay? Now the next step brings in the adjoint, okay? So once you have that, it turns out the adjoint-product, operator-adjoint product $S^*S = SS^* = I$ which is the identity, okay? So it's a powerful result. So far we have not seen something of this type, right? We have not really seen operators of this type. So this also means of course S is invertible and $S^{-1} = S^*$, okay? So we saw one example when you put the orthonormal vectors one after the other, you know? That, for that matrix, S^{-1} is S^*, right? So it's, we've seen this before, but, you know, isometry is the actual name for the operator, when you have $S^{-1} = S^*$, okay? And from here it's also clear that S^* is an isometry, okay? So S is isometry, it's adjoint is an isometry. $SS^* = S^*S = I$. So a lot of interesting things come about because of these properties, okay? So we will quickly prove it and then we will state a few more results and that will be the conclusion of this lecture, okay?

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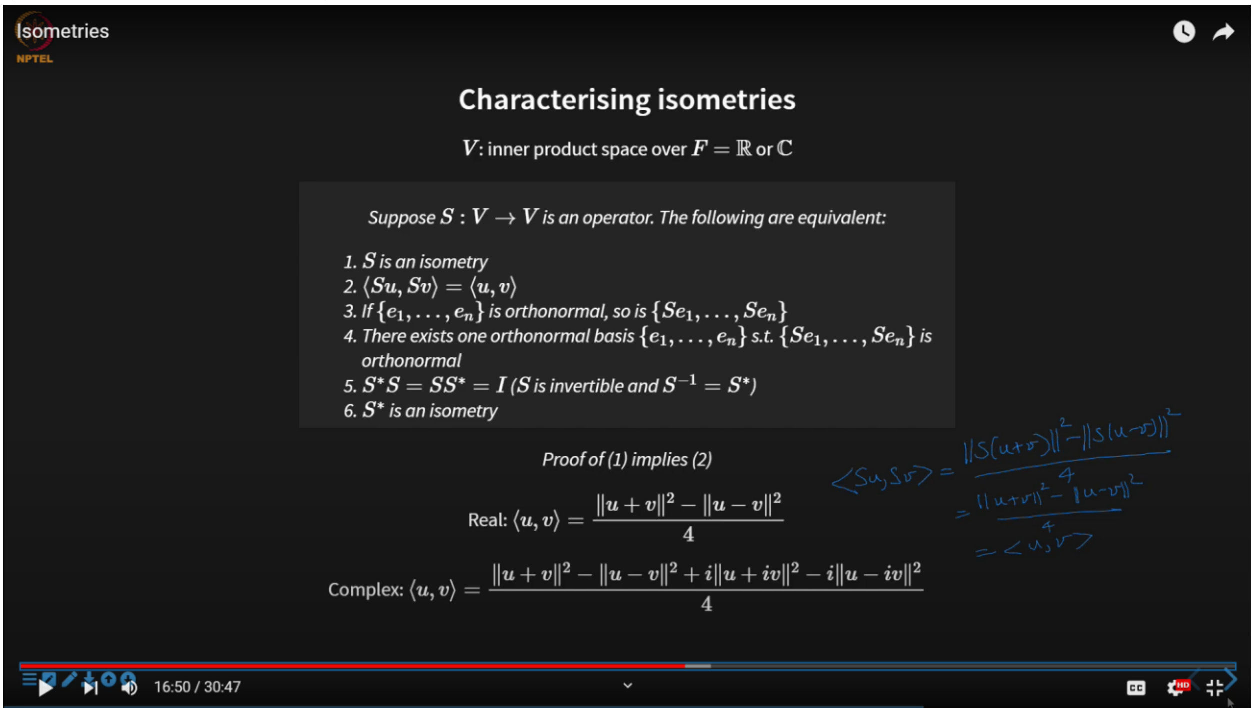

So let us prove it. So 1 implies 2. Why does it, if norms are preserved, why do inner products have to be preserved? The reason is, you can define inner products in terms of norms, right? So we have seen this before in one other proof earlier when we looked at self-adjoint, okay? So if inner products… Inner products can be defined using norms, okay? So given a norm, you can go back to the inner product. So if norms are preserved under S, inner products will also be preserved under S, okay? So I can just show you one little writing, okay? So if you have this, so $< Su, Sv >= \frac{\left(||S(u+v)||^2 - ||S(u-v)||^2\right)}{4}$ okay? In the real case. In the complex case also the same thing works out. And so how did I get $S(u+v)$? $Su + Sv$ is $S(u+v)$, right? And now if S is going to be preserving norms, S is an isometry, right? 1 implies 2. So S is an isometry. So S preserves norms. So that is simply equal to $((u+v)^2 - (u-v)^2)/4$, and that is equal to

$< u, v >$, okay? Same thing in the complex case. So you can show quite easily because inner products can be computed using norms and… Linear combination norms, you know, not something else. If you preserve norms, you have to preserve inner products also, okay? So that's a nice thing to know, right? So angles are preserved by isometries, right? Norms are preserved. And if you have two vectors, the angles between them are sort of preserved by isometries. It's a good intuition to build up in terms of how isometries work.

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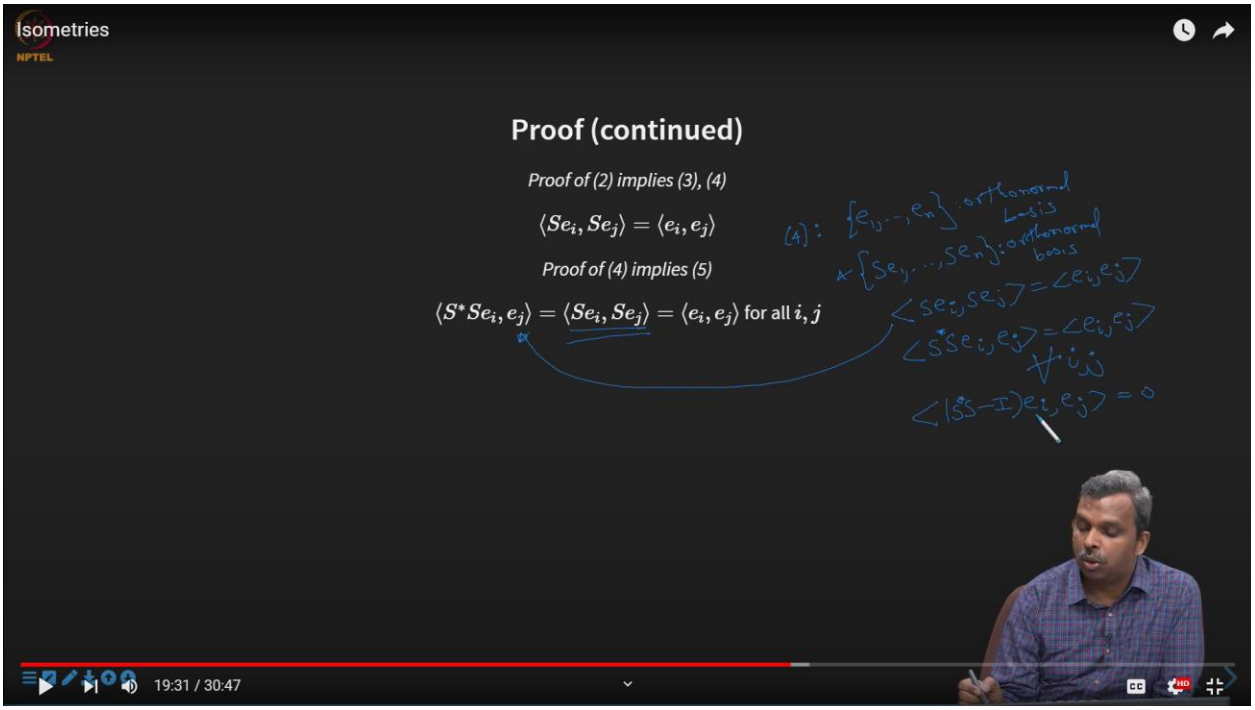

Okay. So, I mean, 2 implies 3, 4 is sort of trivial once you have the inner product being preserved. If e_i are orthonormal, $< Se_i, Se_j > = < e_i, e_j >$. So if e_i, e_j are orthogonal Se_i, Se_j will be orthogonal, everything is okay, right? So it is, the norms are also the same. So 2 implies 3, 4 is quite easy. Hopefully you see 3 and 4 are sort of the same thing, right? If, I mean, any inner product space has an orthonormal basis. So once you have an orthonormal basis, you take that, you operate with S, you're going to get another orthonormal set. So 3 and 4 are sort of the same, okay? Now 4 to 5 maybe needs a little bit of work, it's not too hard, okay? So 4 is what? 4 is: there is an orthonormal basis, so maybe I should write that down… So 4 basically says $\{e_1, \dots, e_n\}$ is an orthonormal basis and $\{Se_1, \dots, Se_n\}$ is also an orthonormal basis, okay? So what happens when I do $< Se_i, Se_j >$, okay? This is my starting point here. That will be equal to $< e_i, e_j >$, right? So that I know, right? So any e_i, e_j take, $< Se_i, Se_j > = < e_i, e_j >$ because this is just orthonormal, right? So this is also orthonormal. This is also orthonormal. This is true. Now this guy we can write like this, okay? So you take one of these S's to this side. So you see

$< S^*Se_i, e_j > = < e_i, e_j >$. So notice this again. So I am getting $< S^*Se_i, e_j > = < e_i, e_j >$ $\forall\ i, j$, okay? For all i, j, this is true, okay? So this is… Okay?

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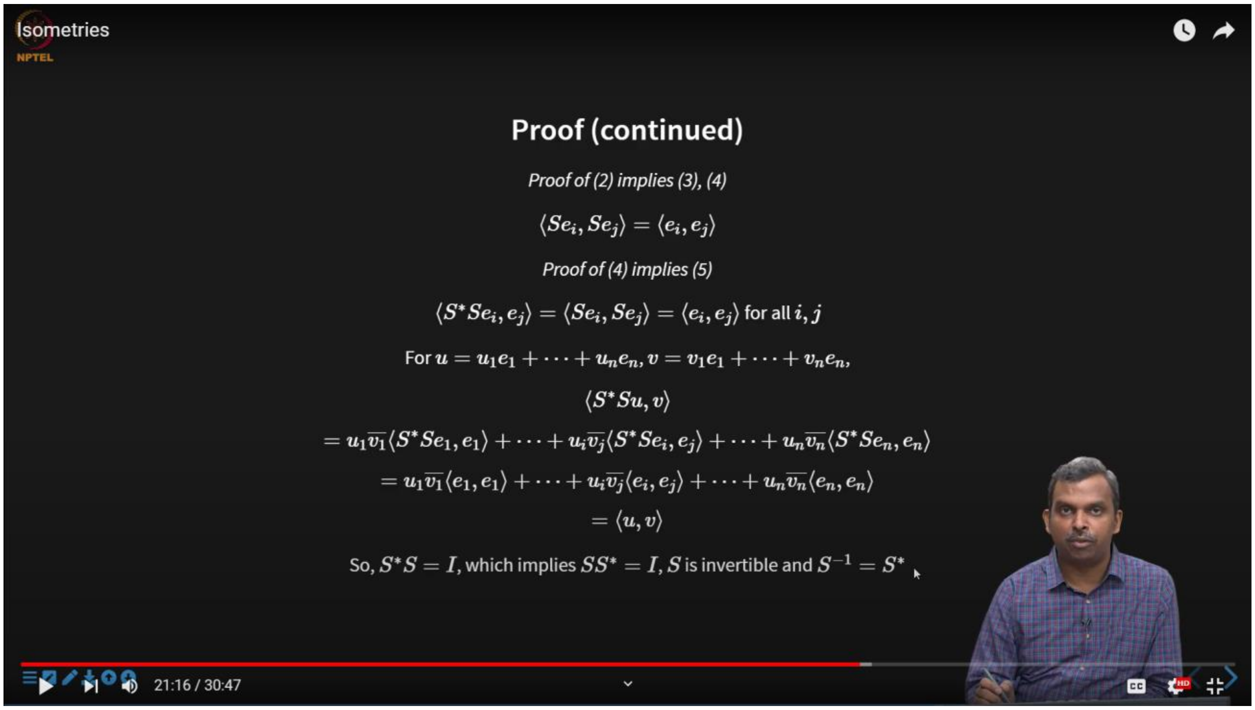

So this is a powerful property. So notice $< e_i, e_j >$. $< S^*Se_i, e_j >$. So you can write it in the other way also if you like. $< (S^*S - I)e_i, e_j > = 0$, okay? So if you want, you can write it like this. So you see this S^*S is going to be equal to I, if you just look at e_i, e_j. But e_i, e_j is actually a basis, okay? So from here you can go to arbitrary u, v if I can show that this, instead of e_i, e_j, I can put u, v here, then I am done. And because e_i, e_j is a basis, that will also work, okay? So that is the basic idea. So let's see how that is done. If you take an arbitrary u which is some linear combination of e_1 to e_n, and another v which is a linear combination of v_1 to v_n, you can look at $< S^*Su, v >$ and then write u and v in terms of this expansion and use linearity and, you know, other properties, simplify, bring it out. You will end up getting $< S^*Se_1, e_1 >$ and then you can just drop the S^*S as long as you have only e_i, e_j here, and then bring in the u_i, v_i inside, okay? And then combine it again, you will get u, v okay? So it's just a simple thing. So you show it for an orthonormal basis. It is true for arbitrary vectors as well. So $< S^*Su, v > = < u, v >$. So S^*S has to be equal to I, right? So for all u, v, this is true. So it is equal to I, okay? So that is the proof to go from 4 to 5, okay? So once S^*S becomes equal to I, if you remember, A and B are operators and $AB = I$, then $BA = I$. So we have seen this is true, right? So this is from… Very old results in basic operator theory. We have seen when we studied invertible operators, we saw that if AB is I then BA is also I. So if S^*S is I then SS^* is also I. So clearly this means S is invertible and $S^{-1} = S^*$, okay? So this is the definition of invertibility. You can go back and check your very early results on operators, I mean without any eigenvector, any inner product or something we proved it, right? So this is true. The inverse is unique and all that we have proved before. So this comes directly from that, okay?

And the last proof is to show that 5 implies 6 and 1. So 5 is, you know, $SS^* = S^*S = I$. And that implies S^* is an adjoint and also that… S^* is an isometry and also that S is an isometry, right? So that is going full circle, right? 5 implies 6 and 1, that is very easy to show, okay? It's not very hard. You look at $||S^*v||^2$ is inner product $< S^*v, S^*v >$ and then S comes this side, SS^* is I, you get that same thing with $||Sv||^2$, $< Sv, Sv >$, $< S^*Sv, v >$ and then S^*S goes away. So this, once you have, you know, 5 is what, $S^*S = SS^* = I$, so once you have that being I, the way the norm becomes same is also true, okay? So this has completed the proof.

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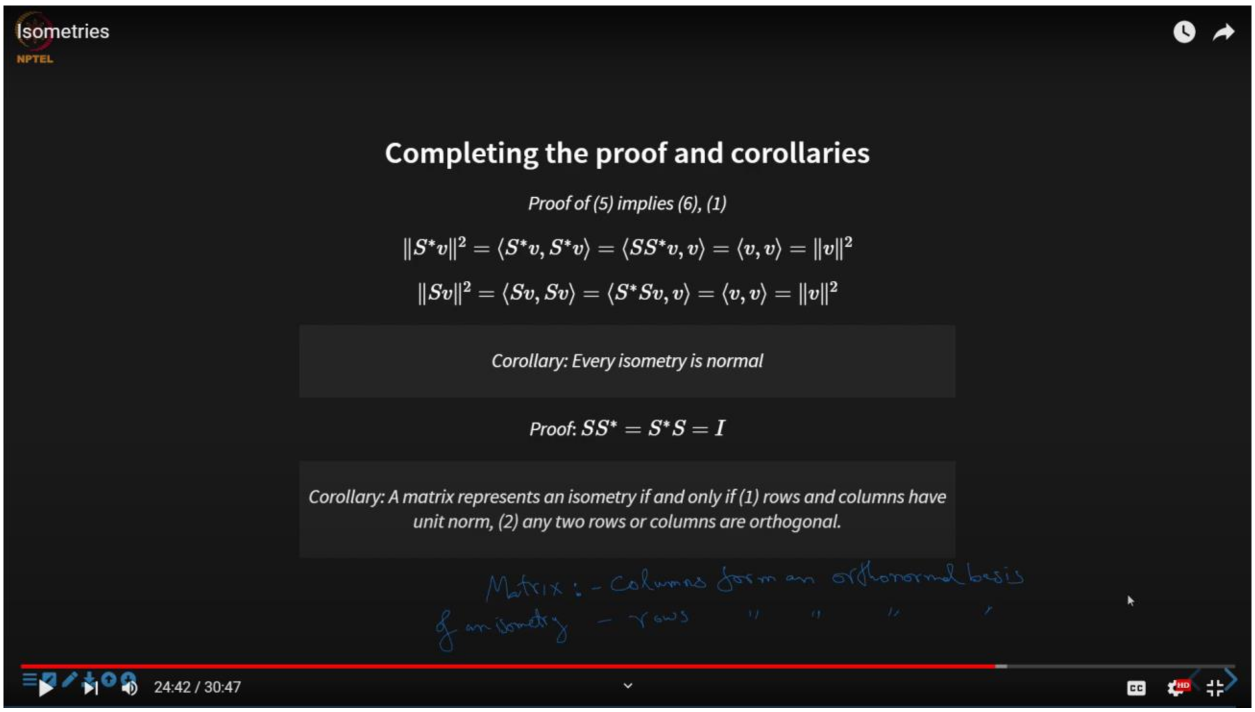

So let me just quickly go back and remind you of all the powerful characterization. So this is the… 5 is sort of the most powerful characterization, right? If S is an isometry. And 2 also is important. If S is an isometry, inner products are preserved, not just the norms. And then you have this nice $S^*S = SS^* = I$, operator-adjoint product is actually equal to identity, okay? So that's a very powerful relationship. And that is sort of if and only if. If an adjoint is also an isometry. Everything works out in that, okay? So this is sort of a complete, nice characterization of what isometries are, okay? So a couple of corollaries. These are very quick and easy corollaries. Every isometry is normal, right? So $S^*S = SS^* = I$. So clearly $SS^* = S^*S$. So that is normal, okay? So it's a little more than normal, but of course it's normal for sure, okay?

And if you think in terms of matrices, okay… So let us say somebody gives you a matrix and you want to find out if it is an isometry or not. What do you do, okay? Matrix represents an isometry if and only if the rows and columns have to have unit norm and any two rows or columns have to be orthogonal, okay? So it is basically the columns should be an orthonormal basis, rows should be an orthonormal basis, okay? If that is true, then it's an isometry, otherwise it's not, okay? So it's a very straightforward way to define it, okay? So you can go back and check all our examples. So you will see that whenever we had an isometry, all the columns will be an orthonormal basis, all the rows if you take them together, you will have an orthonormal basis. Maybe I should write that down, that is easy. So in terms of matrices, all, matrix of an isometry. Columns form an orthonormal basis. Rows also form an orthonormal basis. So basically what is isometry doing, right, if you think about it, once you have this characterization, isometry is simply taking your coordinates and then, you know, multiplying by some orthonormal basis. So basically you're just simply moving to another basis which is in the, which is orthonormal in some sense, right? So it's sort of simple in that way. So this characterization is a very nice and complete characterization for isometry.

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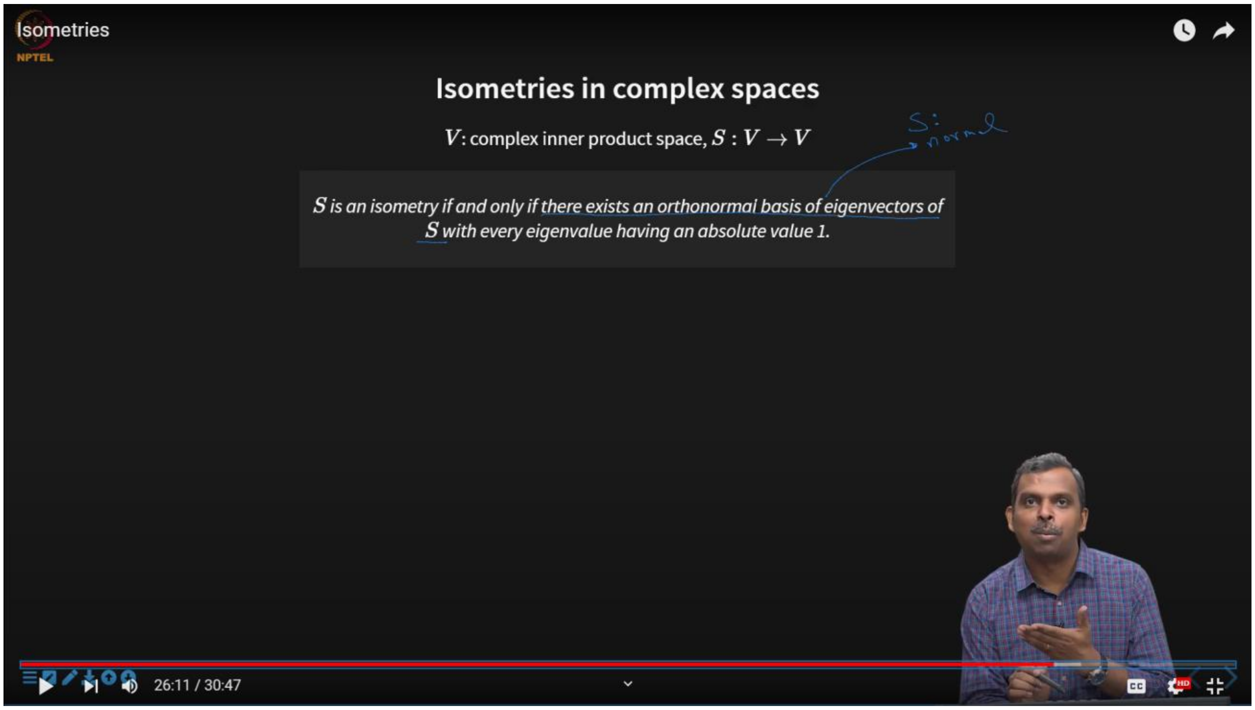

Okay. So finally if you are in complex space, if you are not worried so much about being real, your eigenvectors being real and eigenvalues being real and all that, once you go to complex vector space, you can very easily characterize isometries using another way also, okay? S is an isometry if and only if S is normal, right? Normal meaning there is an orthonormal basis of eigenvectors, right? So that is the same as normal. And every eigenvalue has to have an absolute value 1, okay? So this is, there exists an orthonormal basis of eigenvectors of S. This is the same as normal, right? So if S is normal and absolute value of its eigenvalues, every eigenvalue has absolute value 1, then it is an isometry, okay? If S is an isometry, S is normal and eigenvalues have absolute value 1. If S is normal and eigenvalues have absolute value 1, S is an isometry, okay? Both of these are true. So it's sort of a complete characterization, except that, you know, I mean you have to go to a complex vector space, you have to allow for complex eigenspaces. If you don't want that, then I guess this is not a very complete characterization. If everything has to be real, then maybe you need a little bit more of a thing, right? So, because this orthonormal basis may have complex entries, right?

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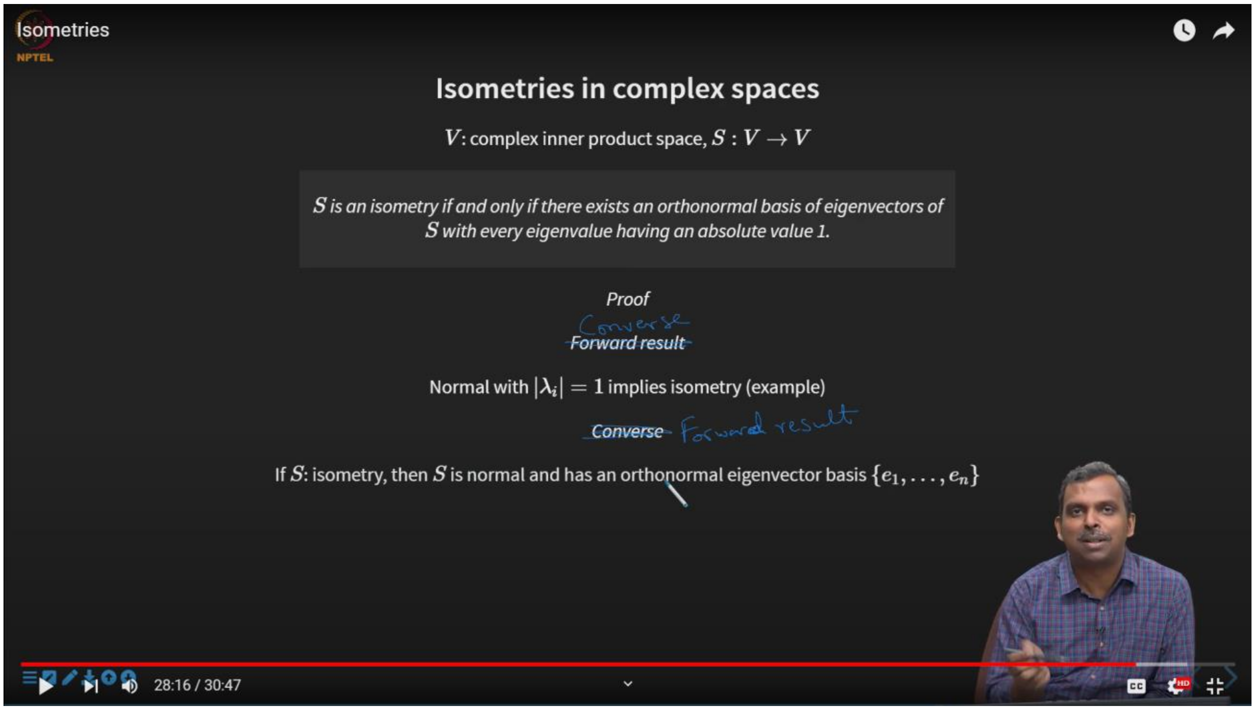

So if you look at the example $[cos(\theta)\ -sin(\theta);\ sin(\theta)\ cos(\theta)]$, if you want an orthonormal basis for it, you have to go to complex, right? Otherwise you're not going to get that basis of eigenvectors. For it, you would go to complex, otherwise you're not going to get it, okay? So if you're okay with going to complex, this works out. If you insist on being real, maybe you need something more. But, you know, this is quite nice, you know? This is very nice and complex is not too bad in most cases, okay? So this is an isometry. A proof is very easy, you know? I mean, forward result: if S is an isometry, we know S is normal and if you have $|\lambda_i| = 1$, we already proved it's an isometry, right? So we did a quick proof for this result when it's a normal operator. Symmetry. For the converse. If S is an… Okay, so I'm sorry, so for the converse, if S is normal, okay… So I'm sorry about this. I think this is not quite correct. So you should not have this. So I think I had the whole thing wrongly written out. I think the proof goes wrong. So this is the converse. Sorry about that. This is the converse and this is the forward result, okay? Sorry. So I wrote it down wrongly. So, it's okay, I think the proof itself is okay. I have just written down the result wrongly, okay? So converse is: if S is normal, as in, if there exists an orthonormal basis of eigenvectors, if it is normal and absolute value is 1, we have already shown, okay? The converse we've already shown. The forward result is: if S is an isometry, then of course S is normal and it has an orthonormal eigenvector basis, right? So this much is true. So the only thing we have to show is: if S is an isometry, its eigenvalues will have absolute value 1, okay?

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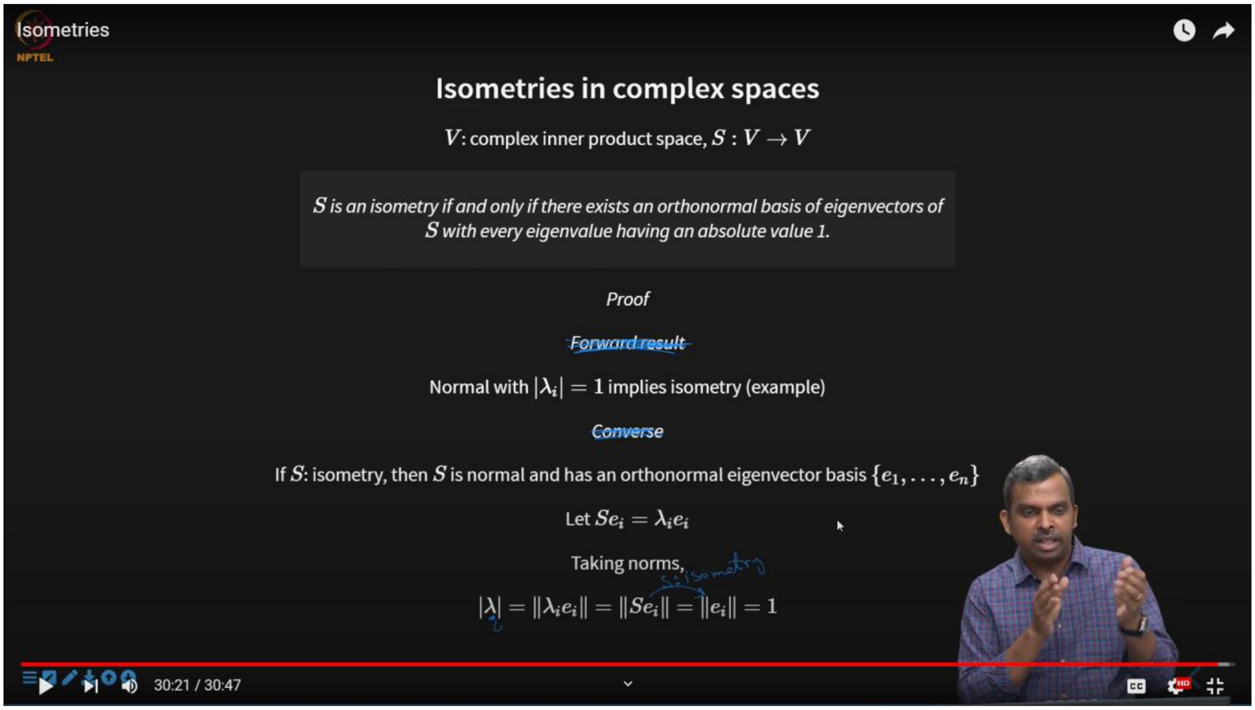

So that is the proof we have not done so far, right? If S is an isometry, then S is normal. I know that. So there will be an orthonormal basis of eigenvectors, okay? That much we have seen. But what about its eigenvalues? Why should its eigen, why should its eigenvalue be 1, okay? So maybe we already showed it. But anyway we can prove it in one more way, just for us to be happy. But this will mean I'll have to rewrite this, okay? So let me just cut it out, okay? So first was the converse, this is the one, you can prove it in yet another way that absolute value of eigenvalue should be 1. So supposing you say Se_i is $\lambda_i e_i$. So these are the eigenvalues. You can take norms. If you take norms on both sides, on the right hand side you will get $\lambda_i e_i$, the norm of that is just λ, okay? Not λ, λ_i. Today is the day of mistakes. Quite a few mistakes are there, okay? So this is λ_i, okay? $|\lambda_i e_i|$. And of course the left hand side is $||Se_i||$. Now S is an isometry, right? How do I go from here to there? S is an isometry. If S is an isometry and it has an eigenvector with eigenvalue λ, then $||Se_i||$ is the same as $||e_i||$ and that is one, okay? So you will get $|\lambda_i| = 1$, okay? So it's very easy to show that, you know, eigenvalues should have absolute value 1. In fact, I mean, you can sort of go back to our example and convince yourself that they should have absolute value 1. Otherwise they won't be an isometry, right? It's the same thing that we are doing here, okay?

So this sort of concludes what I wanted to say about isometries. Isometries are very interesting operators. Basically, if you think in terms of matrices, rows and columns are orthonormal, right? That's a very easy characterization. And that's a complete characterization for isometries. You can think in terms of so many other terms. It should be normal with eigenvalues having absolute value 1 and all that. And another interesting thing is just because it's an isometry, it also preserves inner products, okay? So that also is very useful in practice. The fact that it's an operator that preserves inner product. So people sort of say that if you have a vector space and if you use an isometry on it, you're not really changing anything, right? So nothing is being changed. Inner products are preserved, norms are preserved. So the relationships between vectors are preserved. Everything is sort of exactly the same. But if you multiply with even an invertible operator which is not an isometry, then you know, then you're changing the relationships between vectors, you're changing the inner product, okay? So isometries are very powerful and they have quite a few applications, okay? So that's the end of this lecture. Thank you very much.