Applied Linear Algebra

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Week 11
Classification of Operators

Hello and welcome. So to conclude this week's lectures I thought I would record something, a very short video to describe all these various types of classifications of operators based on various different properties and ideas and sort of present them in one way. It's not complete or anything but still I think hopefully it'll give you a, you know, it'll be useful for refreshing your memory on what these operators are and how they are all related to each other, okay? So maybe it's good to see it.

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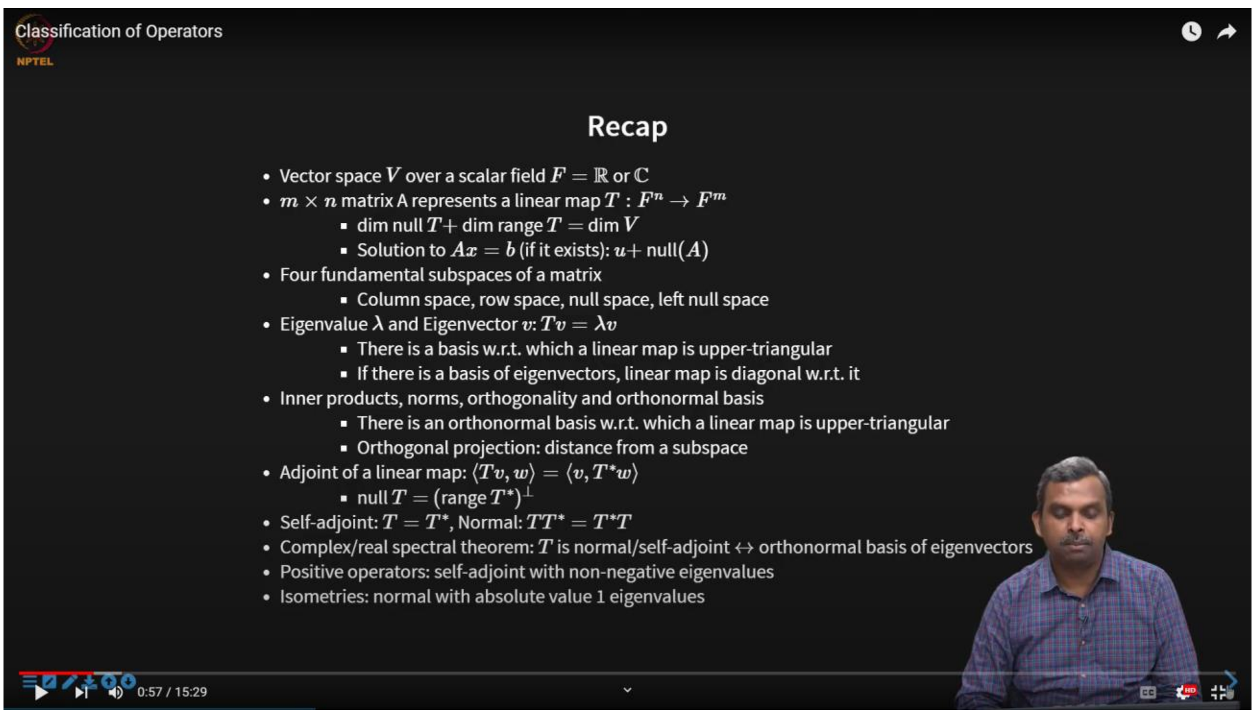

A quick recap. So I think this recap will be useful because some of it, some of what I am going to say will throw you back to what we studied a long while ago. We've been looking at vector spaces over a scalar field which is real or complex. And there is this linear map which is the key object of study for us. And it's represented by a matrix. And null and range space are important to understand what the linear map is. And their dimensions add to the overall dimension. That is a very nice result. Then we can use it to solve linear equations. And then there are these four fundamental subspaces of a matrix and their connections to the operator, it's adjoint and all that. And then eigenvalue and eigenvector played a very crucial role in simplifying the matrix representation and then telling us more about what the operator is doing. Once inner products came in, we used orthogonality, orthonormal basis, simplified quite a bit of things about operators. In particular you know that there is an orthonormal basis with respect to which there is an upper triangular matrix for any operator, okay? And then orthogonal projection is something we studied. Adjoint. All the properties that adjoint has. And then we started looking at these various types of operators and special properties that hold for them. Particularly self-adjoint, normal, positive and isometries. We also studied projections a little bit, but you know, these are the four main operators in inner product spaces. Then they have a lot of connections to each other and very interesting properties and all that. And I will say a few words from a high level about all these 4 types of operators in this lecture, okay?

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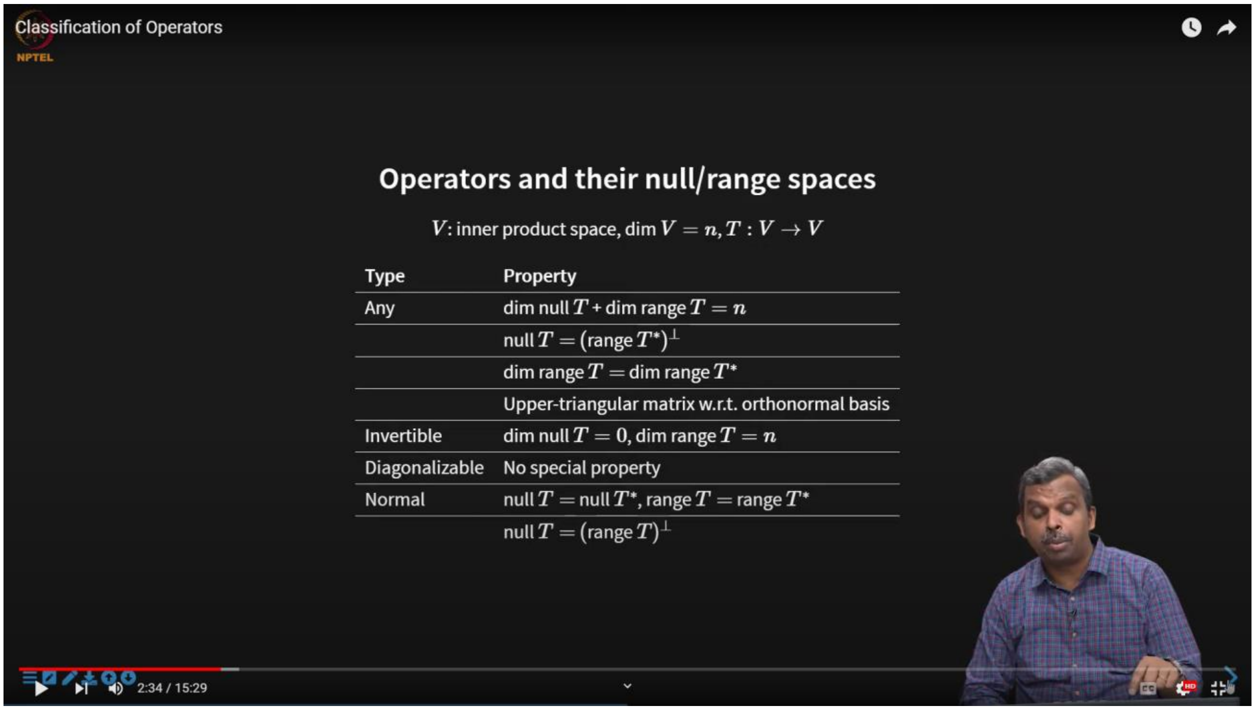

So let's look at the first classification, the first way of thinking of all these things. So you remember we also had this notion of an invertible operator and diagonalisable operator. So this also is very important. And we haven't maybe talked about it quite a bit. Invertible means the, you know, range is the entire space. Then you can go back, right? So that's one way of defining. For operators that's good enough. Diagonalisable is something which, you know, you know that there is an eigenvector basis and it can go to diagonal representation. But in this slide I wanted to talk only about null and range space. And what are the various things we know about, types of operators and their null space and range space and all that. I wanted to collect all of that together. If you take any operator, we know that the dimension of the null and the dimension of the range will add up to n, okay? And then we also have this nice result connecting the operator and its adjoint. The null of T is the range of the adjoint perp, the orthogonal complement. And range(T) and range(T^*) have the same dimension, this is the row-column rank result that we know. And then there is an upper triangular matrix with respect to an orthonormal basis for any operator. So these are good properties for any operator. In particular if the operator is invertible, the range is going to be full dimension, null is 0, right? That's a nice, simple property to have. Diagonalisable, really I mean, null and range, it doesn't limit it in any way, right? So you can have all sorts of null spaces and range spaces and diagonalizability is possible, okay? So there is no special property directly that one can talk about for diagonalizability, okay?

Now once you have normal… So there is this self-adjoint, if T is T^*, then T and T^* do the same thing. But there are these normal operators which, you know, commute with this adjoint. Not exactly equal but they commute. So almost like, you know, similar in some sense, right? So TT^* is T^*T and you see lot of things about T and T^* become the same once it's normal, right? Null space is the same, range space is the same. In fact null is the orthogonal complement of the range for the operator, for a normal operator. So these are all nice points to remember. Now these are all just defining properties. It's not like, you know, I mean… Let me repeat myself. These are properties that these types of operators satisfy. Some of them are if and only if, some of them are not, right? So I think you'd have to be careful about it. You can have null T being $(\text{range } T)^{\perp}$ without the operator being normal. All of these things are possible, okay? So many of these properties, particularly for normal and all are not if and only if, okay? But some are if and only if. Like, for instance, for invertible that's the defining property. And various other things. So I just thought I should collect all these things together in one place in case you need to use it for something. This is how it works, okay? Of course, you know, self adjoint is a subset of normal, okay? And also something that holds for normal will hold for self adjoint. And if you think about it, positive operators are also self adjoint. So all these properties for null space will hold for positive. Isometries are also normal. So whatever properties hold for normal will also hold for it. So that's why I've stopped with normal. All the other guys will come in into that, okay? So this is one type of summary.

The other more interesting sort of look is this sort of colourful slide. And I think usually I don't use many colours in the slides. It's just white and black. But this one I thought there is the genuine need for multiple colours to get the idea through. The classification or understanding in terms of eigenvalues and eigenvectors is what makes the whole thing very interesting for operators. We have this big world of operators. I've drawn like a Venn diagram. Rough Venn diagram to give you an idea of how the containment and all works. We have the world of operators. And within that are invertible operators. In terms of eigenvalues if you think about it, invertible operators have all non-zero eigenvalues right? So that's the invertible operators. And then there are these diagonalisable operators, okay? So now diagonalisable operators, the best way to talk about them is in terms of eigenvectors, right? So there should be an eigenvector basis. A basis for the space full of eigenvectors of the operator. Then that is diagonalisable. Now diagonalizable may be invertible, may be non-invertible. So you see that big, you know, partial intersection with invertible. And there is a… I've drawn it very big because there is a, I mean I have to put many more things there. But it's true that diagonalisable is quite big. It's a big set in terms of operators. There are lots of diagonalisable operators out there, okay? So that's the picture for you.

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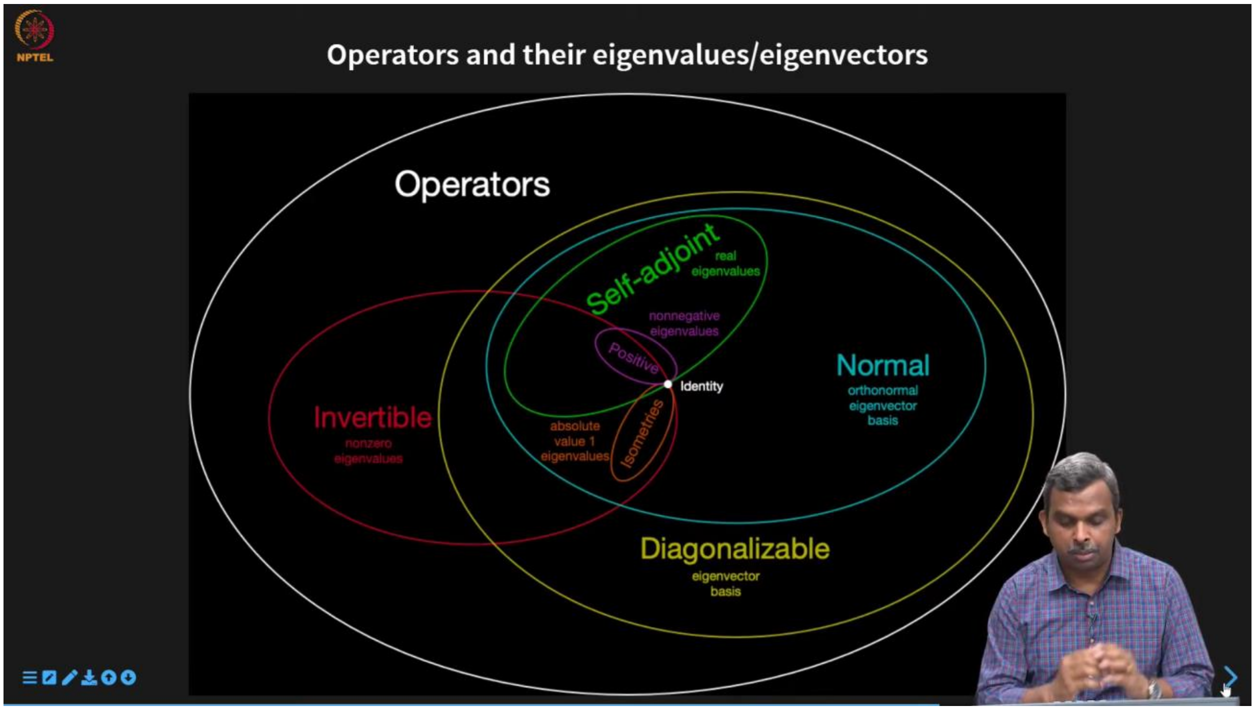

Now inside diagonalisable we go in and we classify way more, you know? Of course there are also lots of operators outside diagonalisable which you might want to classify and all that but we are not looking at that too much. We are looking inside diagonalisable for now, for many of the classifications we looked at. So one big set is this normal operators. So normal operators are very interesting because they are not just diagonalisable, they are diagonalisable with respect to an orthonormal basis, okay? So that's what I've indicated here. So you see normal becomes a subset of the diagonalisable. But normal may be invertible, may not be invertible, okay? So it will also intersect with invertible. But it may be invertible, may not be invertible. So both are possible for normal, okay?

Now inside normal you have these operators called self adjoint operators. These are definitely normal, but the specialization there is that self adjoint operators on top of having an orthonormal eigenvector basis, they are also going to have real eigenvalues, okay? So that is a defining characterisation for self adjoint. One step more. If you have an orthonormal basis and then you have real eigenvalues, then you have a self-adjoint operator, okay? So that's sort of, from normal to self adjoint the specialization is: the eigenvalues are additionally real, okay? Then we studied these positive operators and isometries. Let me come to positive. Positive is quite easy to classify. Among the self-adjoint operators, positive are those which have non-negative eigenvalues. So you go from, you know, orthonormal eigenvector basis you have, and then you specialize on the eigenvalues. Real eigenvalues you go to self adjoint. Non-negative eigenvalues you go to positive. Now because I said non-negative, positive operators may be invertible, may not be invertible. If they have zero eigenvalues, then they will not be invertible. So they lie on both sides of the invertible thing. So you'll have only a partial overlap with that, okay?

And then we studied isometries. Now what are isometries? Isometries are normal and then they have eigenvalues with absolute value 1, okay? Every eigenvalue has absolute value 1. And isometry also, you know, preserves norm, preserves inner product. So it sort of, it has to be invertible, right? So it's clearly invertible. So isometries will lie entirely inside the invertible set, okay? So every isometry is invertible. No eigenvalue is zero, right? So all eigenvalues are non-zero. So it's going to be invertible. Now can isometries be self-adjoint? It's an interesting question. So if an isometry has to be self-adjoint, its eigenvalues should be either +1 or -1, right? So absolute value should be 1. Now self-adjoint means eigenvalues are real. So if you say isometry which is self-adjoint, then almost you are looking at, you know, eigenvalues which are, +1 or -1, right? So only when it's complex it's a little bit more interesting, there's more range. When it's real, when the eigenvalues are real, it's just +1 or -1. You don't have this $e^{i\theta}$ and all that, it's just +1, -1, right? θ is not allowed to vary arbitrarily, okay? So there can be some self-adjoint isometries, but a whole bunch of them you can expect to be not self-adjoint. So that's why I've put a small overlap with self-adjoint instead of a slightly large overlap there, okay? And what about positive? Can you have an isometry which is positive? I think only identity is like that, right? So the identity that way is a special operator, I've shown that. It's everything, right? Everything. It's invertible, it's self-adjoint. Of course it's invertible, it lies on this side of the red thing. It's normal, it's self-adjoint, it's positive, it's isometry, diagonalisable, everything. So identity is sort of there. I think isometry and positive intersect only in identity, okay? Isometry means eigenvalues should be +1 or -1. Positive means eigenvalues are non-negative so -1 even is eliminated. So all eigenvalues are 1, and you have an orthonormal basis, that is identity isn't it? So you have identity coming in there, and that's that, okay? So that's the picture. I don't know if your sense of colour agrees with the colour or not, I don't know if the words and the type of operators and their colours are, you know, compatible according to some code you might have. But, you know, it's clear enough. You can see this picture is reasonable. Maybe I could have made it a little bit more symmetric by adjusting the positive and isometry to have the same size etc. But it's okay, it comes out quite reasonably. So this picture sort of tells you how eigenvectors, eigenvalues are used to nicely classify these various types of operators in relation to their adjoint and clearly you can see how the spectral theorem helps us to really understand what is going on here. Okay? So that's the picture for operators and their eigenvalues, eigenvectors, okay?

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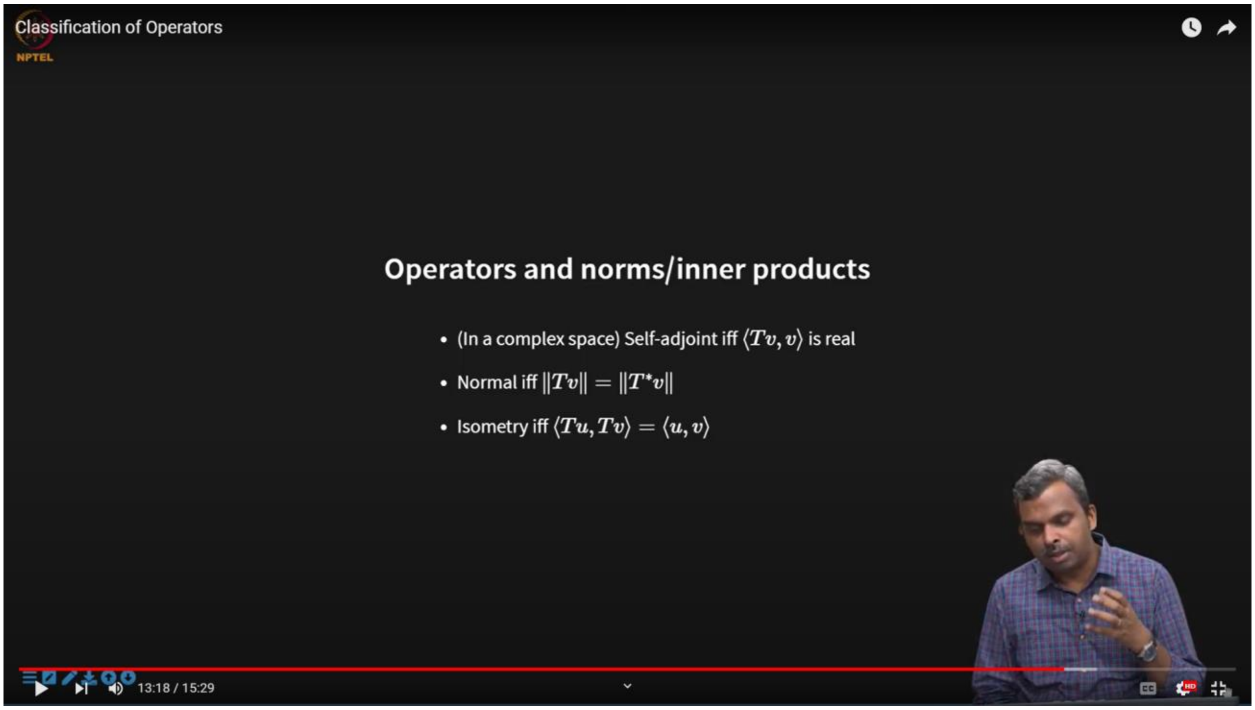

The last piece of information I wanted to talk about is some things which were not covered so far. A few pieces of information which sort of relate to the norms, inner products, but maybe not very obvious or clear in some sense. We saw this nice interesting result that if in a complex space, if you have inner product $< Tv, v >$, this quadratic form, if it is real, then it has to be, then the operator has to be self-adjoint. Both ways, right? It's if and only if, so it's a very interesting characterisation of self-adjoint. And then we also saw normal if and only if. If the norm of Tv and the norm of T^*v are the same. See, remember, self-adjoint means $Tv = T^*v$, right? So T and T^* are exactly the same. So normal, T and T^* need not be the same, $TT^* = T^*T$, but their norm has to be the same, okay? So it's so strong, norm is quite a strong constraint on the operator T. And for isometry, we saw that, you know, norm before the transform and after the transform have to be equal. But more interestingly, for any two vectors, the inner product has to be preserved by the isometry. So that's an isometry. So these are nice, interesting properties. And we also saw this other very interesting property which helps us prove the spectral theorem and all that, right? So normal means eigenvectors corresponding to two distinct eigenvalues are orthogonal, okay? So that's at the heart of everything. I didn't quite put it down. But, you know, we spoke about the spectral theorem before. So it's sort of hidden in there, okay? So hopefully these three slides sort of gave you a, you know, just a high level view on how this operator classification and all of that worked. And hopefully that concluded the last couple of weeks, this sort of summarises all that we have seen so far in terms of classifying operators. So going forward, in the next week, we will study two types of decompositions of operators, okay? The first one is called polar decomposition. It is sort of interesting, but maybe not, I mean maybe it has applications, I have to look at it more closely. But the second one is called the singular value decomposition, okay? And today if you rank applications of Linear Algebra, singular value decomposition will come way, way up on top. It's used in so many different areas. Communications, you know, compression and learning. So many other areas. So singular value decomposition gives you a way to understand an operator, understand a large matrix and all the relationships between the entries there. And we will spend some time to understand how the SVD works and why it is so widely popular. And it gives a very nice and simple way to understand operators. So at the end of it, when you look at SVD, people, you know, are sort of disappointed, saying you should have taught SVD first. But, you know, it takes a little bit of time to work towards that. So I think SVD is good topic to sort of conclude our lectures in Linear Algebra. So next week we'll do SVD and something called Polar Decomposition, okay? Thank you very much.