Applied Linear Algebra
Prof. Andrew Thangaraj
Department of Electrical Engineering
Indian Institute of Technology, Madras

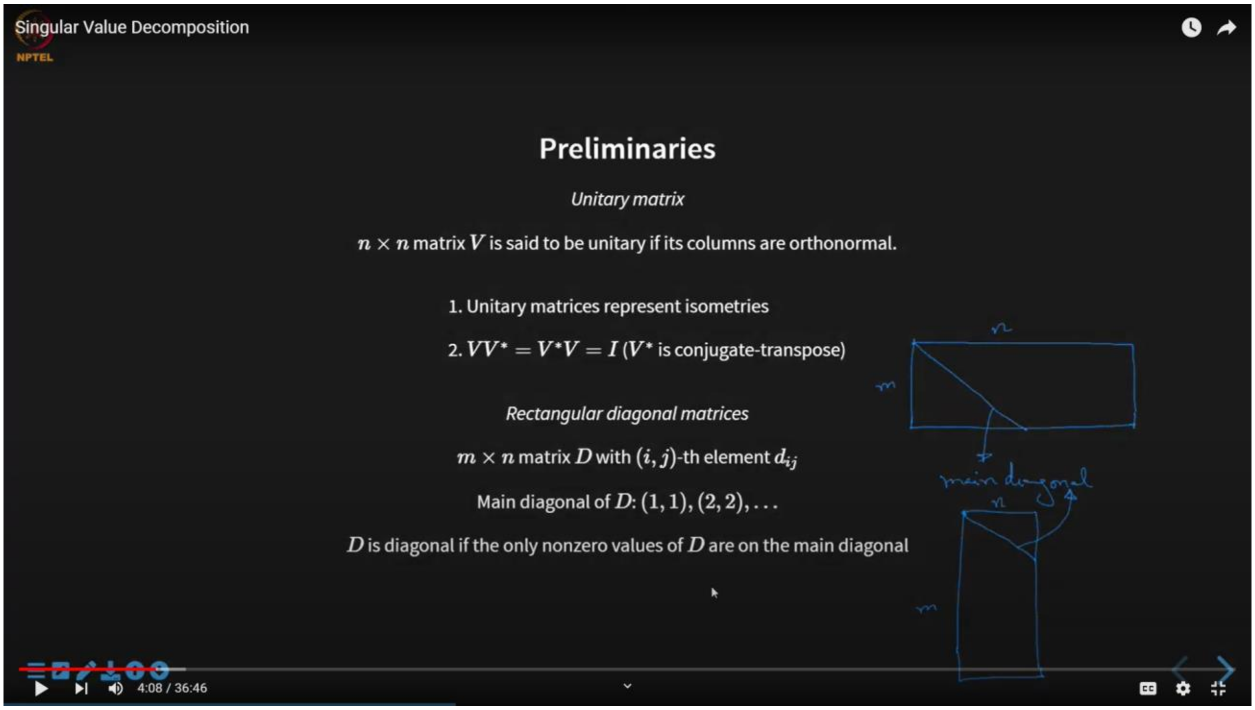
Week 12
Singular Value Decomposition

Hello and welcome to this lecture. We will study what's called singular value decomposition in this lecture. In the previous lecture, we saw definitions of singular values of an operator, of a linear map and singular vectors of a linear map, right? Right singular vectors and left singular vectors. And there was this very interesting relationship between the map and its right singular vectors and its left singular vectors and their singular values, right? So that was something we did not prove in the previous lecture. We just saw it. And we will prove that in this lecture, okay? So it is mainly a proof of what is called Singular Value Decomposition in this lecture, okay? I'll skip the recap. We saw it in the previous lecture. The main idea is that this definition of singular values and vectors which is basically eigenvalues and eigenvectors of T^*T, okay? Singular values and singular vectors of T are basically eigenvalues, eigenvectors of T^*T and TT^* also, right? The left singular value is that of TT^*, okay?

Okay. A few preliminaries before we formally describe singular value decomposition. I will describe it in two ways, one with respect to matrices. It's very popular to study singular value decomposition purely as a matrix sort of result. We'll also see an operator sort of version and finally we'll prove it in the operator version. That's where we'll prove it, okay? So a few basic ideas. Many of these ideas you must already know but I am just trying to reinforce them because these will show up in the definition. And you may not be surprised when you see it in the definition. First is that of a unitary matrix. We've already seen this before. We've seen isometries and we saw that unitary matrices represent isometries, unitary operators, right? These are isometries. The basic definition is: an $n \times n$ matrix V is said to be unitary if its columns are orthonormal, okay? So another way of writing it is VV^* must be the same as V^*V and that should be equal to the identity matrix I, okay? So V^* is basically conjugate transpose. From here you also see that V^* is V^{-1}, right? So that's also something that's important, okay? Inverse is the same as conjugate transpose, right? This V^* is conjugate transpose and inverse is the same as that. The product is one. So these are unitary matrices. And unitary matrices, like I said, represent isometries. So isometries are pretty much… I always say they don't do much in some sense, right? So they don't change your landscape in anything, in a significant way, right? They take, you know, they preserve norms, they preserve inner products, you know? It's almost like a rotation. So you're not changing much when you do an isometry in some sense. It's sort of like an equal thing, okay? So that's one way of thinking about it.

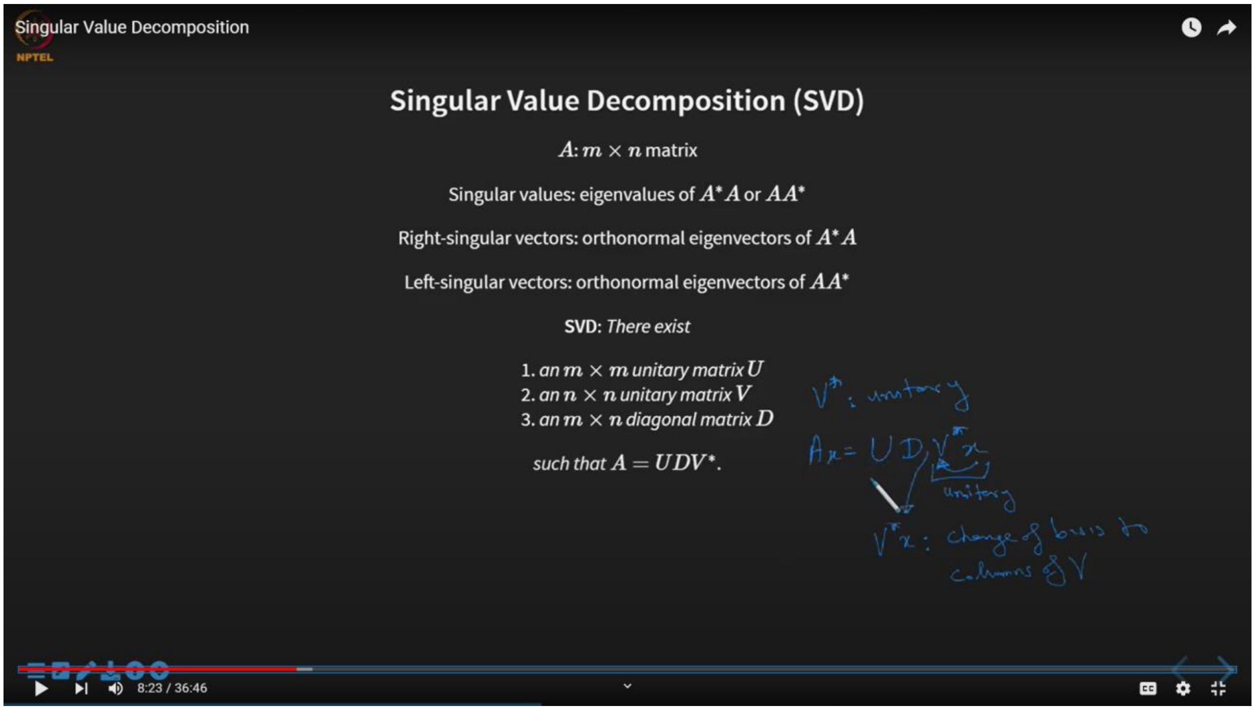
Another notion we'll need is this notion of rectangular diagonal matrices. We know what square diagonal matrices are, right? So we know the main diagonal, it is easy to talk of diagonals with a square matrix. But with rectangular matrices, how do you describe diagonals? It's not very different. So even if you have a rectangular matrix, simply drawing a rectangular matrix here, this would be the main diagonal, okay? So this is the main diagonal, okay? Basically $(1,1),(2,2)$, so on. So even if you have a tall matrix like this, the definition of diagonal is the same, okay? So if you have $m \times n$, this is diagonal. If you have $m \times n$, m being larger also, that's diagonal, okay? So this is the main diagonal. $(1,1),(2,2)$, etc. okay? So when do I say a rectangular matrix is a diagonal matrix? It's only if the non-zero values, if they show up only on the diagonal, then the rectangular matrix is also called the diagonal matrix, okay? So this is a convention that we will use. Both these unitary matrices and such rectangular diagonal matrices will show up in the description of the singular value decomposition, okay? So that is why we need this. So these are preliminaries. Hopefully it's clear, okay?

(Refer Slide Time: 04:08)



So here is the singular value decomposition in terms of matrices, okay? So this is also a very popular way in which it's described anywhere that you read it. So I think it's good to see it with matrices first and then we'll go to the operator view and then we'll prove it, okay? So if you have an $m \times n$ matrix, we know that there are singular values for this which is the eigenvalues of A^*A or AA^*, right? So both of these we know the non zero eigenvalues are the same and the zero eigenvalues will differ depending on the rank and size and all that, okay? Then you have right singular vectors. These are the orthonormal eigenvectors of A^*A, okay? Now, I mean, there is no, there is some ambiguity about uniqueness here. I mean, so orthonormal eigenvectors when you have repeating eigenvalues, it's not unique, isn't it? So you can have multiple types of orthonormal eigenvectors. And so right singular vectors also are defined in that fashion. You can have different sets of right singular vectors, okay? Same thing with left singular vectors. These are orthonormal eigenvectors of AA^*, okay? So A^*A and AA^*, okay? So since m can be different from n, these are all different things in some sense, okay? That's the setup.

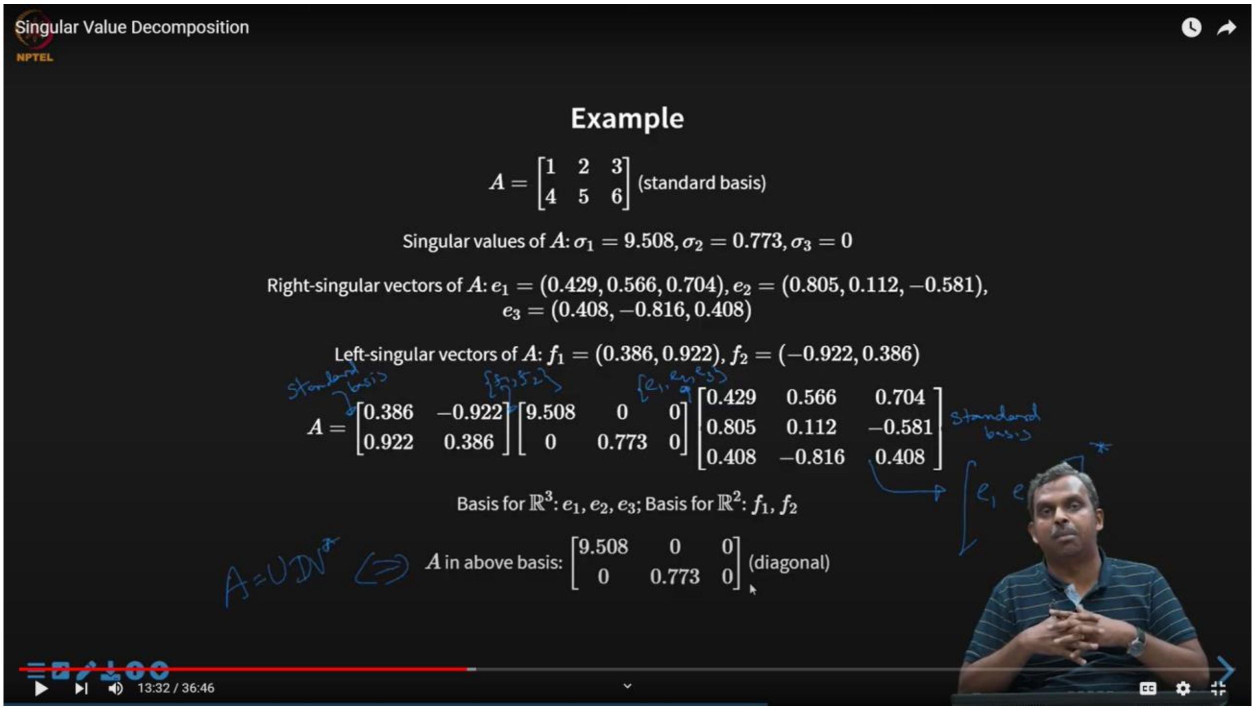
(Refer Slide Time: 08:23)



And the SVD is basically a decomposition of A into the product of three matrices, okay? So SVD singular value decomposition basically states that there exists an $m \times m$ unitary matrix U and $n \times n$ unitary matrix V and an $m \times n$ diagonal matrix D such that A can be decomposed or written as a product of three matrices. And all three, right? So UDV^*, okay? So you know V^* is the conjugate transpose or the inverse of V. And then you have D and then you have U, okay? So it's a product of these three matrices. So now if V is unitary, V^* is also unitary, right? Okay? V^* is also unitary. So when you write A as the product of these three matrices and when you apply, you know… So when you do U, so when you do Av, it's or maybe Ax, its UDV^*x, okay? So when you apply A on x, or x is input to this transform A, and then you do Ax to get the output, you can equivalently do it first by multiplying with V, okay? And this guy is just, you know, it does not change x that much, it's just a unitary transform, okay? So at this point, okay, the vector that you would have, right, V^*x is sort of a change of basis to columns of V, okay? So that is what this V^*x is going to do. And then you do DV^*x and D is diagonal, right? So it's just going to be scaling of the coordinates at this point. And then you multiply by U. And what is multiplication by U again?

It's again a change of basis, isn't it? Change of basis to the, so you've gone to a different basis here. So then you multiply by U. So remember this is gone to from, you know, from $\mathbb{F}^n \to \mathbb{F}^m$, right? So after multiplication by D it is only an m length vector, right? And that m length vector, I don't know. It gets multiplied by U now, there is a further change of basis by this unitary matrix U. And what does unitary matrix do? Nothing much, right? So it's sort of like… I shouldn't say nothing much, should be careful here, so it's just sort of rotates or something like that. It doesn't change the the metric or norm doesn't change, so it just gets multiplied by that. So any A, however complicated it may be, always decomposes like this. Unitary multiplied by diagonal multiplied by unitary, okay? So that is the idea. So once you see that, you will see how interestingly one can work with this. So it's very very promising and it has so many different applications etc. But this is the main SVD result, okay? So any matrix A can be written as a product of UDV^* where U and V are unitary of suitable dimensions $m \times m$ and $n \times n$ and D is diagonal, okay? Diagonal, rectangular diagonal matrix $m \times n$, okay? So that's the interesting part.

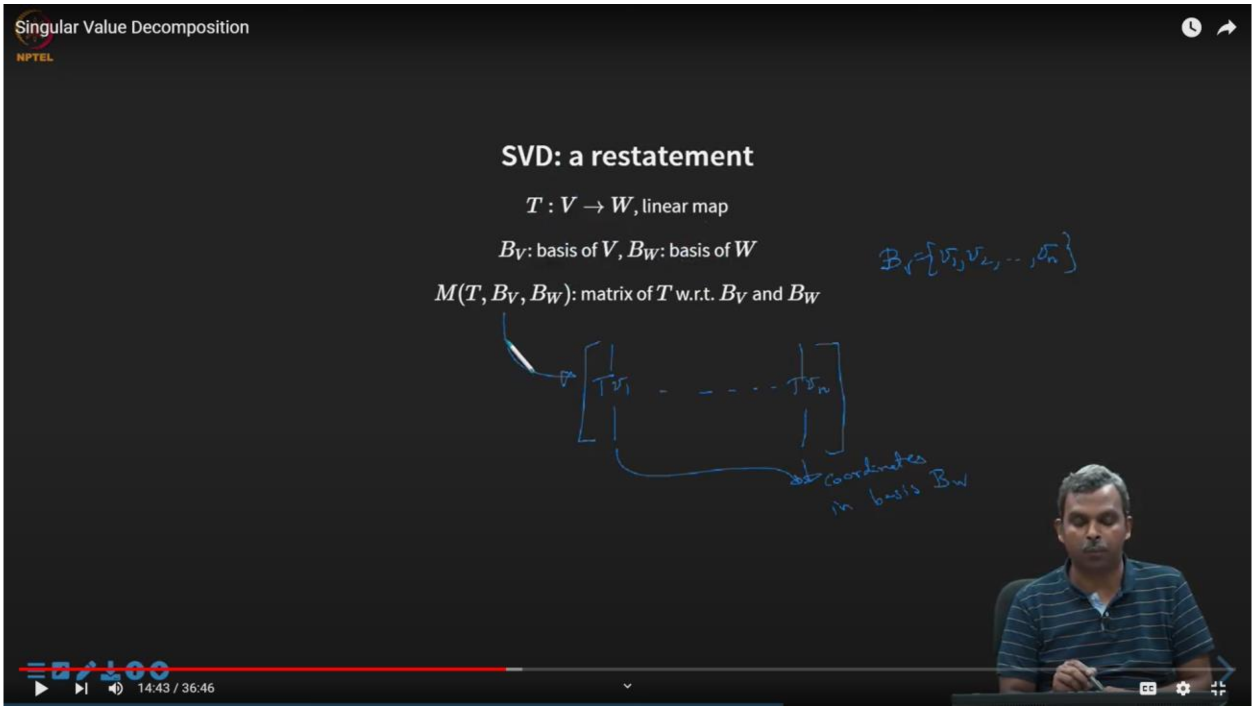
(Refer Slide Time: 13:34)



So once again what are these U, V and D? For U, the columns are left singular vectors. For V, the columns are the right singular vectors. And for D, it has the singular values of A on the main diagonal, okay? So this also is an important result, okay? So you have the existence of these three. And how do you find these three? D simply needs the singular values. V needs the right singular vectors and U needs the left singular vectors. That's it, okay? So simple enough in a way to look at, okay? And you get the answer you want, okay? So this is SVD in terms of matrices. You look up any book or website or something, this is how SVD is described. And this simple, I mean, we will see more more detailed applications later on and you will see this kind of decomposition is very very very powerful. So see if you have a self adjoint matrix or self adjoint operator A, then, you know, you can write it as VDV^*, okay? So that is really very powerful. And we saw so many different applications of that. Now here the U has changed, right? So in fact this result you can also use it for a square matrix. Supposing A is an $n \times n$ matrix and maybe it is not self adjoint, then what happens? You can write A as UDV^*. It's not VDV^*, UDV^* and U is still unitary, okay? So it's not by much except that, you know, you view it in a different axis which is all orthonormal. The operator becomes diagonal, okay? So that's the powerful statement here, okay? So this is SVD in terms of matrices.

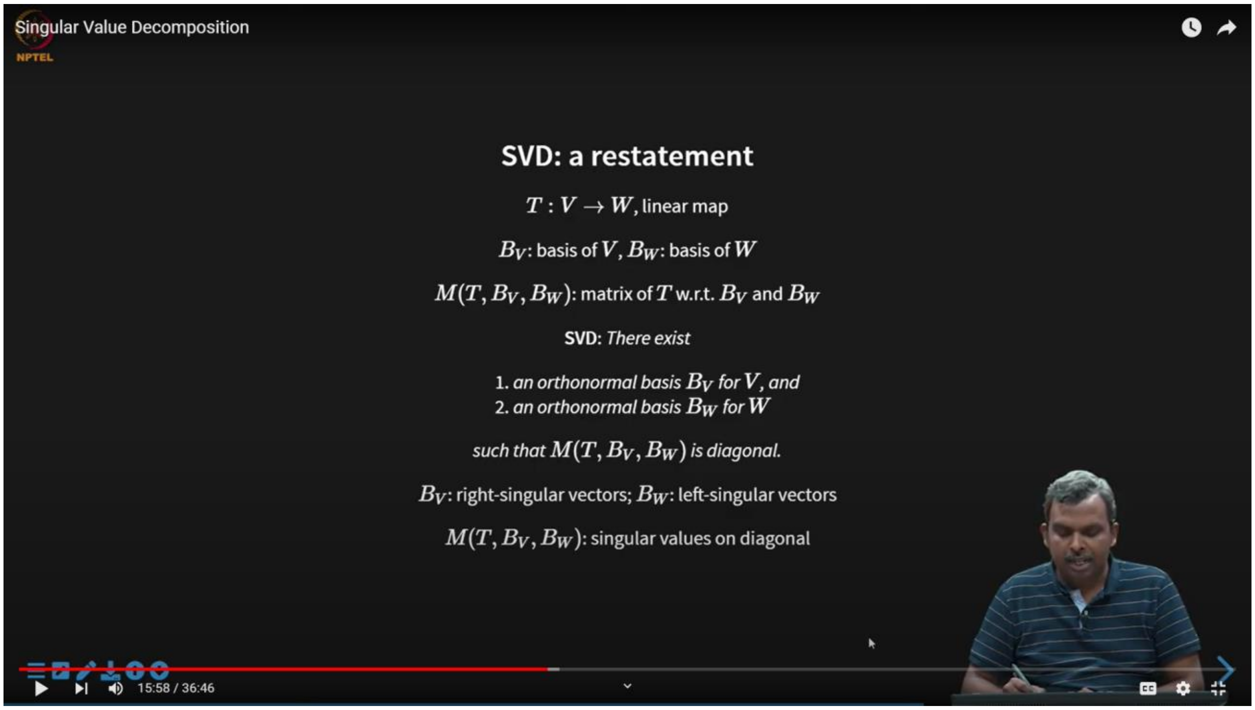
And let us see a quick example to firm up matters here. So this is an example we have seen before, okay? This 2×3 matrix $[1\ 2\ 3;\ 4\ 5\ 6]$. We computed the singular values, we computed the right singular vectors, left singular vectors. And then using this you can check that A will work out as this product, okay? So we have seen this before. This is how A will work out to be, okay? So you can check this again. Hopefully this is correct. So I have not made a mistake here. So you will get A to be the product of these three things, okay? So notice something that is very interesting. If I change basis, right… So this A was in the standard basis. Let's say standard basis is for $\mathbb{F}^3$ and standard bases for $\mathbb{R}^3$ and standard bases for $\mathbb{R}^2$, okay? So it's in the standard basis. Now if I change basis of $\mathbb{R}^3$ to $\{e_1, e_2, e_3\}$, okay, remember this matrix is $[e_1\ e_2\ e_3]^*$, right? So conjugate transpose. So e_1 comes on the first row, e_2 comes on the second row, e_3 comes on the third row, right? So this is that, okay? So this is like the inverse of that. So it takes from standard basis… So here the coordinates will be $\{e_1, e_2, e_3\}$ basis, okay? That okay? So then you get an answer which is in $\{f_1, f_2\}$ basis, isn't it? So if you shift the basis to $\{f_1, f_2\}$, here the answer is in $\{f_1, f_2\}$ basis. So here you go back to standard basis, okay? So in terms of coordinates, this is how you can think about it. You have an input coming in standard basis. When you multiply by this unitary matrix $V\ *$, you go to the bases $\{e_1, e_2, e_3\}$. The coordinates are in that basis and you do your linear map, right? So it is just a scaling at this point. You will get the coordinates in $\{f_1, f_2\}$ basis and then from $\{f_1, f_2\}$ basis you go back to the standard basis by multiplying by U, okay? So this is the description. So if you change bases to $\mathbb{R}^3$ and $\mathbb{R}^2$ and look for the matrix of A in that basis, then you simply get a diagonal matrix, okay? So both these are equivalent as A being written as… This and this are equivalent, right? So A being written as UDV^* and A in the basis of columns of U for our $\mathbb{F}^m$ and then the columns of V for $\mathbb{F}^n$, it being just diagonal, both of these are exactly equivalent, okay? So that's sort of an example which sort of illustrates how SVD works in practice. So now this can work for even an arbitrary matrix. For any matrix you want, you find the singular values, find the right singular vectors, left singular vectors you have this decomposition. You go to the orthonormal basis dictated by the singular vectors then your linear map A becomes diagonal, okay? So that is the power of this SVD.

(Refer Slide Time: 14:44)



Okay. So now we will do a restatement in terms of operators. We saw the example before, it's sort of motivated by that. Let us say you have $T: V \to W$ being a linear map. Then you have a basis for V, basis for W and then you can think of a matrix for T in this basis. So what is the matrix of T in this basis? How will you write it? So supposing B_V is $\{v_1, v_2, \ldots, v_n\}$. What is this? This is $\{Tv_1, \ldots, Tv_n\}$. Each coordinates in basis B_W, okay? So all these things are coordinates in basis B_W, right? So this is your matrix $M(T, B_V, B_W)$, right? So this is the matrix. So given a basis for V and basis for W, you can come up with a matrix. So typically people take the standard basis for V here, standard basis for W and give you a matrix A, right? So but for an arbitrary basis also this is what you do, okay? Hope this is clear, okay? So now what does SVD tell you? SVD tells you that there exists an orthonormal basis B_V and an orthonormal basis B_W such that $M(T, B_V, B_W)$ is diagonal, okay? So it will be generally a square matrix, a rectangular matrix, but that rectangular matrix will be diagonal, that's what this SVD tells you. Moreover, B_V is basically the right singular vectors, B_W can basically be the left singular vectors, a set of left singular vectors, B_V is a set of right singular vectors. And this $M(T, B_V, B_W)$ not only is diagonal, but it has the singular values on the diagonal, okay? So that's more interesting results about what the SVD says, okay? So this is a restatement of SVD in terms of operators and this is what we will prove, okay? So we will prove this in the next few slides. Proof is going to be fairly involved. It is one of the longest proofs maybe we have had in this class. It just breaks up the ideas into smaller segments. None of the ideas themselves are complicated. It's just viewing them from the right point of view and carefully tracking what happens, okay? So let us get into the proof, okay?

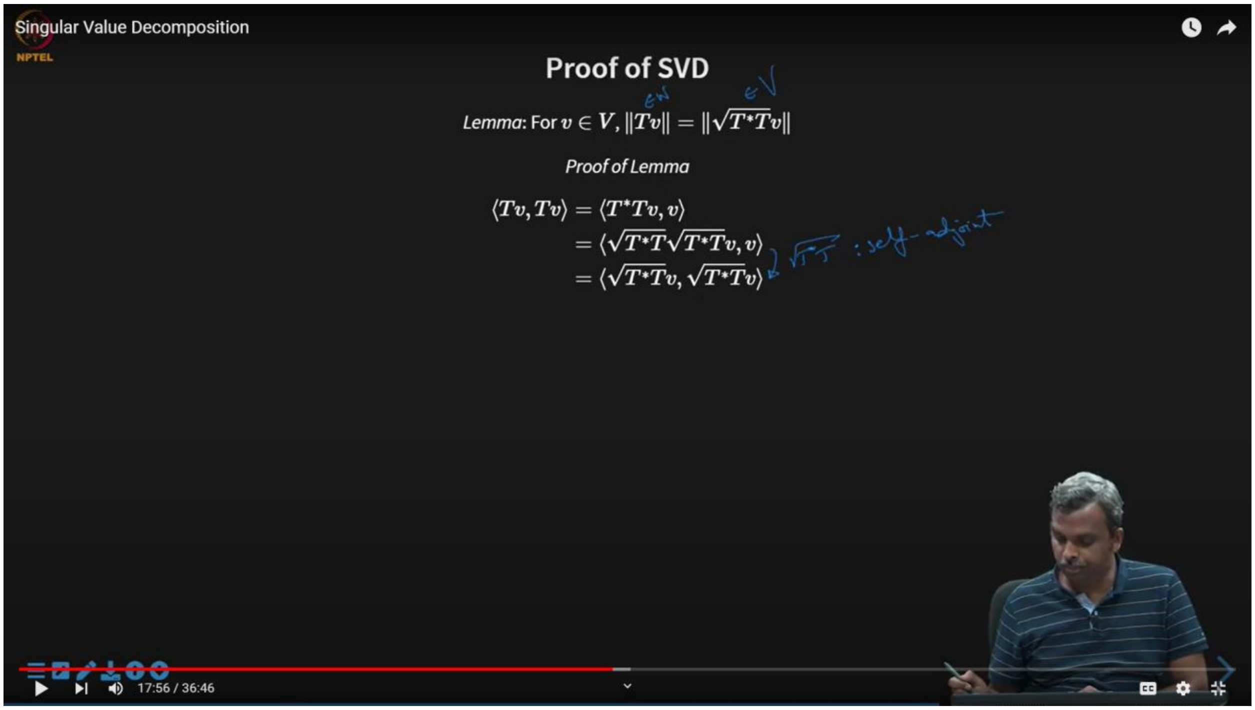
(Refer Slide Time: 15:58)



So a crucial role is played by T^*T and $\sqrt{T^*T}$, okay? So a very important lemma in the proof is the following, okay? So I have always been commenting about how T and $\sqrt{T^*T}$ are sort of the same in some sense, right? So I keep saying that all the time. So they give you a sense of being the same. So we saw from the null space, range space and all that, a lot of them are similar. The null spaces are the same, range spaces are the same. So there's lots of similarity. Range space is not the… You know what I mean. So null space is the same. So it's all, it's got a lot of similarity, these two operators, right? T and $\sqrt{T^*T}$.

So here is this lemma which again brings out one more very interesting result, right? The norm of Tv equals norm of $\sqrt{T^*T}v$, okay? So when when you hit both T and the $\sqrt{T^*T}$ with the v… Remember this Tv, okay? So it is important to remember this Tv actually belongs to W and this guy belongs to V, right? So these two belong in two different vector spaces but their norms are the same, okay? And you can sort of quickly see that proof. The proof is not very difficult at all. This is a very easy proof. Just any algebraic manipulation proof is very easy, isn't it? So you just bring this to this side. T^*T. Write it as square root, square root. And how did I do this? What is the final step? Reason for the final step? This is because, how did this happen? This is because $\sqrt{T^*T}$ is self adjoint, right? Since it's self adjoint. if you push it to this side, you get the same operator. So the norms are the same, okay? So this is the proof that these two are equal, okay? So this is very important. The fact that, you know, Tv and $\sqrt{T^*T}v$ have the same norm, this will play a crucial role in SVD and its proof. Okay. So let's take the range of T which is the set of all Tv $\forall$ $v \in V$, right? This is the range of T. And range of T we know is a subset of… Again do not forget this is a subspace of W, right? And what about range of $\sqrt{T^*T}$? This is again set of all $\sqrt{T^*T}v$ for $v \in V$.

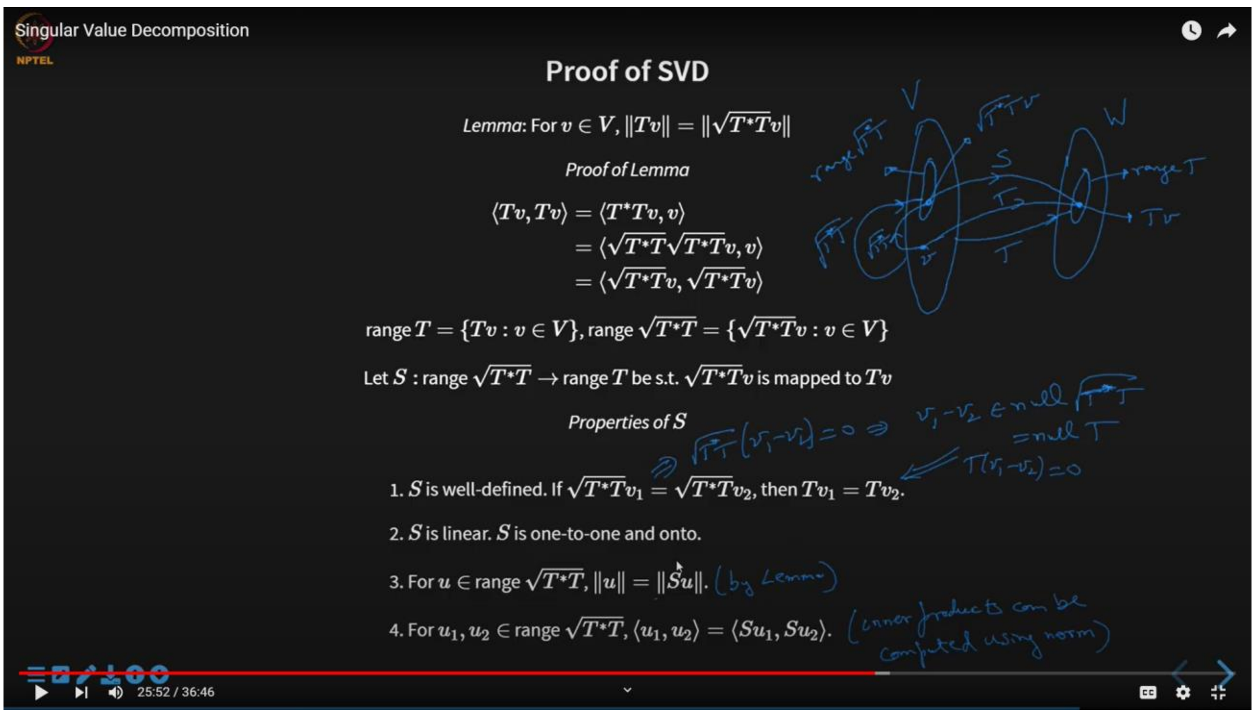
(Refer Slide Time: 17:56)



And this is a subspace of V, isn't it? So this is a subspace of V and both of these we know have the same dimension, right? So that's one of the results we saw in the previous lecture. These two have the same dimension. And now we have seen that, you know, Tv and $\sqrt{T^*T}v$ have the same norm etc. and… So it turns out one can define this very interesting map S from range of $\sqrt{T^*T}$, from this set, to this set, range of T. How will I define this interesting little map S? I will simply take $\sqrt{T^*T}v$ and map it to Tv, right? Remember once again, every element of range of $\sqrt{T^*T}$ is of this form: $\sqrt{T^*T}v$ for some v. As you keep changing v, this thing will presumably change, okay? Same thing happens here. As you keep changing v, this Tv also changes. I will take the $\sqrt{T^*T}v$ and map it to Tv, okay? So remember $\sqrt{T^*Tv}$ belongs to V. I mean it's in that vector space. Tv belongs to W, it's in the other vector space. So this is the mapping that I am going to do. S goes from range $\sqrt{T^*T}$ to range T, okay?

Now this S has so many, very many interesting properties. I am going to just put down all those properties and then maybe draw some other pictures to describe what is going on, okay? So notice what is going on here. So you have V and you have W. You have range T. So this takes you to, T takes you from here to the range. And then you have, how will $\sqrt{TT^*}$ work, okay? So it will have some other thing here. This is going to be range $\sqrt{T^*T}$, okay? It takes you from, you know, $V \to V$, right? So it goes from $V \to V$. Maybe I should write it like that. $\sqrt{T^*T}$. And that will be the range, right? It will go to this from V, it will go directly to that, okay? So this is the sort of picture if you want. In case you like pictures, you can keep that in mind. And what is this mapping? So if you have a v here, v went to something like this under T, right? So this would be Tv. And this v would go to, sorry, this v would go to some place here under $\sqrt{T^*T}$, right? So this point here would be $\sqrt{T^*T}v$. And what is S going to do? S is going to take this guy to this, okay? So this is that picture. Hopefully this picture is clear to you, okay? So you have V and W. T is a mapping that takes you from V to range of T inside W. $\sqrt{T^*T}$ is a mapping that takes you from V to range of $\sqrt{T^*T}$ which is inside V. Now if you take a particular vector v, T will take you inside range T, $\sqrt{T^*T}$ will take you inside range, the corresponding range there. This S is basically a map that maps this guy to that guy, okay? After the multiplication, okay? So this is a picture if you want you can keep in mind.

(Refer Slide Time: 25:52)

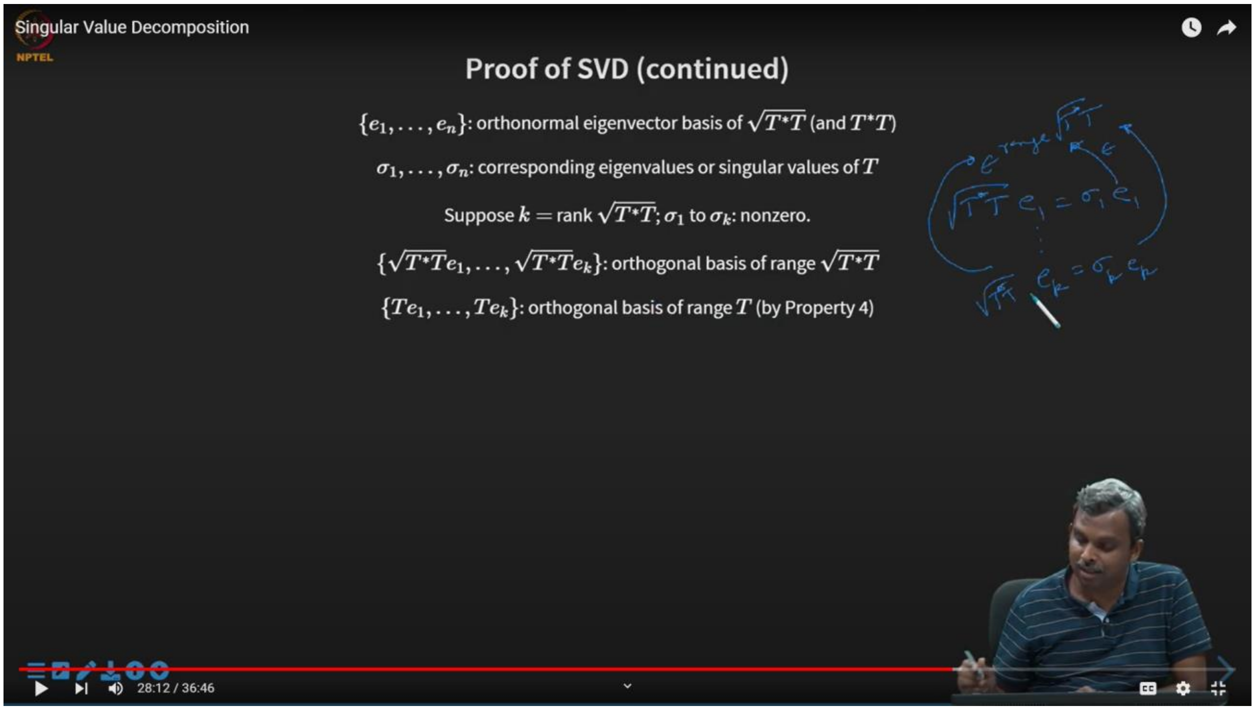


So the first thing is: S is well defined, okay? If you have two different vs, v_1 and v_2 that went to the same point, okay? Then they will go to the same point under T also, okay? So this is something you can show. So it is not very difficult. So this implies $\sqrt{T^*T}(v_1 - v_2) = 0$ which implies $(v_1 - v_2)$ is in null $\sqrt{T^*T}$ which is equal to null T and that implies this, right? So these two are equal, the null T and null $\sqrt{T^*T}$ are the same. So null of T, $(v_1 - v_2)$ belongs to null T. So $T(v_1 - v_2) = 0$. $Tv_1 = Tv_2$. So this map is well defined, okay? So if you have two different vectors going to the same point under $\sqrt{T^*T}$, they also go to the same point under T, okay? So it is well defined, okay?

So this also shows you this kind of a manipulation. Okay S is linear is not too bad to see. The other thing you can quite quickly see is that S is one to one and it's also onto, okay? So it's sort of like a one-to-one and onto map, it's injective surjective, it's like a, it's almost like an isomorphism in some sense, right? So between these two things. So that's what S is. These are easy to prove. I'm not going to go into great detail here. But this you can hopefully see why this is true. The dimensions are also the same, no? So the range of $\sqrt{T^*T}$ and range of T have the same dimension. So this is also easy to see, okay? So from the lemma. So this is just by lemma, okay? From that lemma you can see this result, right? So if you have a u which belongs to range of $\sqrt{T^*T}$ and that u by S goes to some point here in range of T, that u and Su have the same norm, right? So that's the restatement of this result, right? Of the same lemma. u and Su have the same norm, okay? S is linear, u and Su have the same norm, that is very interesting, isn't it? So when you have something that is linear and having the same norm, we know that this S will also preserve inner products, okay? So, why? Because norms come from inner products or inner products come from norm, both of these are true. As long as S is linear you know only summation $u_1 + u_2$, $u_1 - u_2$, S times that will show up in the norm definition. So the inner products will also be the same. So this is sort of how we showed for isometries when they preserve norms, they also preserve inner products, right? So this S is like an isometry from range $\sqrt{T^*T}$ to range T. So it will also preserve inner product, okay? So I am not going to prove this result, I'll just write something. Inner product can be computed from norms, okay? I leave it like that. So this proof is easy enough to do. We have seen it in one other case. For the isometry case we have seen this. The same idea you can use to show that this map S preserves the inner product. This is very very crucial. So notice all that we have done here. We have taken $\sqrt{T^*T}$ and then we have taken T. And we now have almost an isometry with… Well we don't quite define it as an isometry. Isometry is between one vector space to itself. This is like from a subspace to another subspace. We have a map which is linear, it preserves norms, it preserves inner products, okay? So this is very crucial. So once you have this, we can do something very interesting with it and the proof of the SVD will follow, okay? So this is important. A few ideas but the crucial idea is that the $\sqrt{T^*T}$ and T are very similar in some sense, right? So and that's what this is establishing clearly.

Okay. So let's continue with the proof. We have done most of the hard work. Now it is just a question of carefully arranging everything. It will work, okay? So now $\sqrt{T^*T}$ is also self adjoint, okay? Of course T^*T is also self adjoint. And both of them will have a nice common orthonormal eigenvector basis, right? So I will take that as $\{e_1, \dots, e_n\}$, okay? The corresponding eigenvalues will be the singular values of T, right? $\sqrt{T^*T}$, if you take the eigenvalues of $\sqrt{T^*T}$, they will be the singular values of T. I will write them as σ_1 to σ_n. Now let's suppose this k is the rank of $\sqrt{T^*T}$. It need not be full rank, isn't it? It can have some lower rank. So let's say k is the rank. So when k is the rank… Remember it's a self adjoint matrix. So σ_1 to σ_k will be nonzero and after that you will have 0, right? There will be exactly k non-zero eigenvalues. And after that it will be all 0, okay? So that is how for rank and the eigenvalues are connected. So this is how it will be. This is the first step, this setup is easy to do.

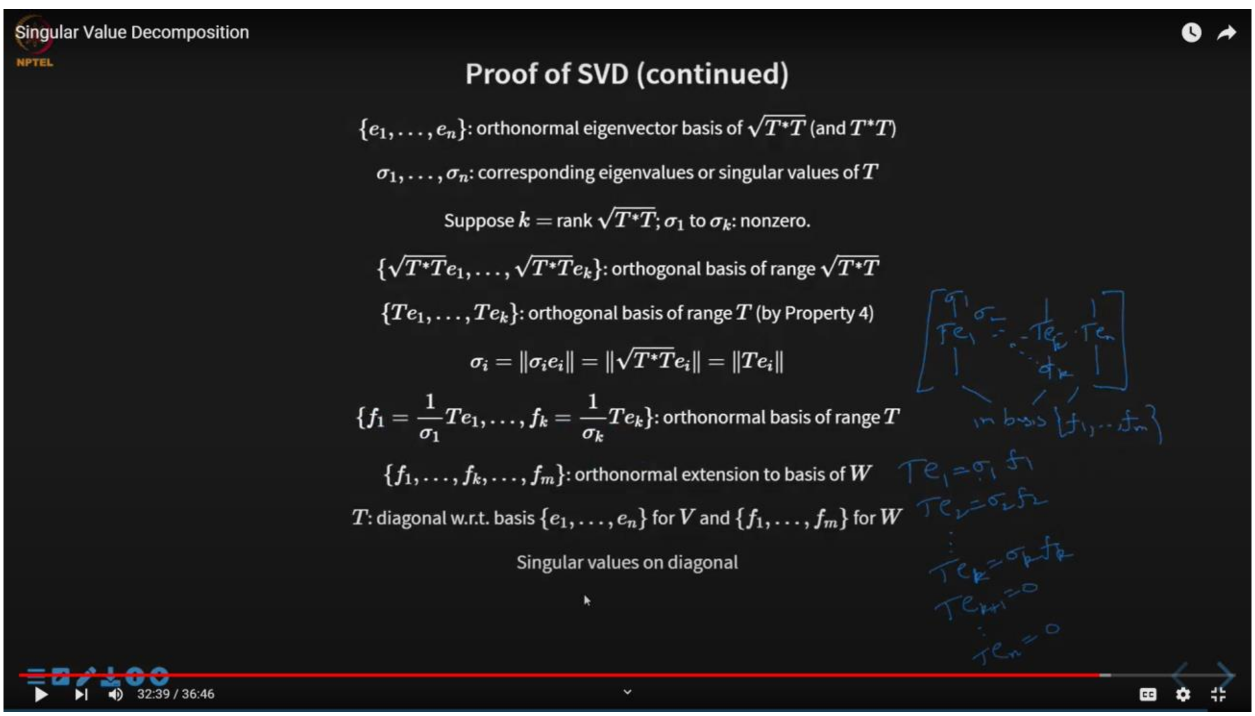
(Refer Slide Time: 28:12)



Now we will move on to the connection between range $\sqrt{T^*T}$ and $\{e_1, e_2, \ldots\ e_n\}$ okay? So remember e_1 through e_k are eigenvectors of $\sqrt{T^*T}$, right? So what will happen? So maybe I should write this out. $\sqrt{T^*T}e_1$ for instance, is $\sigma_1 e_1$, okay? So this clearly belongs to range $\sqrt{T^*T}$, okay? So e_1 also belongs to the range of $\sqrt{T^*T}$. That is quite easy to see. And this will be true all the way till e_k, right? The same thing is true. Oh sorry, the other way. T^*Te_k is $\sigma_k e_k$. And so this also will belong to range T. This will also belong to range T. So you have k as the rank of $\sqrt{T^*T}$ and here you have k orthogonal vectors, okay? So remember if I took e_1 to e_k, e_1 to e_k will be an orthonormal basis of range $\sqrt{T^*T}$. Once I multiply by $\sqrt{T^*T}$, I will only get an orthogonal basis, because, the reason is the norm of these guys… I'll write down that in the next step. You will see the norm of each of these guys is actually slightly different from 1. I mean, so maybe I should show it to you. So you see that the norm of each of these guys, $\left\|\sqrt{T^*T}e_1\right\|$ is basically σ_1, it's not it's not 1, okay? So you can't say orthonormal, it’s orthogonal basis of T^*T.

So now I do this S map, right? So notice what I have done here. I have done the S map. What will happen to $Te_1, \ldots\ , Te_k$, right? So this will also become an orthonormal basis of range T. Why is that? This is by property 4. What is property 4? The inner products are preserved by S, okay? So when you have this set being orthogonal in range $\sqrt{T^*T}$, this set Te_1 to Te_k will be orthogonal in range T because this S map that we are doing, going from $\sqrt{T^*T}e_1$ to Te_1 preserves norms, preserves inner products. Since it preserves norm, the norm of Te_i and the norm of $\sqrt{T^*T}e_i$, right, both of them have to be equal to σ_i, okay? So they will all be, the norms will be the same and they'll be σ_i and they'll also be orthogonal, okay? So this is the crucial idea, okay? So you have an orthonormal basis for the entire vector space V. Some of those vectors form an orthonormal basis for the range $\sqrt{T^*T}$. You simply scale them suitably to get to the $\sqrt{T^*T}$ form, just scale by σ_1 or something. And then you use the S, you go to range T from range of $\sqrt{T^*T}$, okay? So that is very crucial. So you get an orthonormal basis. You go to orthogonal basis, you apply S, you go to orthogonal basis of range T, okay? So now how do I get an orthonormal basis from the orthogonal basis? You simply divide by the norm. And the norm is σ_1 etc. So you divide, okay? And you call this as f_1 to f_k, okay? Notice how we have gotten this f. You got an orthonormal basis for the original vector space V, you assumed a certain rank, you went to the orthogonal basis for range of $\sqrt{T^*T}$. And then you did the S map, went from $\sqrt{T^*T}e_i$ to Te_i. And you realize that will also be an orthogonal basis for the range T. Then you can divide by the magnitude you have, right? Magnitude of these guys that you have again by that lemma.

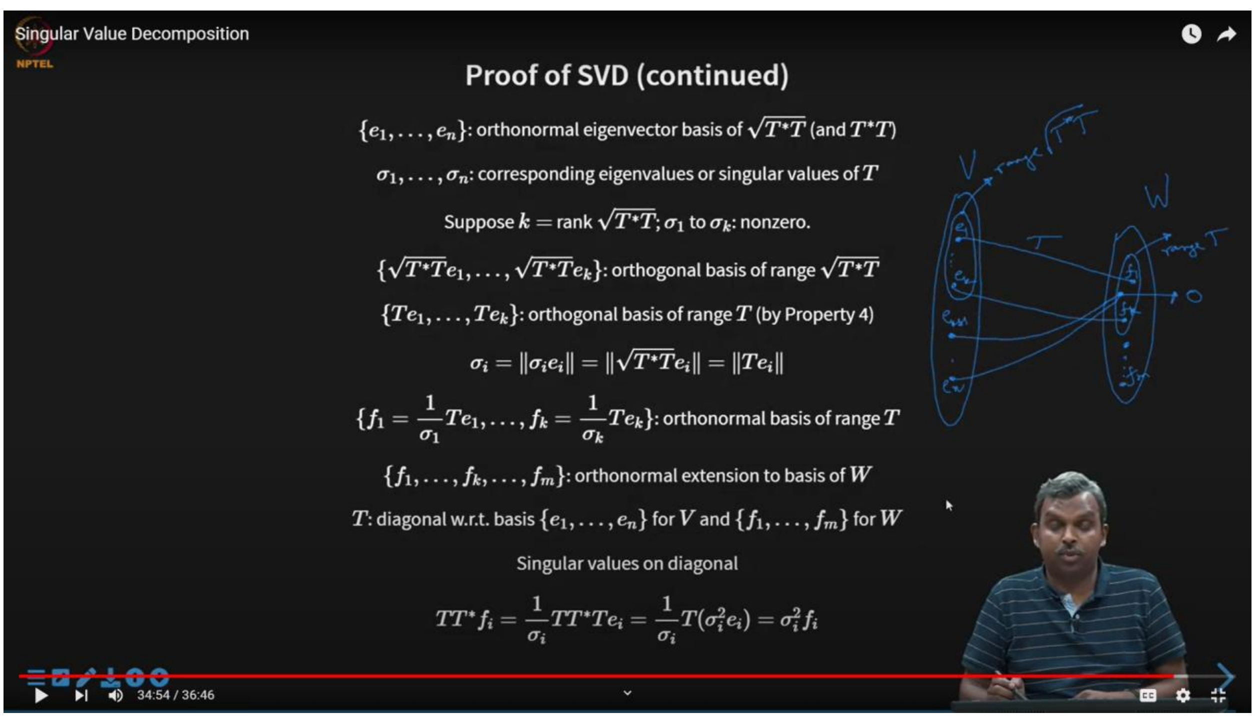
(Refer Slide Time: 32:29)



So you get this wonderful result. You have an orthonormal basis for the range of T which is simply a mapping of e_1 to e_k under T. So that is the nice idea. This you can call as f_1 to f_k. And once you do that, you can extend this f_1 by f_k to an orthonormal basis for the entire W. Find the orthonormal basis for range T in this fashion and then you extend it to all of W and this will give you all that you want, right? So in this basis if you pick the basis e_1 and the basis f, you see that Te_i, okay… So why is this diagonal? Te_1 is $\sigma_1 f_1$, Te_2 is $\sigma_2 f_2$, right? Te_k is $\sigma_k f_k$. What is Te_{k+1}? All of them are zero, right? Okay? Why is that? $\sqrt{T^*T}e_{k+1}$ is 0. So this T will also be 0, okay? So this tells you that if you form a matrix for this basis, right, T will be diagonal with respect to this basis. You have to write down $Te_1, \ldots, Te_k$, I'll put here then Te_n, okay? How do you express this? This just to be expressed in the basis, in basis f_1 to f_m. And you see Te_1 is just $\sigma_1 f_1$, so it will be just σ_1 here, σ_2 here and so on till σ_k here. So it will be diagonal. Everything else will be 0. So T becomes diagonal with respect to this basis. And the singular values show up exactly on the diagonal. So this relationship is crucial. Te_1 is $\sigma_1 f_1$. So that's the crucial relationship that you have, that we have proved, okay?

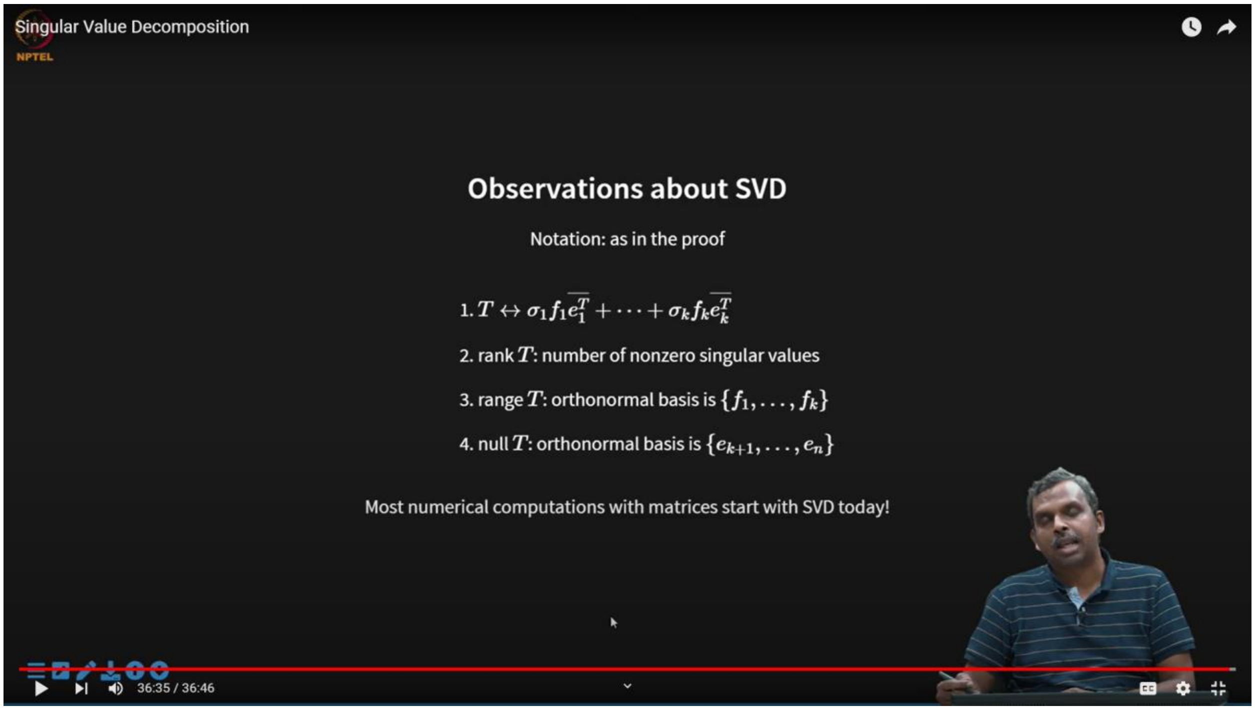
So the last part we have to show is that these fs should also be left singular vectors for T, right? How do you show that these are left singular vectors? I mean show TT^*, if you multiply TT^* by f_i, you will get, you know, f_i is simply $\frac{1}{\sigma_i}Te_i$, okay? So you plug that in, you multiply, you get T^*Te_i. What is T^*Te_i? $\sigma_i^2 e_i$, right? So it's like an eigenvector for T^*T, right? And then you get $\sigma_i^2 f_i$, right? So this T will come in. $\left(\frac{1}{\sigma_i}\right)Te_i$ is f_i. So $\sigma_i^2 f_i$. So TT^*f_i is $\sigma_i^2 f_i$. So f_i are our left singular vectors for T, okay? So that is the idea.

(Refer Slide Time: 34:56)



So what has happened overall is: if you want a picture, okay, so you have V, we have W here. There is an orthonormal basis $e_1, \dots, e_k, e_{k+1}, \dots, e_n$. These guys are inside range $\sqrt{T^*T}$, okay? What happens when e is applied to these guys? You have range T here. When T is applied to these guys, this goes to f_1, all the way till this going to f_k, okay? And then all these other fellows under T go to zero, right? So zero will also be here. They all go to zero, okay? So this f_1 to f_k ends up being an orthonormal basis. That was my property of that nice little relationship, that map S which which was like an isometry from range $\sqrt{T^*T}$ to range of T. And then you extend this guy to f_m, okay? You extend this. So now you have an orthonormal basis, orthonormal basis which gives you a diagonal for T. That's the idea, okay? So it's a crucial little, interesting little insight into how SVD works, okay? So that is the end of the proof and we have shown this existence of two orthonormal bases under which any linear map becomes diagonal, okay?

(Refer Slide Time: 36:37)



So final observations about the SVD, okay? So the notation is like in the previous proof. The first observation is: given that you have the product of, you know, f_1s, diagonal, e_1s, e_1^T etcetera, you can write T equivalently as this product, okay? So this is same as, you know, when you had the spectral theorem for self adjoint operators. We could write it in this form, right? f_1 would be equal to e_1, right, you would have got $e_1\overline{e_1^T}$ etc. Now instead of e_1, you will have f_1. It's the same three matrix product with diagonal in the middle, column, row etc. So this is that product. You can quickly show rank of T is the number of non-zero eigenvalues, non-zero singular values, okay? Range T is basically the orthonormal basis f_1 through… Orthonormal basis for the range T is this f_1 to f_k and orthonormal basis for null T is e_{k+1} to e_n. So this SVD gives you all that you need to know about the linear map, okay? So in fact most numerical computations, you give any numerical package a matrix, $m \times n$ matrix, it will first compute the SVD and all the other things it will compute based on that. In fact you can project to the range, right? You find an orthonormal basis for range T. So that gives you the projection, right? Everything you want you can compute using the SVD, okay? So and the SVD has wonderful numerical methods today for computation. So it plays a very important role just in terms of numerical implementations and etc. And also several other applications. So so many interesting applications for SVD. In the next lecture we will see a couple of very interesting applications for SVD and that will be the conclusion of the course, okay? Thank you very much.