**Applied Linear Algebra**
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**Week 01**
**Basis and Dimension**

Welcome once again to this lecture. We are going to move ahead and study more about vector spaces. Once again we've been, we defined a basic abstract notion of a vector space and then used only aspects of that definition to develop the area more. In the previous lecture we saw, you know, linear combinations, span, subspaces and you know, how to deal with subspaces etc. Some various examples, ways of thinking about them and very importantly linear dependence, linear independence, what does that mean, all of that we saw. With the basic definition we're going to build further and study about basis and dimension. So this is quite crucial and an important notion in a vector space. So let's go ahead with it, okay?

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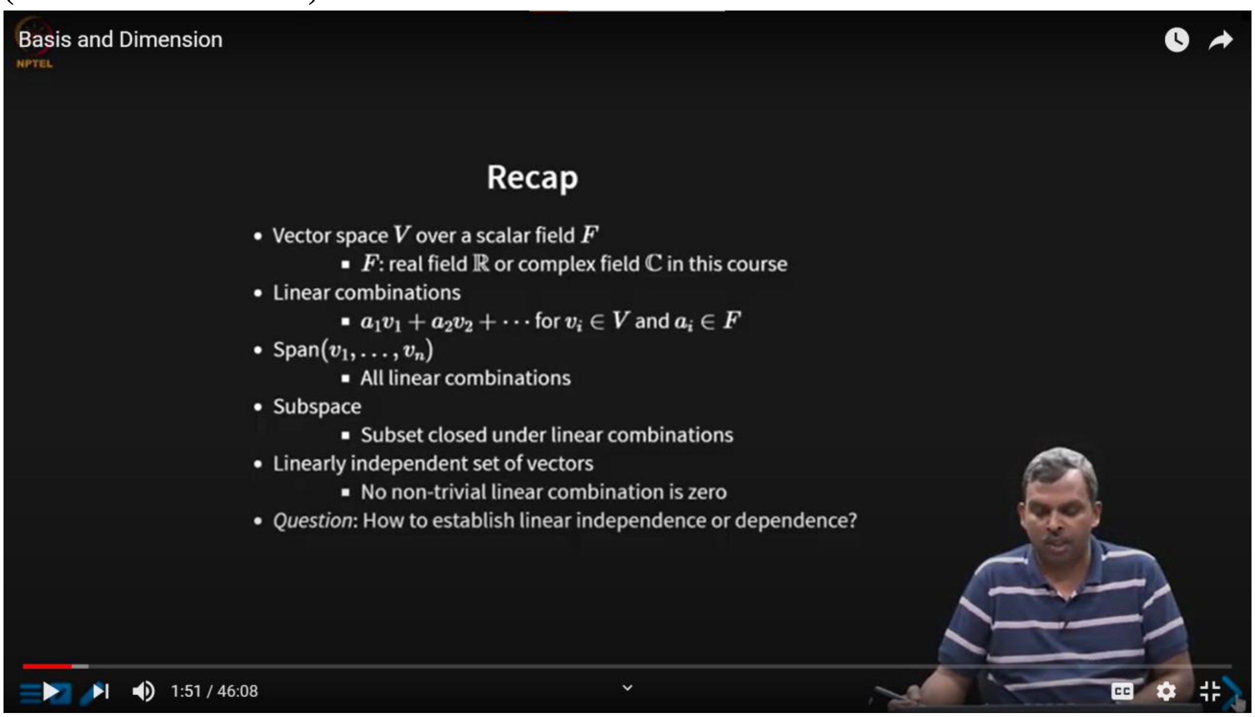


A quick recap. We studied this in the last lecture, particularly linear combinations $a_1v_1 + a_2v_2$, things like that. Plus then we looked at span. Span of a set of vectors is all possible linear combinations. Then we saw subspace, the most important idea of subspace. How when you have a subset of a vector space which itself is closed under addition and multiplication, scalar multiplication, you have a subspace, right? And then this notion of a linearly independent set of vectors, as in other than the trivial combination which gives you the zero vector, no other combination can give you a zero vector, okay? So these are important ideas. We did have this question on how to establish linear independence or dependence. We didn't fully answer that question. That's still, you know, lurking around. It's not going to go away, we'll eventually see that there are interesting methods to do such things, okay? So let's proceed.

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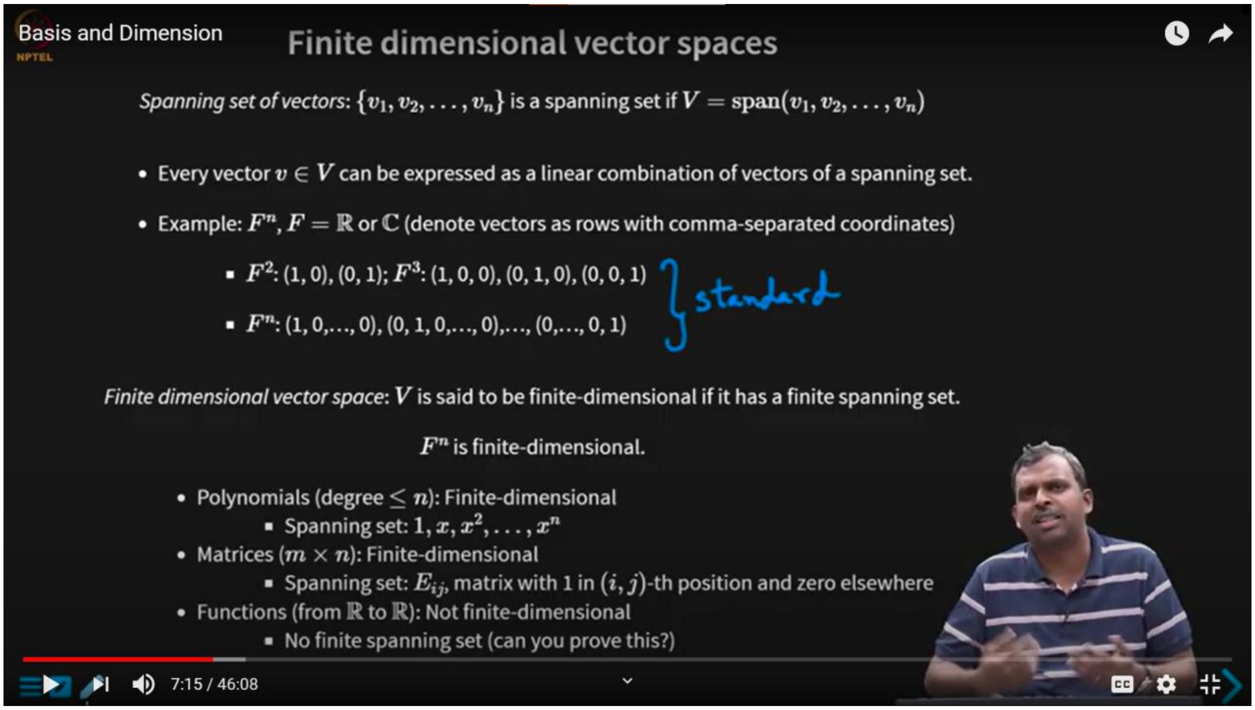


So this slide is quite important, there's lots of detail in it. I'll walk you through one after the other. The first thing we will define is something called a spanning set for a vector space, okay? So this is an important definition. A set of vectors $v_1$ through $v_n$ is said to be a spanning set for the entire vector space if... The very obvious sort of definition for a spanning set - what is spanning set? A set that spans the whole space, right? So if you say a spanning set for a vector space, what you mean is - if you take the span of these vectors, span of $v_1, v_2$ $to$ $v_n$, you should get the entire vector space. No vector should be left out in the span, okay? If you have a set of vectors like that, then you have a spanning set of vectors, okay?

So look at the definition once again, it's there on the top of the slide. if you have the span of $v_1$ through $v_n$... A set of vectors $v_1$ through $v_n$ is called a spanning set if the span of those vectors is the entire vector space. So we saw some examples of this before, I will quickly give you other examples etc. okay? So basically what it means is - every vector in the vector space can be written as a linear combination of vectors from the spanning set, right? So that's what it means. It's sort of a restatement of the definition in some sense, okay? So we can give you examples.

And in these examples I will start denoting vectors, instead of like, you know, the column vector description that we had before, a big matrix like thing occupying a lot of space, we'll reduce space and write vectors as rows, okay? So just imagine that these are actually columns if you want, okay? So these, for simplicity, I'll write it like that. So you notice here for $\mathbb{F}^2$ and $\mathbb{F}^3$ it's very easy to come up with a spanning set, right? For $\mathbb{F}^2$ (1, 0), (0, 1) is a spanning set. $\mathbb{F}^3$ (1, 0, 0), (0, 1, 0), (0, 0, 1) is a spanning set.

But remember these are not the only spanning sets, right? So for $\mathbb{F}^2$ we saw other examples, right? So you remember, in one of the earlier examples, we saw that (1, 2), (2, 5) is also a spanning set, right? So those two vectors end up spanning the entire $\mathbb{R}^2$ for instance, right? So there are many spanning sets, but it's easy to come up with one spanning set. Quite quickly you can come up with one spanning set. In general for $\mathbb{F}^n$ I can come up with a spanning set - one followed by zeroes, zero one followed by zeroes, like that, right? The $i^{th}$ vector will have one in the ith position and zero elsewhere, okay? So that is the spanning set. These type of spanning sets have a name - these are called, so these are sort of standard in some sense, right? Like I mentioned there are many other spanning sets, but these particular type of spanning sets where you have just ones and zeroes, you know, $i^{th}$ position has one everything else is zero, that kind of thing is called a standard sort of spanning set, okay? That's nice to know.

Now we come to the important definition of a finite dimensional vector space, okay? A vector space $V$, we will call it finite dimensional if it has a finite spanning set, okay? If a finite number of vectors span the whole vector space, that vector space itself is supposed to be finite dimensional, okay? We haven't yet told you what dimension is, I will say that in some time. Dimension is something we have not yet defined, but we will define finite dimensional in this form - if it has a finite spanning set, then the vector space is finite dimensional. So clearly $\mathbb{F}^n$, $\mathbb{F}^n$ this vector space with you know, n coordinates written one below the other or one after the other in a row, that vector space is clearly finite dimensional because there are n vectors which completely span the space.

Okay so that's a nice enough example for us to have. There are also other things which are finite dimensional. You see the polynomials of degree less than or equal to one - that's finite dimensional. I have given you a spanning set there. Matrices, $m \times n$ matrices, they are also finite dimensional. You can come up with the matrix once again. In the ij-th position it's one, everywhere else is zero, right? If you take all such matrices together, they will clearly span the set of all matrices. So that is that is also finite dimensional.

But here is an example. Out of all the examples we saw, the last example was these functions from real numbers to real numbers, right? This ends up not being finite dimensional, okay? There is no finite spanning set, think about how to prove it. The proof is sort of already in this page, you can go back and look at the polynomials and argue that - how if you have all type of functions, then your polynomial need not be finite degree, right? And when you have, you know, polynomials of any degree, clearly you are not going to have a finite set of vectors which span the whole space, right? So you can prove this in that fashion. So already we have seen out of the examples, some of them, most of them seem like they are finite dimensional. But at least there is immediately, there is one example which is not finite dimensional, okay?

And this $\mathbb{F}^n$ is a special sort of finite dimensional vector space, it turns out, we will see later, that any finite dimensional vector space has to look like $\mathbb{F}^n$, okay? So that's like, sort of like it's a canonical standard sort of finite dimensional vector space, okay? So these finite dimension vector spaces are very important. For the rest of this course we will focus only on finite dimensional vector spaces. Most of the results we have will be for finite dimensional vector spaces, okay? So these are very important.

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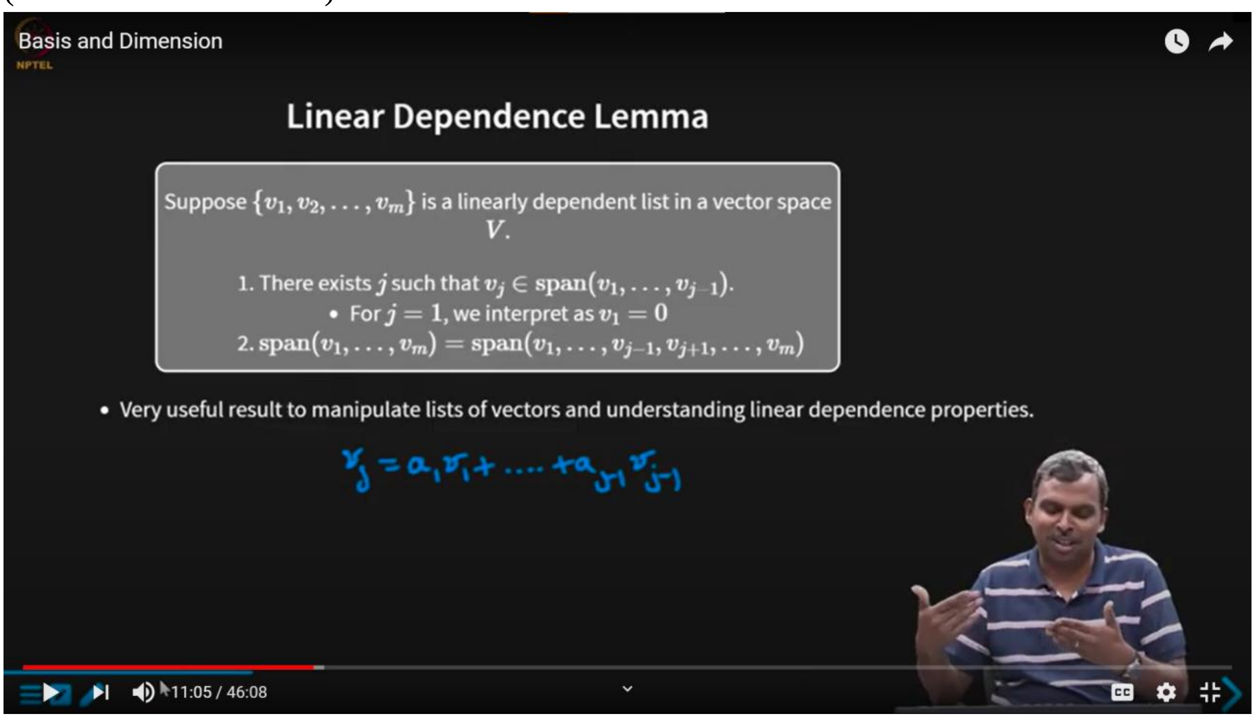


Okay. So now we slowly start taking steps towards technical aspects of this course. We have already seen mostly definitions. And here is a lemma which talks about linear dependence in vector spaces. And this is sort of a technical lemma. What do I mean by technical lemma? It's a lemma that's useful to prove other things. By itself, it will look like a simple result, okay? You may not look at it... You might look at it, read it and think - okay, what is the big deal in this result? So it is a very fair reaction, lot of people will have that reaction. But it is very useful, okay? So you will see in various other results. I will end up, at least in this class, using this lemma again and again and again. And it's a very powerful result to have. So many, many results in many areas and mathematics develop like that. There will be like one basic lemma, technical sounding lemma, which by itself may not mean much but eventually you'll see it's so useful in proving results, okay?

So let me read out what this linear dependence lemma is, okay? Supposing you have a set of vectors $v_1$ through $v_m$ and supposing they are linearly dependent, okay? So it is a linearly dependent list of vectors in some vector space $V$. So it turns out whenever you have a set of linearly dependent vectors, there is an index j, okay? Such that $v_j$, that is the j[th] vector belongs to the span of all the vectors that came before it... What does that mean? $v_j$ can be written as a linear combination of $v_1$ through $v_{j-1}$, okay? So let me write that down, that aspect of it. So $v_j$ equals, for that particular j, $v_j$ is some $a_1 v_1 + \cdots + a_{j-1} v_{j-1}$ okay? So there exists $a_1, a_2\ to\ a_{j-1}$ such that this is true, okay? So that's the linear dependence lemma.

Of course you have this sort of a bizarre situation - if j ends up being 1. What is the meaning of j being 1? $v_1$ is a linear combination of nothing, right? So nothing comes below it, before it, that we have to interpret as $v_1$ being zero, okay? So we'll sort of do a trivial definition for that. So quite often in mathematics this will also happen, you know? You will write a result and then there will be this trivial corner case for which you have to make an artificial definition, okay? So that always happens. So in this case, there is this trivial case that - if suppose $v_1$ ends up being zero, right? Everything else is linearly independent and $v_1$ ends up being zero, then the j here is actually 1 okay? And then when j is 1, you have to, you know, account for it as 0. So this special case we will just deal with it in that fashion, okay?

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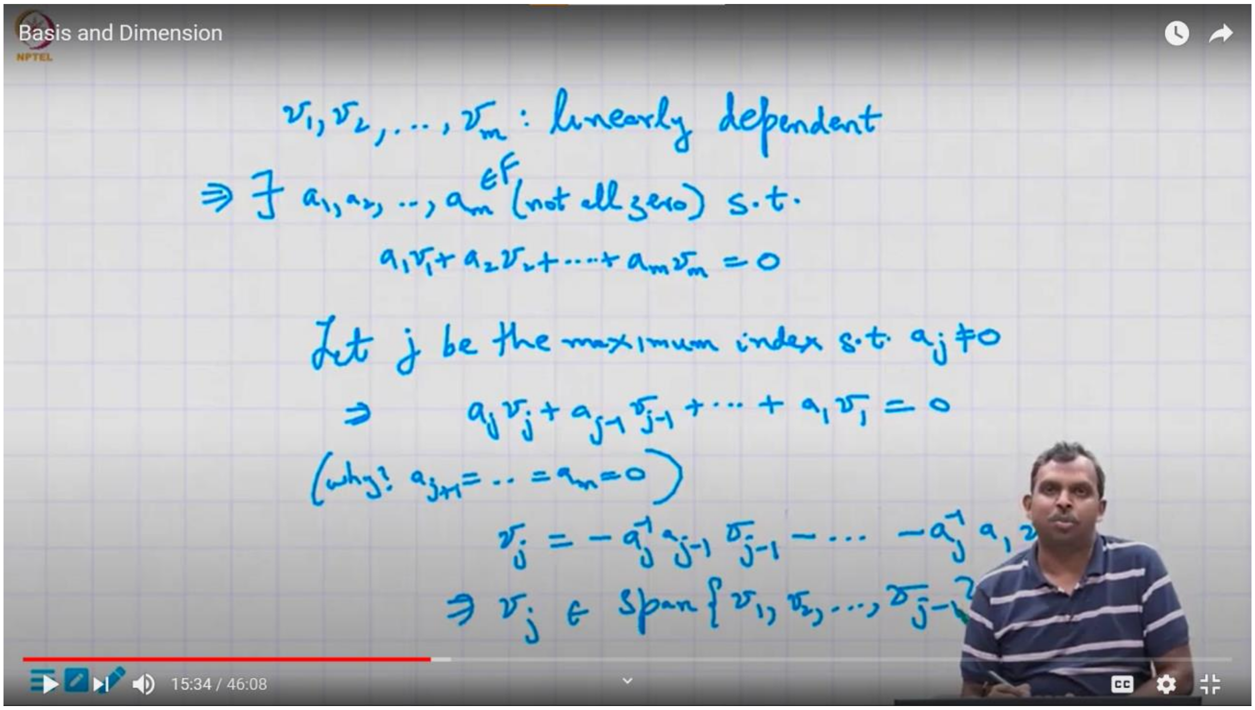

Not only that, that's the first result. It says $v_1$ to $v_m$ if somebody claims they have a linearly dependent set of vectors and they order it like that, they say first-vector, second-vector, third-vector, like that, you can find one $j^{th}$ vector which will be a linear combination of the j-1 before it, okay? That's the result. It's a very simple sounding result, you'll see in the proof that it's also a very simple thing to prove. It's not very complicated, but it's very powerful. You can use it in so many interesting ways and you'll see when I use it to prove other results, you'll see that it's very powerful.

Okay now look at the next part, okay? Next part is actually quite easy to visualize, right? So once you have $v_j$ being a linear combination of $v_1$ to $v_{j-1}$, if you drop $v_j$ from your list of vectors, the overall span remains the same. That is not very difficult to imagine, right? Span is all possible linear combinations. If you already know $v_j$ is a linear combination of $v_1$ through $v_{j-1}$ you might as well drop it from the list. You know that you can generate $v_j$ if you want and everything else is okay. All right so that's the second part of the result. We will prove the first part alone, it is not very hard to prove. You see the proof is quite easy. I will write down the proof in the white board, okay?

So notice $v_1, v_2, \ldots, v_m$ are linearly dependent, right? Okay? This has been given to you. That implies what? There exists $a_1, a_2, \ldots, a_m$ not all 0, okay? So that is what the linear dependence means. Such that $a_1 v_1 + a_2 v_2 \ldots$ Okay, so where do they all belong? They all belong to the field $\mathbb{F}$, right? $a_2 v_2 + \cdots + a_m v_m$ equals zero. By the way I do not keep repeating it in every lecture, but we will always assume we are dealing with the vector space $V$ over a field $\mathbb{F}$, and this $\mathbb{F}$ is usually $\mathbb{R}$ or $\mathbb{C}$. But in these kind of results, it can be anything, okay? So that's the way to think about it. Okay, so now among this $a_1$ through $a_m$, there will be a maximum index element which is not 0, okay? I will let j be that value, okay? So let j be the maximum index such that $a_j$ is not equal to zero, okay? There has to be a number like that. After all, there are only $a_1$ to $a_m$, okay? Maybe $a_m$ is non-zero, okay? If $a_m$ is non-zero, j will be equal to m. Now if $a_m$ is zero, you go to the next one before it, $a_{m-1}$ okay? And see if $a_{m-1}$ is non-zero. If it is not zero, then j will become $m-1$. If that is also zero, you go to $m-2$, like that okay? So from the right, or from the largest possible index, you keep coming down till you hit the first non-zero guy. You know that will always happen, right? Because at least one of them is not zero or non-zero. So you will hit that and that one becomes your index j, okay?

So now what happens? So now once you write it like this, so that will imply $a_j v_j + a_{j-1} v_{j-1} + \cdots + a_1 v_1 = 0$. Why is that? Because see, once you say j is the maximum index such that $a_j$ is not equal to zero, $a_{j+1} = \cdots = a_m = 0$, right? So in this summation, all of them will disappear. And $a_j$ is not equal to 0, so I can multiply by $\frac{1}{a_j}$ okay? Since $a_j$ is not equal to 0, I can multiply by $\frac{1}{a_j}$, I will get $v_j$ equals... I will move everything to the other side. $-\frac{1}{a_j} * (a_{j-1} v_{j-1}) + \cdots$. I should put minus here. $v_{j-1} - \cdots - \frac{1}{a_j} a_1 v_1$, okay? And we are done, okay? So this implies $v_j$ is in the span of $v_1, v_2, \ldots, v_{j-1}$, okay?

So like I said, the proof itself is really simple, it's easy to write down. You just look at it and argue that there should be a largest possible index, then after that it should be less. So it is a very easy argument. So we see that this is true, okay? So this innocent sort of looking result in some sense is actually very useful to derive very non-trivial results about vector spaces and you know, linear independence, dependence, basis, dimension etc. So when I actually prove it, you will see it, but this is a result that's good to know, okay? If you have a linearly dependent list, there is one element which is going to be a linear combination of everything below it, before it, okay? There is always something like that, okay?

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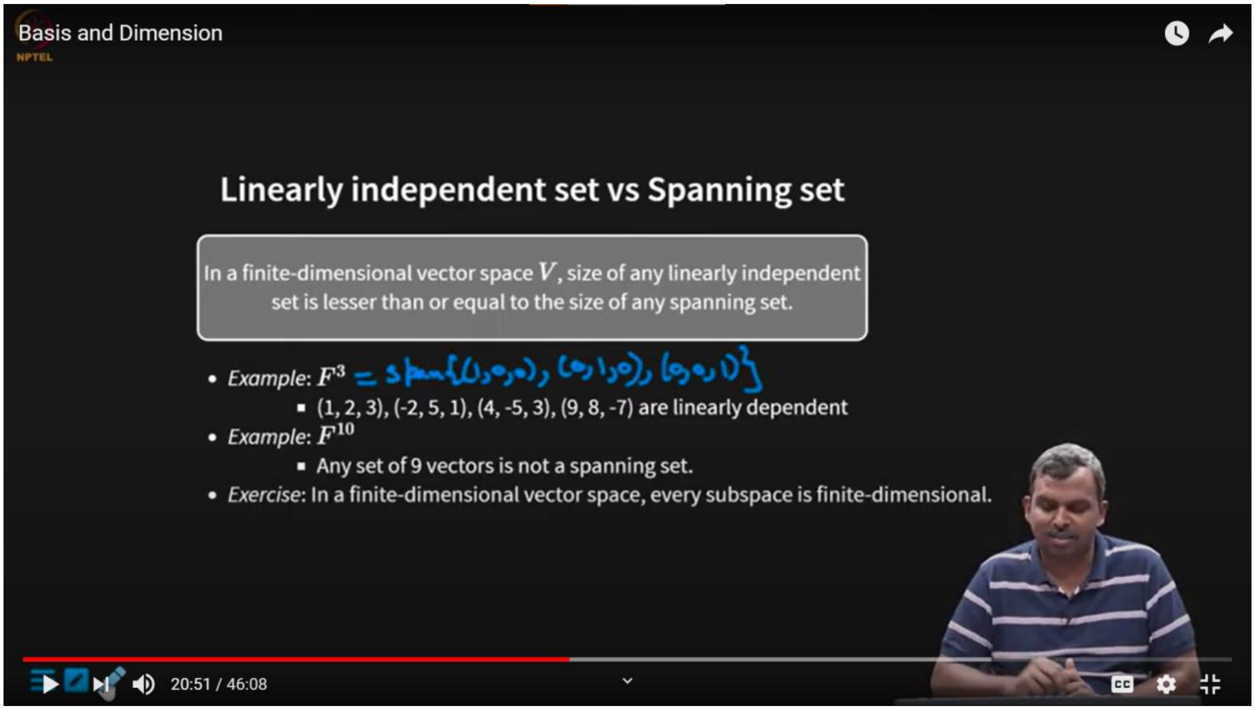


All right, let's move on. This is linear dependence lemma. I will use it again and again and again. Please remember that, okay? Linear dependence lemma. Okay. All right, so now we have defined different types of vectors, okay? A list of vectors satisfies different types of properties. Two important properties we have defined so far. One is the linearly independent property - you can have a list of vectors and you can say they are linearly independent. Another property we defined just about a little while ago in this lecture, is this spanning set property. You have a list of vectors and it ends up being a spanning set for the whole vector space. What does it mean to say spanning set? Every vector in the vector space is a linear combination of vectors from the spanning set, okay? So you have two different types of sets. One says it's linearly independent, no linear combination can, no non-trivial linear combination can give you a zero. Another is a spanning set. Every vector is a linear combination of this.

Now it turns out the size of these two sets are related in some way, okay? There is ordering in the size and that is the result here. This result is very important. In a finite dimensional vector space $V$, the size of any linearly independent set has to be lesser than or equal to the size of any spanning set, okay? So on the face of it, maybe some of you are surprised by this result. Why should the spanning set have a connection with the linearly independent? The linear dependence lemma is what will show you the connection. But we will come to that soon enough. When you see the proof, you will understand. But this is a powerful result to know, okay? If you have a spanning set, okay? It will have a certain size, there will be a certain number of elements in it. Because it is a finite dimensional vector space, there will be a spanning set, right? Any number of vectors more than the number of vectors in the spanning set have to be linearly dependent, right? So that's what's nice about this result.

Look at that, the example for $\mathbb{F}^3$ there, okay? $\mathbb{F}^3$ means you have a spanning set with three vectors, isn't it? $\mathbb{F}^3$, let me write it down, $\mathbb{F}^3$ equals span of $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ isn't it? All right? So we know that there is a spanning set for $\mathbb{F}^3$ with just 3 vectors. So if I give you a set of vectors, set of four vectors, right? Then they cannot be linearly independent. Why is that? Because if they were linearly independent, it would violate this result that you have, right? The size of a linearly independent set has to be lesser than or equal to the size of any spanning set. I have a spanning set of size 3, which means the maximum number of linearly independent vectors in $\mathbb{F}^3$ has to be only three, okay?

So look at the vectors here. You might think about linear combinations that result in, non-trivial linear combinations that result in zero. Maybe you can try to find it. But ahead of time, I know that there has to exist a non-linear, non-trivial linear combination giving you zero, just because I have four vectors and I have a spanning set of size three, okay? So that's what's nice about it. Same thing you can push, right? So for instance if you have $\mathbb{F}^{10}$, right? $\mathbb{F}^{10}$, you can argue that any set of nine vectors, will not be a spanning set. Why is that? Think about that, okay? If you take the standard set of $\mathbb{F}^{10}$, right? $(1,0,0,0,\ldots)$ $(0,1,0,0,\ldots)$, $(0,0,1,\ldots)$ like that, that is a linearly independent set, isn't it? You can show it's a linearly independent set. So which means any spanning set should have at least 10 vectors. So you cannot have a spanning set with nine vectors for $\mathbb{F}^{10}$, okay?

So these are the nice results that one can show once you have this result. In fact look at the final exercise. It's a nice exercise for you. You have to show that in a finite dimensional vector space, every subspace is also finite dimensional, okay? Think about why that has to be true, right? Every subspace should also be finite dimensional. As in if you have a vector space with a finite spanning set, okay? Then every subspace also has a finite spanning set, okay? You can prove that using this result. Think about how you would prove it, okay? So it is not too difficult to think about the proof.

But it's important to write down carefully, okay? So it's a nice little challenge to do this result, okay? So that's an exercise for you, you have to try it, see if you can think of a proof for that.

But what I will do in this lecture is to show you a proof of this result, okay? So it is a slightly complicated proof. If you have not seen too many of these abstract proofs before, it might be a little bit confusing. But stay with me, try and see if you can follow the proof. It is a very nice elegant argument, it uses the linear dependence lemma very strongly, okay? So let me write down a proof for this, okay? So let us say $u_1, u_2, \ldots, u_m$ is a linearly independent set, okay? And let us say $v_1, v_2, \ldots, v_n$ is a spanning set, okay? So we need to show what? $m \leq n$. This is our goal, isn't it? That's what we want to show, okay? So what we will do is this. This following sort of trick, okay?

I will consider this list $u_1, v_1, v_2, \ldots, v_n$, okay? So this is a list of vectors. It's got $u_1$ and then it has $v_1$ to $v_n$. Is that clear? Is that okay? Hopefully that is clear, okay? What type of set is this? My claim is this is linearly dependent, okay? All right. So think about why this is linearly dependent. Why is that? Because $v_1$ to $v_n$ is a spanning set, okay? So $u_1$ can definitely be written as a linear combination of $v_1$ through $v_n$, right? $u_1$ is also a vector in the same vector space, and $v_1$ through $v_n$ spans the entire vector space. So $u_1$ can be written as a linear combination of $v_1$ through $v_n$ and that makes it a linearly dependent set. So now by linear dependence lemma, there exist $j$ such that $v_j$ is a, belongs to the span of $\{u_1, v_1, \ldots, v_{j-1}\}$, okay? So now remember, $u_1$ belongs to a linearly independent set. $u_1$ cannot be zero by itself, right? So this, by linear dependence lemma, if you use, there has to be a $v_j$ which is a linear combination of these things, okay?

So what I will do is, I will drop the $v_j$ from the list, okay? And I know my span is, still remains the same. So I will have this nice result that $\{u_1, v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$ is a spanning set, right? Okay I said... Okay hopefully you saw the argument here. I can drop the $v_j$ and I can still have the same span, isn't it? Okay, so now I start with the spanning set, and what will I do? I will add $u_2$ now to this list, okay? So let us take $u_2, u_1, v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n$. This is linearly dependent, okay? The same argument as before, okay? You have a spanning set. If you have a spanning set, if you add another vector to it, you will definitely get a linearly dependent set, okay? Now you use linear dependence lemma, okay? Again there is, there has to exist a $j$ such that... Or maybe I will use $j_1$ such that $v_{j_1}$ one belongs to the span of vectors before it. Why is that? Okay? Linear dependence lemma says $j_1$ has to be like that.

But then remember $u_2, u_1$ are linearly independent. So $u_1$ cannot be in the span of $u_2$, right? So $u_2$ came in. But $u_1$ cannot be in the span of $u_2$ because $u_1, u_2$ are linearly independent. So when you use the linear dependence lemma, I know that there is a $j$, there is a vector which is in the span of all the vectors before it. But that vector has to be the $v$, right? It cannot be from the $u$'s, right?

So this $j_1$ has to be from among the $v$'s. So what do you do? You drop $j_1$ also, drop $j_1$ and add $u_3$ okay?

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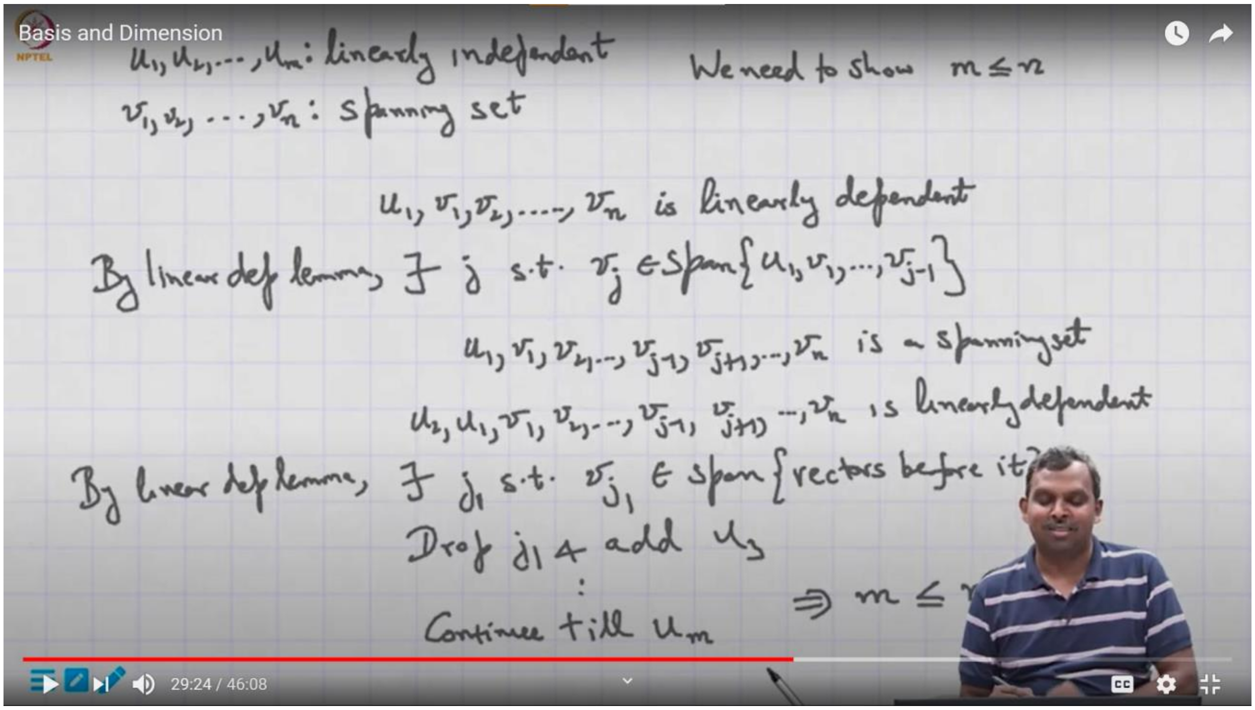


So like that, you keep proceeding, okay? You drop $j_1$, you will have a spanning set. You add $u_3$, you will get a linearly dependent set. You use the linear dependence lemma, it cannot give you any vector in the $u$'s because $u$'s are linearly independent. It has to give you a $v$, okay? And then you drop that, add a $u$, add a $u$, add $u$, you keep on doing that in every step. In every step you dropped a $v$ and added a $u$, you dropped a $v$ and added a $u$, and by the linear dependence lemma you can keep on going till you exhaust all the $u$'s, okay? Okay? Continue till $u_m$ okay? Now what did I do? In every step… There were a total of $m$ steps, right? In the first step I dropped $v_j$ and added $u_1$. In the second step, I added $u_2$ and dropped $v_{j_1}$, and the third step I added $u_3$ and dropped another $v$. Every time when I added a $u$, I dropped a $v$ okay? Without dropping a $v$, I could, I did not add a $u$. So what does that mean, right? And finally I was able to do it all the way up to $u_m$. So that implies $m \leq n$, okay? Think about that once again, it was slightly, maybe complicated, set of sequence.

So there is a stepwise way in which we are proceeding. In every step we are adding a $u$ and dropping a $v$, adding a $u$ dropping a $v$, adding a $u$ dropping a $v$, and I am able to go up all the way up to $m$. Which means in the spanning set, there must be at least $m$ vectors. Otherwise I would have gotten stuck somewhere, okay? And since I did not get stuck, there is no chance for me to get stuck, I can keep on adding $u$'s and dropping $v$'s, the number of vectors $m$ has to be less than or equal to the number of vectors $n$, okay?

So this wonderful little proof which uses the linear dependence lemma and the argument like adding the $u$’s in the left, okay? And then saying that a vector has to be a linear combination of everything before it, okay? And that's the nice little argument that comes out, and you see that the number of linearly independent vectors has to be upper bounded by the number of vectors in the spanning set, okay? A nice little proof, if you have not seen these kind of proofs before, it is probably very interesting and maybe a bit mysterious. But this is a very nice proof to show how size of a linearly independent set has to be upper bounded by the size of a spanning set, okay? So this is what I mean when I say, you know, a very innocent looking lemma which, you know, says something very obvious, can be used in a non-trivial way to get a very nice result which helps you come up with a lot of nice ideas, okay?

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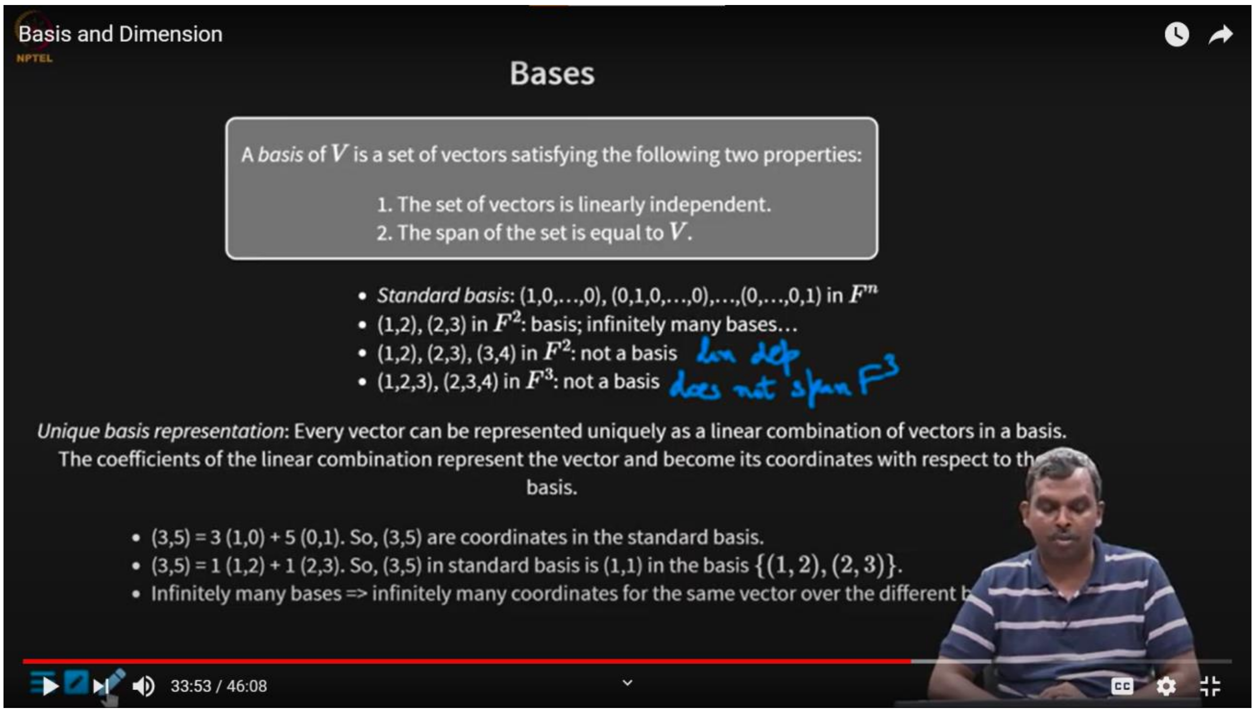


So why don’t you try this exercise, it's a nice exercise. Think about how you will show that in a finite dimensional vector space, every subspace is also finite dimensional, okay? So it's a very interesting little argument, you may have to use some careful, nice argument to show that, you know, that there will be a spanning set for the subspace, or you know some... You can start with some interesting ideas like that, okay? So that's the idea, okay? So anyway, we will discuss during one of our regular meetings how to do such proofs, okay?

This leads us to the next definition of basis. Basis is a very important idea in linear algebra. So we defined the spanning set for the vector $v$, vector space $V$. We also had this notion of a linearly independent set in vector spaces $V$, and there was a relationship between these two. Now a basis for a vector space is going to sort of be both a spanning set and a linearly independent set, okay? So that's the basic idea behind a basis, okay? So that's the definition in a box for you. A basis of a vector space $V$ is a set of vectors that has two properties - the span of that set has to be equal to $V$ and the vectors themselves have to be linearly independent, okay? So if you find a set of vectors like that - they're linearly independent and they span $V$ then that's called a basis. It's a special set of vectors in a vector space $V$, okay?

So I've given you some examples right below that. There's the standard basis. The standard spanning set that we gave before in $\mathbb{F}^n$, that's called the standard basis. But there are many other interesting bases, okay? In $\mathbb{F}^2$for instance, $(1,2),(2,3)$. $\mathbb{F}^2$ $(1,2),(2,3)$ is a basis. But in $\mathbb{F}^2$ (1, 2), (2, 3), (3, 4) is not a basis, okay? Why is that? Okay, so this is not a basis. Why? Because it is linearly dependent. okay? So the set will span the whole $\mathbb{F}^2$, okay? Spanning condition is satisfied, but the linear independence is violated, okay? It is not linearly independent anymore. What about this one? $(1,2,3)$ and $(2,3,4)$ and $\mathbb{F}^3$, okay? So it's not a basis. It is linearly independent but it does not span $\mathbb{F}^3$, okay? So think about how you can prove it, okay? You can find... How do you show that a set of vectors do not span $\mathbb{F}^3$? You have to find a vector which is not in a, not a linear combination of $(1,2,3)$ and $(2,3,4)$, okay? So take that as an exercise, try to find a vector which is not a linear combination of $(1,2,3)$ and $(2,3,4)$, okay? So you can do that.

So now what is interesting about a basis is this unique basis representation, okay? So this is really, really interesting and important. So once you have a basis, it turns out if you take any vector, okay? A basis is a spanning set, so if you take any vector, you can write it as a linear combination of the vectors in the basis, right? What is unique about a basis is - you can write it only in one way, okay? So if you take any vector and if you express it as a linear combination of vectors in the basis, it is possible to do it uniquely, okay? As in, you can have only one possible linear combination that will give you the vector $v$. If you do another linear combination you will get another vector, okay? But for one particular vector, you will have only one linear combination which gives you... So this gives you, gives rise to this, you know, vector representation of, a list of coordinates representation for a vector, okay? So if you have any vector, you take a basis from a finite dimensional vector space, you take a basis, okay? And then you write the vector as a linear combination of the vectors in the basis. So now you take the coefficients of those linear combination, list them out as coordinates. You get a coordinate system for the vector space, okay? So this is the idea behind this unique basis representation. It's very, very important.

So for instance, down below I have given a couple of examples. If you take the basis $\{(1,0),(0,1)\}$ and you look at the vector $(3,5)$ you can clearly write it as $3(1,0) + 5(0,1)$. You can also take the other basis $\{(1,2),(2,3)\}$ and the same $(3,5)$ can be written as $1(1,2) + 1(2,3)$. So you see that for the same vector (3, 5), if you change the basis, you get a different coordinate representation, okay? So this is an important thing to keep in mind. So you have any number of bases, so if you think of the 2D plane and you think of a vector, you have to think of the vector as being the same and as you keep changing the basis, the vectors' coordinates may change but the vector itself remains the same. So this picture of a vector independent of the basis is something very important. Usually all vectors are described in the standard basis, so once you describe the vector in the standard basis, the vector itself becomes the same, and if you change the basis the vector remains the same. The coordinates may change, okay? So something important to think about. And as you change bases, the coordinates can change for a vector. But given a coordinate system, given a basis, the coordinates get fixed, okay? So that's what's really important, okay? So we'll see more and more about bases and a lot of properties for bases and lots of interesting things as we go along, okay?

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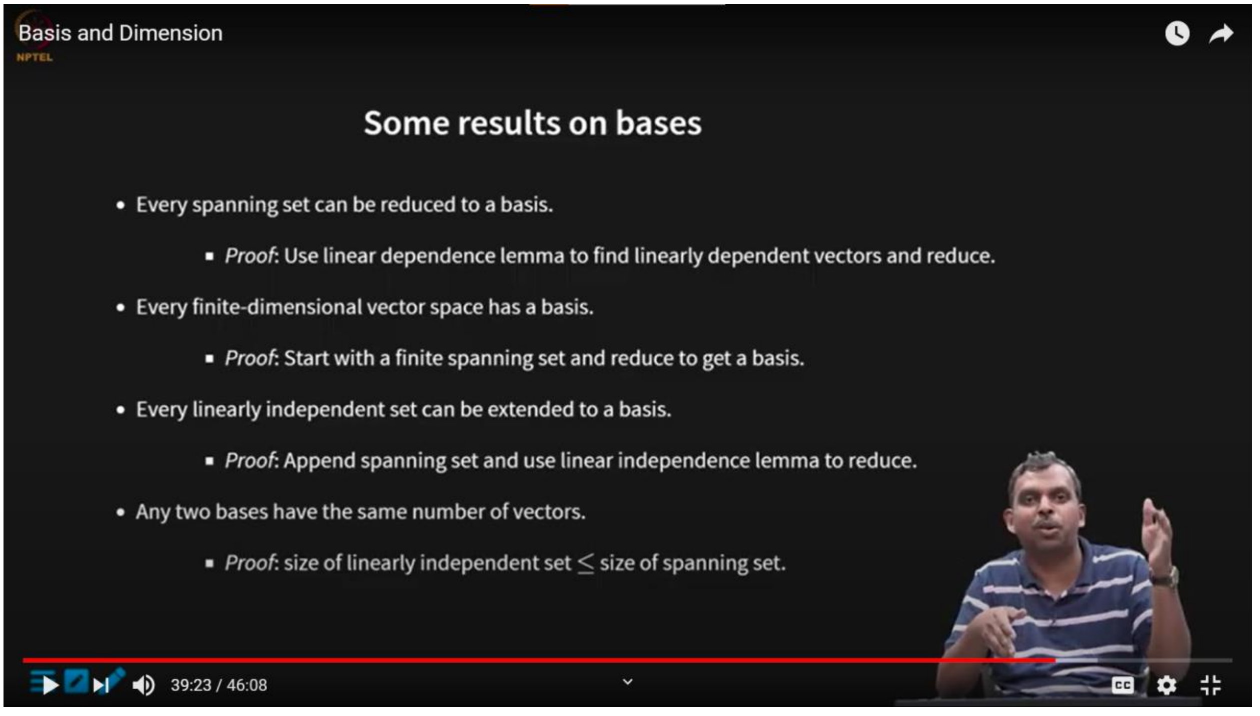


So here are three-four results already coming out for basis, okay? The first result, okay, I'm not going to write the proof in detail. You can see the textbook for a detailed proof. I'll just give you a motivation for where it comes from. The first result for basis is - every spanning set can be reduced to a basis. So now remember, if I give you a spanning set, already one of the requirements for basis is fulfilled, right? For basis, there are two requirements. It has to be a spanning set, it has to be linearly independent. Once I give you a spanning set, one requirement there's a tick mark, right? So it's already a spanning set. Now the only thing I have to make sure is - it could be linearly dependent. The vectors in the spanning set could be linearly dependent. Now what can I do? I bring in my linear dependence lemma. If the vectors are linearly dependent, use linearly linear dependence lemma to find the $v_j$ which is in the span of the previous case. You throw it out, span remains the same, so spanning set property is retained and then again you check if it's linearly independent. If it is not linearly independent. use the linear dependence lemma to throw away one vector, okay? So algorithmically how you do it is something we'll see later. but at least from a technique point of view, from a theory point of view, it's clear that if somebody gives you a spanning set you can reduce it down to a linearly independent set. Once you get a linearly independent set with the same span as the original set, you have a basis, okay? So every spanning set can be reduced to a basis, okay? That's the idea.

So the next result is very interesting. So once you know… So is there a basis, right? Is it guaranteed that every vector space should have a basis, okay? I told you there should be a spanning set, there should be a linearly independent set. Will there always be, for any vector space will there always be a basis? It turns out if the vector is finite dimensional, it will have a basis. Why is that? It's very easy to see. Just use the previous result, you know? A finite dimensional vector space has a finite spanning set. Every spanning set can be reduced to a basis. So you will have a basis, okay? So every finite dimensional vector space has a basis, okay?

And the next result is also interesting. Every linearly independent set can be extended to a basis. What is this extension, okay? So if somebody gives you a linearly independent set, okay? You take the entire spanning set for the vector space, add it to this set, okay? You will get a big set of vectors, your linearly independent set and the entire spanning set. Now you start using the linear dependence lemma one after the other. You throw away vectors from the spanning set till you get. right, a linearly independent set which is a spanning set. It's also a basis, it's also linearly independent, right? So that becomes a basis.

All right, so that's the argument for it. You can write it down in a very careful way so the... Just like... This is very similar to the previous proof that I wrote down. I don't want to write down a clear proof once again. If you have a linearly independent set you can append the entire spanning set to it, okay, right? And then... Or any other basis to it, and then use the linear dependence lemma. But you know since you started with the linearly independent set, the first few things will not really show up in your linear dependence lemma, it will show up only afterwards. So only the new spanning set that you added will get chopped down, chopped down, chopped down till you get to a linearly independent set. And that one becomes a basis. And it had your original independent set as part of it, okay? So maybe in your exercises and assignments we’ll give you some questions for doing this, okay?

The final result is what's most important. Remember a basis is both a spanning set and a linearly independent set, okay? What do I know about size of a linearly independent set? It has to be less than or equal to spanning set, okay? But suppose now you have two bases, okay? Somebody gives you two different sets which are both bases for the same vector space, okay? Each one is a linearly independent set, each one is a spanning set. So the first size has to be less than or equal to second size and the second size should also be less than or equal to the first size. So if you have two numbers, both have to be less than or equal to each other. What is the only way it can happen? Both have to be equal to each other and that is done, okay? So the last result is very, very interesting. Any basis, any two basis set of vectors has to have the same number of vectors, it cannot be different. If you have a vector space, a finite dimensional vector space, you cannot have a basis with 10 elements and another bases are 12 elements, okay? It violates the basic conditions that need to be satisfied. So any two bases have the same number of vectors, okay?

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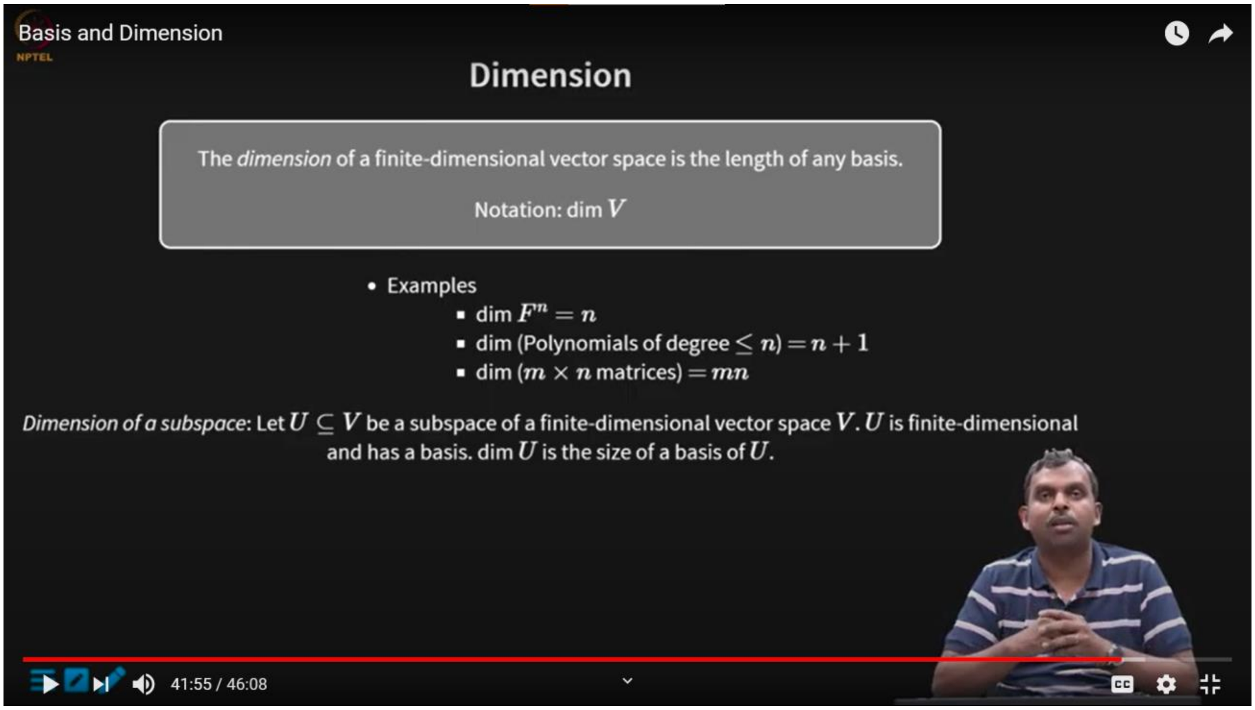


So interesting, isn't it? How do you start with some abstract notion of vectors, define some basic rules and define more things. more things, interesting things and finally you end up with very nice sounding results such as this. Okay, so this is the power of abstraction. All these results hold for any finite dimensional vector space, okay? So that is the nice thing about approving these kind of results. Okay. So next we come to dimension. So we saw that any two bases have the same length, okay? This leads us to the definition of dimension. The dimension of a finite dimensional vector space is the length of any basis, okay? So this number is very important and we will have a notation for it. We will call it dim, next to that $V$. okay? So anything I write like that becomes dimension of the vector space $V$, okay? So that notation is important.

So we can see some examples here. It's easy to write down dimension of $\mathbb{F}^n$, okay? The standard vector space is actually $n$, you can come up with the basis with $n$ vectors. And dimension of polynomials of degree $\leq n$ is actually $n + 1$, right? So if you say polynomial degree $\leq n$, the basis has $n+1$ vectors $1, x, x^2, \ldots, x^n$. What about $m \times n$ matrices? The dimension is $mn$, okay? The basis has $mn$ elements okay? So just like vector space has a dimension, the subspace also has a dimension. What is subspace? Subspace is nothing but a vector space itself, right? It's a smaller vector space which is living inside a larger vector space, right? So it's also a vector space.

So since… We also had a result. I gave that to you as an exercise. If the bigger vector space is finite dimensional, then any subspace is also finite dimensional. It's very intuitive, it's easy to see but you can write down a rigorous proof also, okay? So it has a basis also, and dimension of... $U$ is the dimension of the subspace so... Okay, so for subspaces also we will define the same notion of dimension, okay? So the notation will be the same. So if you have a vector space $V$ its dimension is dim $V$. A subspace of $V$ also we will use the same notation, okay? dim $U$ is the dimension for that, okay?

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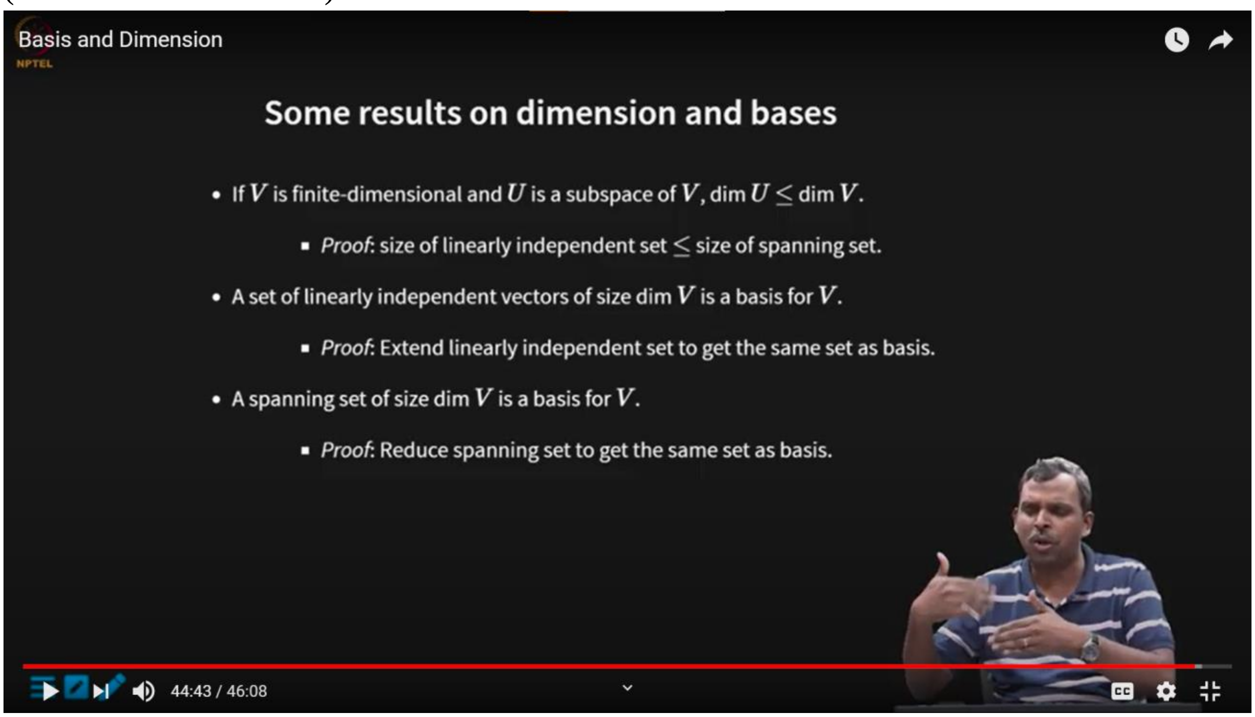


Okay so here are a few results on the dimension. First result is - if $U$ is a subspace of $V$, dimension of $U$ is less than or equal to dimension of $V$, okay? So it's sort of obvious in some sense, but you can prove it using the result that size of a linearly independent set has to be less than or equal to size of a spanning set, okay? The next thing is a set of linearly independent vectors of size dim $V$ is a basis for $V$, okay? Notice this interesting thing. Once you know the dimension of a vector space, any linearly independent set of size equal to dimension of $V$, okay, will automatically become a basis. What does that mean? It can span the whole set, okay? So in $\mathbb{F}^3$, if I take three linearly independent vectors, I have a basis. It also spans the whole set, okay?

Right, so how do you prove this result? You take your set of three, a set of, you know, linearly independent vectors of size dim $V$, then append the entire spanning set to it and you use the linear dependence lemma. Reduce, reduce, reduce, extend it to a basis. Of course it can be extended to a basis. Once you extend it into a basis, the size has to be dim $V$. Anyway originally you started with dimension $V$. Those are the only vectors that will be left, okay? So it's a very clean way to rigorously prove this result that if you know the dimension of the vector space $V$, right, you just take linearly independent set of size dim $V$. You have a basis automatically. It will automatically span the whole space it cannot leave any vector, okay? Nice result, isn't it?

Same thing about a spanning set. If you have a spanning set of size dim $V$, that will also be a basis. If a set of vectors of size equal to dim $V$... What do I mean by size? The number of vectors is equal to dimension, dimension of $V$ and it spans the whole space. So in $\mathbb{F}^3$ for instance, if you have three vectors which can span the whole space, they have to be linearly independent. So they become a basis. How do you do that? Again you use the same thing. You take the spanning set and reduce it to a basis, okay? Originally you started with dim $V$. When you reduce to a basis you will also have dim $V$ which means nothing got reduced. It has to be linearly independent, okay? So these are all interesting, small, you know, small ways in which you use previous results to build on the existing things and get more and more interesting results, right?

So this is how a lot of abstract mathematics proceeds. You start with the basic definition, then you have some lemmas. You prove some basic result and then you use it more, use it more, use it more, use it more, figure out more ways of using it to get more and more non-trivial results, okay? So notice how these results are becoming more and more non-trivial. In any vector space if you have a linearly independent set of size equal to dimension, it's already a spanning set. If you have a spanning set of size equal to dimension it has to be linearly independent also, okay? So nice results to know, okay?

Okay so here are three problems. I'm not going to solve them for you in this lecture, but please try them. It's very important to get your understanding of everything right. So here are three subspaces that I've defined for $\mathbb{F}^3$ and for $\mathbb{F}^5$. I've given various conditions, you can check that these are subspaces. I want you to find a basis for these subspaces, okay? So how do you find a basis for the subspace? Think about the conditions you need, right? How do you find a basis for a subspace? It has to be linearly independent, it has to span. So you have to check both. Come up with a list of vectors, right, which will be linearly independent, which will also span the whole space, okay? So take up these problems, try them, we will discuss them when we have a chance to meet live. Okay, thank you very much.

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