Applied Linear Algebra
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Week 01
Sums, Direct Sums and Gaussian Elimination

Hello. Welcome again to this lecture. In this lecture we will talk primarily about sums, and what are called direct sums of subspaces. This is a very crucial idea. When I introduced subspaces, I mentioned how you can sort of do divide and conquer when you want to understand a vector space. And maybe it's too big, you want to think of a smaller subset of it which behaves like a vector space, which is a subspace, and then try and understand that. And presumably, you know, sort of divide the vector space into all these subspaces and once you've understood the subspaces, you can end up understanding the vector space itself, or how it works and various things. So subspaces play a crucial role, and particularly the sums and direct sums also play a crucial role in breaking down things for us.

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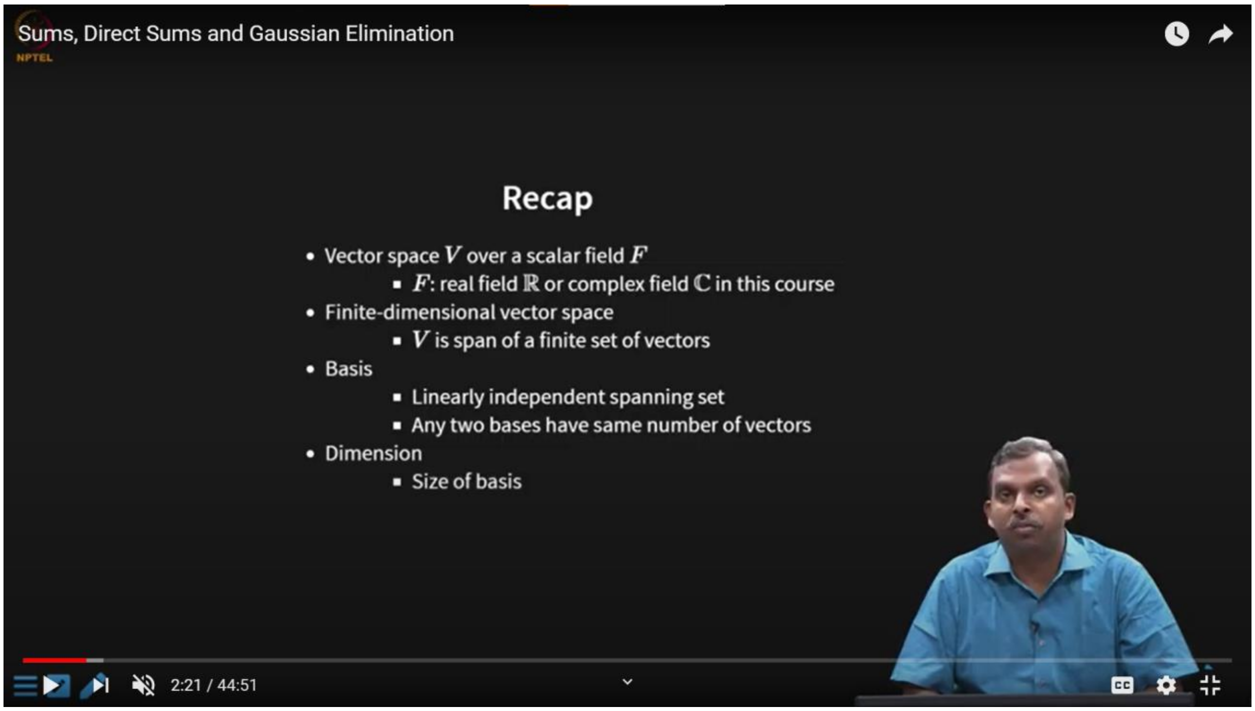

I also mentioned Gaussian Elimination. I will come back to it towards the end of the lecture, maybe a slide or two to describe that process. Gaussian Elimination, primarily we'll study to... It's sort of an algorithmic method which is very common. In the numerical examples that we will see, it is useful for me to ask you questions, and ask questions about linear dependence, independence, and for you to easily answer. So this is the main subject of this lecture, let's go on.

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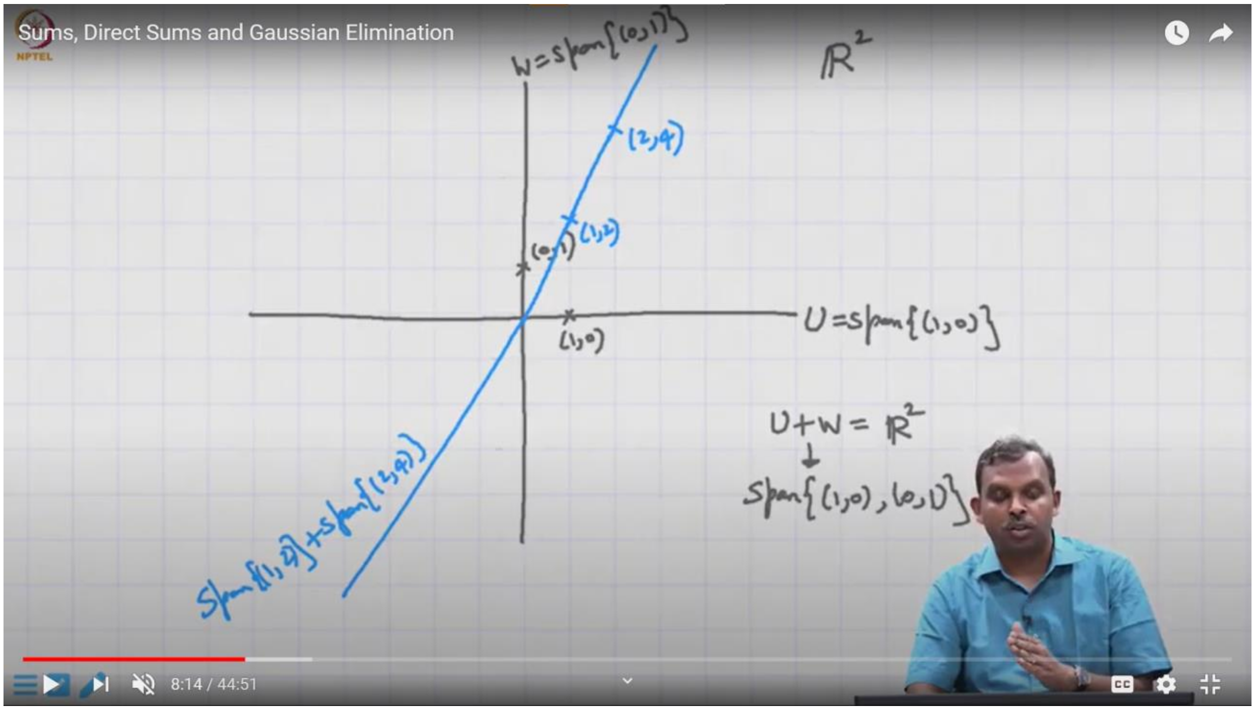

Okay, so I think I had a quick recap. Yeah, there you go. So once again we will be talking about vector spaces V over a field $\mathbb{F}$ and most of our results in, even in this lecture, applicable for a general field $\mathbb{F}$. But specifically in this course mostly, when I say field $\mathbb{F}$, I am thinking of the real number field $\mathbb{R}$ or the complex field $\mathbb{C}$. We will be looking at finite dimensional vector spaces mostly, where, in which case, the vector space V becomes a span of a finite set of vectors, right? And then we introduce the notion of basis and dimension in the previous lecture. Basis is basically a linearly independent spanning set, okay? Set of vectors which are linearly independent and they also span the space, okay? And then we saw this result that any two bases have the same number of vectors, and that gave us lots of interesting results about how to find the basis and what properties it has etc. And then finally, dimension of a finite dimensional vector space is basically the size of a basis. Number of linearly independent vectors which span the whole space, okay? So that's a quick recap. Let's jump into sums and direct sums of subspaces, okay?

So the sum itself is actually very easy to define. Supposing... It can in fact be defined even for two subsets, not necessarily subspaces. Supposing you have two subsets U and W in any vector space V let's say, the sum $U + W$ is defined like shown in the slide, okay? So $U + W$ is basically - you take one vector from U, one vector from W and add them, okay? So you can do it in so many different ways, right? You can pick any vector you want from U, any vector you want from W, add and you get another vector. So you put together all these vectors, then you get a set of vectors. That set of vectors is called $U + W$, okay?

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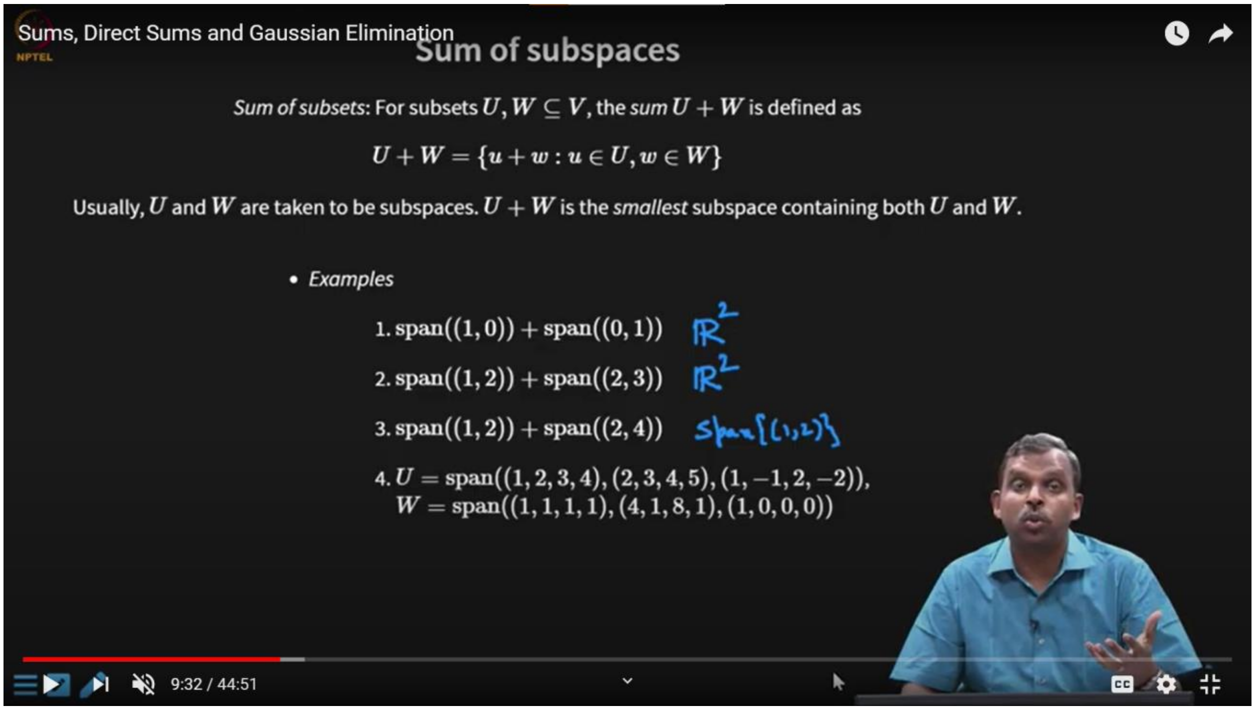

So it's sort of a natural definition, when you say one vector plus another vector, there is only one vector here, one vector here, you add them, you get another vector. Now the first vector, you can pick from a set U, another vector you can pick from a set W, so when you can add all possible combinations like this, you will get a much bigger set. And then that set we call $U + W$, okay? So sort of a natural extension. So usually when people think of sums of sets... It's interesting to think of U and W as being subspaces themselves, okay? So U and W, we are going to take them to be subspaces, and in which case $U + W$ will definitely be a subspace, okay? If U and W are subspaces, $U + W$ will be a subspace. In fact, it will be the smallest subspace containing both U and W. So these are things that... This word smallest is something that you need to prove, and that that is something you can think about. What it means etc. So if you want a subspace to contain both U and W, it turns out $U + W$ will end up being the smallest such subspace. We won't prove those results, just for you to think about. Why that means something, okay?

A bunch of examples here, okay? So I have taken a lot of examples from $\mathbb{R}^2$, and maybe one which is a little bit bigger to illustrate to you what can happen. So maybe I should go to this picture here and show you these illustrations in $\mathbb{R}^2$. So let us once again draw our axes… So this whole course is an exercise for me on how to draw straight lines properly. Okay, so you can see I have not done a very good job here, so let me draw... One second. My goodness, it's going all over the place, so let me draw from here to there. Let's see, this is better, yeah. That's better. Okay, so that's the y-axis and then you have the x-axis, okay? There you go, that's the x-y plane, the $\mathbb{R}^2$ vector space, the way we have seen it.

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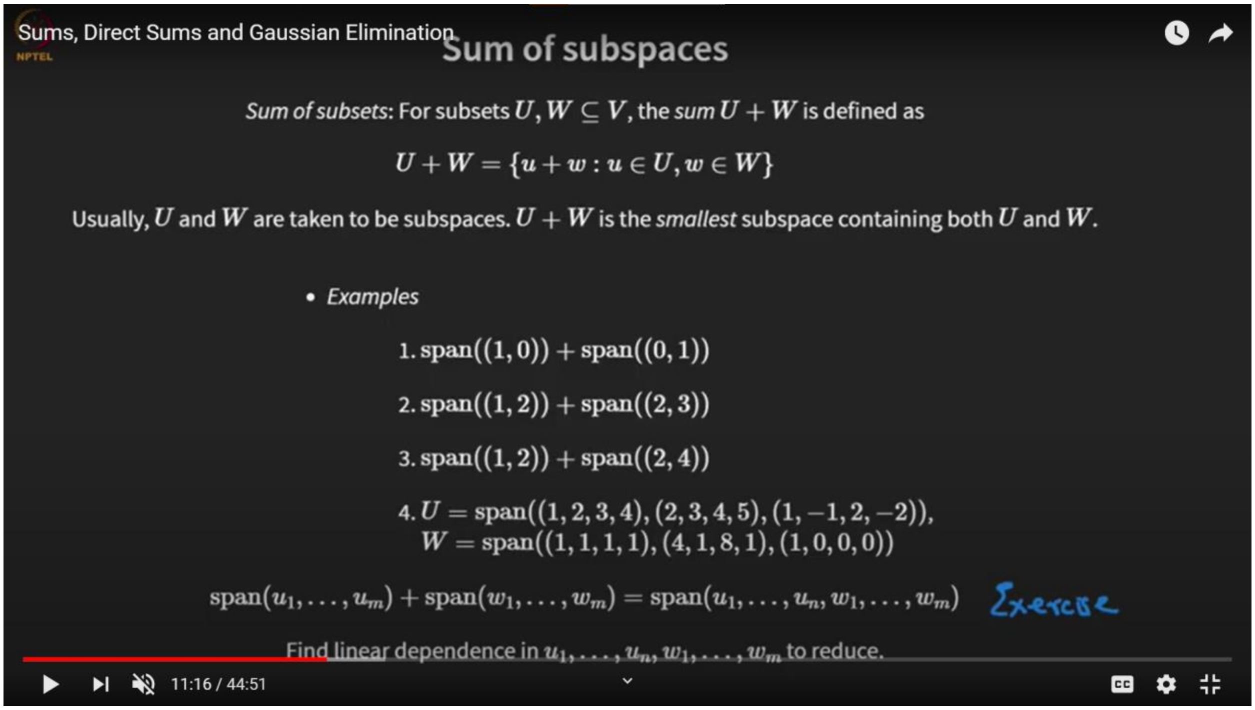

I'm talking about two vectors first. It's $(1,0)$ and $(0,1)$, okay? So my U is basically the span of $(1,0)$ in the first example, and then my W is the span of $(0,1)$, okay? So what will happen when I do $U+W$, okay? So you can sort of imagine. So any vector in the span of $(1,0)$ is any multiple of $(1,0)$. Then any vector in the span of $(0,1)$ is any multiple of $(0,1)$. So you multiply $(1,0)$, multiply $(0,1)$, what will you get? You will get the span of both the vectors together. So this just becomes $\mathbb{R}^2$, okay? You can think about why that is true. Maybe an intermediate step for this is basically - this is the span of $(1,0)$ and $(0,1)$, right? So you just put the two things together, whether you, you know, think of the span first and then you do the plus, or you put both vectors together and think of the span together, it's the same thing, right? So you multiply and then add them, you get this, okay?

There are other examples that have also been given here. $(1,2)$, $(2,3)$, okay? So all of these things will also end up being... So this will also end up being $\mathbb{R}^2$, this will also end up being $\mathbb{R}$... This will, what will this end up being? So maybe this is an interesting example. I should show you. Okay, so these two are $\mathbb{R}^2$. What about this one? $(1,2)$, $(2,4)$. So let us go back to this board, then look at $(1,2)$. $(1,2)$... so maybe I should draw this in blue. $(1,2)$ and then $(2,4)$. Notice $(2,4)$ is a multiple of $(1,2)$, so I know both of them are in the same subspace, right? So there $(1,2)$ and

$(2, 4)$ are linearly dependent. right? So if you draw a line through the origin, it's going to go through both $(1, 2)$ and $(2, 4)$. So if you look at span of $(1, 2)$, that would be this straight line and that straight line also contains $(2, 4)$, okay? So maybe I didn't draw it very well, but this is nothing but a span of $(1, 2)$ and it is also the span of $(2, 4)$. In fact if you do even span of (2, 4) you will end up getting the same thing, nothing much will change, right? So because $(1, 2)$ and $(2, 4)$ end up being linearly dependent, the span of $(1, 2)$ will contain $(2, 4)$ itself and the span of $(2, 4)$ will also contain $(1, 2)$. So they are all just the one set. So when you do a plus, you don't end up getting the whole $\mathbb{R}^2$, okay? So that you can see. So this is simply span of $(1, 2)$, right?

So look at the fourth one. It's a little bit more tricky. Now I have gone to $\mathbb{R}^4$, okay? So you have this… So you have this, no? Larger possibilities. So what I've done here is I've taken a U which is a span of three vectors in $\mathbb{R}^4$, right? $(1, 2, 3, 4), (2, 3, 4, 5)$ and $(1, -1, 2, -2)$, okay? Three different vectors in $\mathbb{R}^4$, they are going to span this subspace U. Then I have a subspace W which is spanned by three other vectors, okay? $(1, 1, 1, 1), (4, 1, 8, 1), (1, 0, 0, 0)$, okay? And then now if I ask the question - what is $U + W$, okay?

I mean, you can start seeing why this is a more difficult question to immediately answer, right? So even for $\mathbb{R}^4$, we are having to think a little bit, have a method which would work. Now supposing you go to my, you know, big example of $\mathbb{R}^{1000}$, and then hundred vectors in U, hundred vectors in W. And then you know, span of hundred vectors, span of hundred vectors, and you do the sum of those two. You are going to have a big problem in your hands numerically. How to establish those things is difficult, okay? So we need a good method, and the method is actually not very hard. You can see that in general, span of a set of vectors plus span of another set of vectors is basically span of all those vectors put together, okay? So you just collect all of them together, you will get the span. So now if you want to understand what that is, you take those set of vectors and you find linear dependence, and reduce, okay?

So span of u_1 to u_m is given to you. Span of your w_1 to w_m is known to you. So how do you find the sum of those two subspaces? You simply take the list of vectors together and collect them together, and find the span of all of them, okay? So this is not very difficult to show, I will leave this as an exercise for you, okay? So you should try to prove it, okay? So please try to prove this. It's an interesting problem to think about, it's not very hard. I think if you try to write it down, you will get it. So once you know this description, and supposing you have a method to implement your linear dependence lemma, right? You remember your linear dependence lemma? So once you, if you know that the list of vectors is linearly dependent, you know that there is one vector that will be dependent on all the previous ones. You can throw it out, throw it out like that. If you have some way of implementing that, so you can go and find out the linear dependent vectors, throw them out and finally end up with the basis for the sum, okay? So this is possible, this would be the most general way to find span, okay?

So that is the bottom line here, right? Okay, so that's all I want to say about the sum of two subspaces. So you have to think about this a little bit more. So for instance, if you have the real line. Subspaces are lines through the origin, you take any two different lines and add them, you end up getting the whole space, right? So it's almost as if the $\mathbb{R}^2$ is, can be decomposed into two lines, in some sense, right? So you can add them. Same with $\mathbb{R}^3$ also. $\mathbb{R}^3$ you can take three different lines through the origin, and as long as they are not, you know, dependent in some way, you will end up getting the whole $\mathbb{R}^3$, okay? So you can do these kind of decompositions, you can think of the whole space, and along certain directions indicated by subspaces, along some subspaces. Then add them up together and form the space. Same way to think of $\mathbb{R}^3$ as - you know the 2D plane, plus the z-axis, right? So you can think of it like that. That also works.

So lot of decompositions like that are possible. In fact, infinitely many decompositions you can do, no? Instead of just the x-y plane, you can go to some other plane and then take the normal through it, as we studied. Then this plus that would work. You can also in fact decompose the plane itself as a sum of two lines in so many different ways. So you have so many different ways of decomposing a vector space into sums, okay? And it's all interesting to study, okay? In fact you can even do more interesting things. You can take two planes, okay? Two planes can add to give you the whole space, right? $\mathbb{R}^3$ for instance, okay? So all these kind of interesting decompositions are possible, okay?

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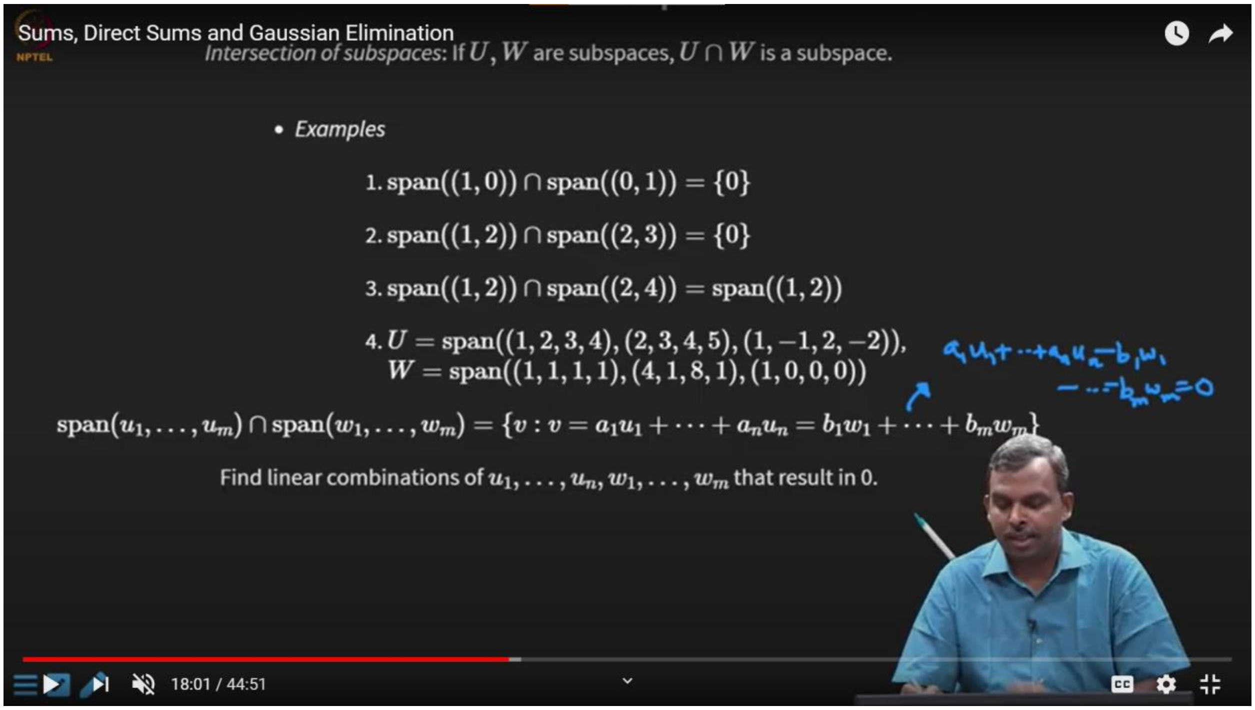

So let's move ahead. So the next interesting thing to study and sort of get our head around is this notion of intersection of subspaces, okay? Supposing you have two subspaces. $U \cap W$ is also a subspace, okay? So this is an exercise. Again, we won't prove this in this class, but it's quite easy to imagine why it should be. So if you have two subspaces, individually inside them they are closed under linear combination. If you take the intersection, of course they will also be closed in the linear combination. You can prove it rigorously, write down a proof if you like, that $U \cap W$ is also a subspace. And I've shown you here a lot of examples to illustrate what happens, okay? So this is sort of reminiscent of the previous example we saw in the summation. If you take span of $(1, 0)$ and intersect it with span of $(0, 1)$, you see that you get the trivial subspace 0, okay?

So couple of things to remember. So this 0 is a trivial subspace, okay? Zero-dimensional subspace. It has only the vector zero, and also you should remember V itself, the entire vector space itself, is also a trivial subspace, okay? The full vector space is also a subspace. So usually when we think of subspaces, we are usually not thinking of zero and V itself, but technically they are valid subspaces, okay? So you should keep that in mind, okay? So this $(1, 0)$ and $(0, 1)$, you can imagine from your picture. x-axis, y-axis, they intersect only at zero. Same thing with $(1, 2), (2, 3)$, they intersect only at zero. But we saw before that this $(1, 2)$ and $(2, 4)$, right? They are actually both the same subspace, and their intersection ends up being another subspace, okay?

Let's move on to this slightly more interesting question. I have $\mathbb{R}^4$ here, and U is a subspace spanned by three vectors, okay? They are given here. $(1, 2, 3, 4), (2, 3, 4, 5), (1, -1, 2, -2)$. Same as the previous slide, okay? And W is another subspace that is spanned by three different vectors. They are given here, okay? I think, let's just use that, okay? So this is spanned by three different vectors. So the question that is being asked here is - what is the intersection of U and W? How do you go about finding intersection of subspaces in a methodical, you know, sort of numerical way? How do you find it, okay? So it doesn't seem very clear that… There are some things that you need here. It's not very obvious, at least, how one can immediately find out. Because, see, there are all sorts of linear combinations, no? I mean, I should find linear combinations of the first three which will also be linear combinations of the second three, right? Right? So that's the way to think about it.

So it turns out, we will see later on how to precisely solve these kind of problems. But I want to motivate the problem from this point of view. So you see simple problems in linear algebra require some tools and methods which we do not have yet, okay? So we have to develop all of those, okay? So here is the description of how to do intersections generally. Supposing somebody gives you a vector space as a span of u_1 through u_n, and another vector space W which is span of w_1 through w_m, and you have to compute the intersection. You have to find vectors v such that v can be written as a linear combination of the u's and, also as a linear combination of w, okay?

So this is the equation that you sort of have to solve for the coefficients. So you find linear combinations of u_1 through u_n that result in zero, right? So what do I mean by that? So you can sort of rewrite this as $a_1 u_1 + \cdots + a_n u_n - b_1 w_1 - \cdots - b_m w_m = 0$, okay? So if you find a linear combination of u's and w's, okay, which result in 0, then you have a vector in the intersection, okay? So you can sort of use this idea and then find the intersection. So you need to find linear combination. So this thing of finding linear combinations, finding dependent linear combinations, as in, given a set of vectors, how do you find which of them are linearly dependent, how do you implement the linear dependence lemma in some sense, right? So that is a very important numerical method that we need to have, it's showing up again and again. When I wanted to find $U + W$, it showed up. When I want to find the bases, when I want to reduce it to the basis. Now here, when I want to find the intersection of two vector spaces, right? Two subspaces, you are immediately needing these kind of linear combinations. Given a set of vectors, can you find a linear combination, non-trivial linear combination which will result in zero, okay? So these kind of questions will crop up again and again in numerical problems, and having a method for this is very important. We will see later on how, in this lecture itself, how Gaussian Elimination provides the starting point for a method like this, okay?

So this is intersection of subspaces. It's easy to define and it's numerically... We don't yet know how to find it, but there is a method. We can eventually get to it in this class, okay? So that is the intersection. Now it turns out, there is a very interesting connection between intersections, sums, and all that. And that connection goes through this notion of a direct sum, okay? So the most interesting sum of subspaces is this direct sum, okay? So how does direct sum work? It is actually, basically, a very simple definition. So we already saw the sum of two subspaces, right? Given two subspaces U and W, $U + W$ is the sum. We know how to do it. Now there can be this special case where the intersection of U and W is the trivial zero subspace, as in U and W are sort of linearly independent, mostly, okay? Vector and a non-zero vector in U is sort of not non zero... I mean cannot be formed from the basis for W, you know? I mean there is no intersection, there is no non-trivial intersection. No non-zero vector is common to both U and W, okay? So that kind of a situation is special when you do sums.

So when you have U and W and they do not have any intersection other than the trivial intersection 0, okay? Then the sum $U + W$ is given a special name, it is called a direct sum, okay? So we usually denote it with a U $\oplus$ to illustrate that it is one of those exclusive sums, the direct sum, where, you know, U and W have nothing in common other than the zero. By the way, any two subspaces have to have zero in common, right? Because U has zero, W has zero, so the intersection should have zero in common. So that is the smallest intersection possible in some sense, right?

Okay, so here is the most interesting result in this idea of direct sums, okay? So that's put in this box for you here. So supposing you have a vector space V. It is finite dimensional and somebody gives you any, some subspace of V. U is some subspace, okay? It can be any subspace. It turns out there is another subspace W of the same vector space, of course, such that the vector space itself can be formed as a direct sum of U and W, okay? So this I sort of call as a, you know, subspace decomposition idea, okay? So you have a vector space. If you can identify a subspace somehow, which is interesting to you, it always turns out that the entire vector space is actually a direct sum of U and some other subspace W, okay? So there is always another subspace so that they have a trivial intersection and they direct sum to give you V, okay?

So this again, you know, sort of solidifies our idea of studying vector spaces. By looking at subspaces, okay? Think of two different subspaces and think of different subspaces, different ways in which you can decompose. And this gives you a solid way to say the entire vector space itself can be sort of formed as direct sum of subspaces, okay? So the proof itself is not very hard. It's in your book. I will write down, maybe, a couple of steps. I have given you the idea here as two points, and that is what we will develop.

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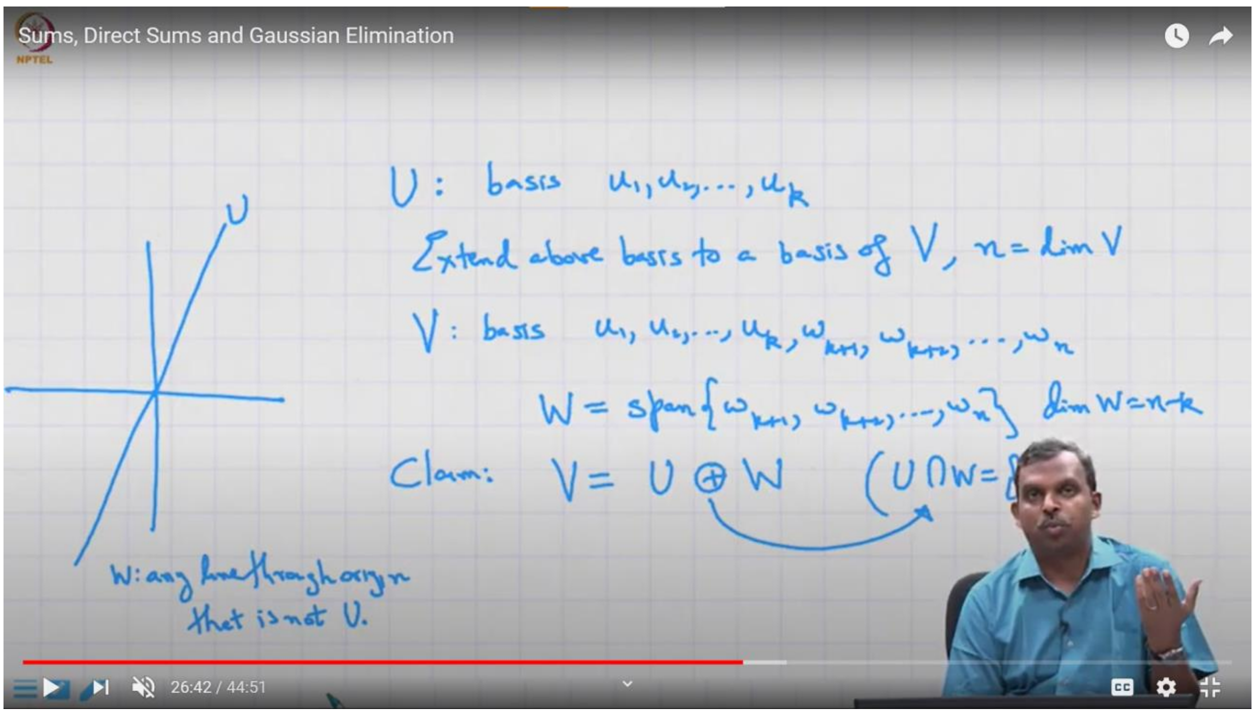

Supposing you have a subspace U, right? U will have a basis and you take that basis and you extend it to a basis of V. We know that's always possible, right? So any linearly independent set can be extended to a basis of V. Of course you take... The subspace basis is also linearly independent. So you can extend it. So let me just write down a couple of lines in the proof. So U is basically, has basis, let us say $u_1, u_2, \dots, u_k$, okay? So we will extend above basis to a basis of V, this is crucial, of the entire vector space V, okay? So then V will have a basis $u_1, u_2, \dots, u_k, w_1$, okay? Maybe, write like this. $w_{k+1}, w_{k+2}, \dots, w_n$, okay?

So what is n? n is dimension of V, okay? We are assuming it's finite dimensional, okay? So we know this is always possible. You can extend a set of linearly independent vectors u_1 through u_k. It's linearly independent, It is also a basis for U. I can take that and extend it to a basis of V. Now the claim is - I can also define W as the span of $w_{k+1}, w_{k+2}, \dots, w_n$. So notice that the dimension of w is $(n-k)$, right? $(n-k)$ linearly independent vectors, they span the W. So that gets there, okay? Now the claim which is not very hard to prove is $V = U \oplus W$, okay?

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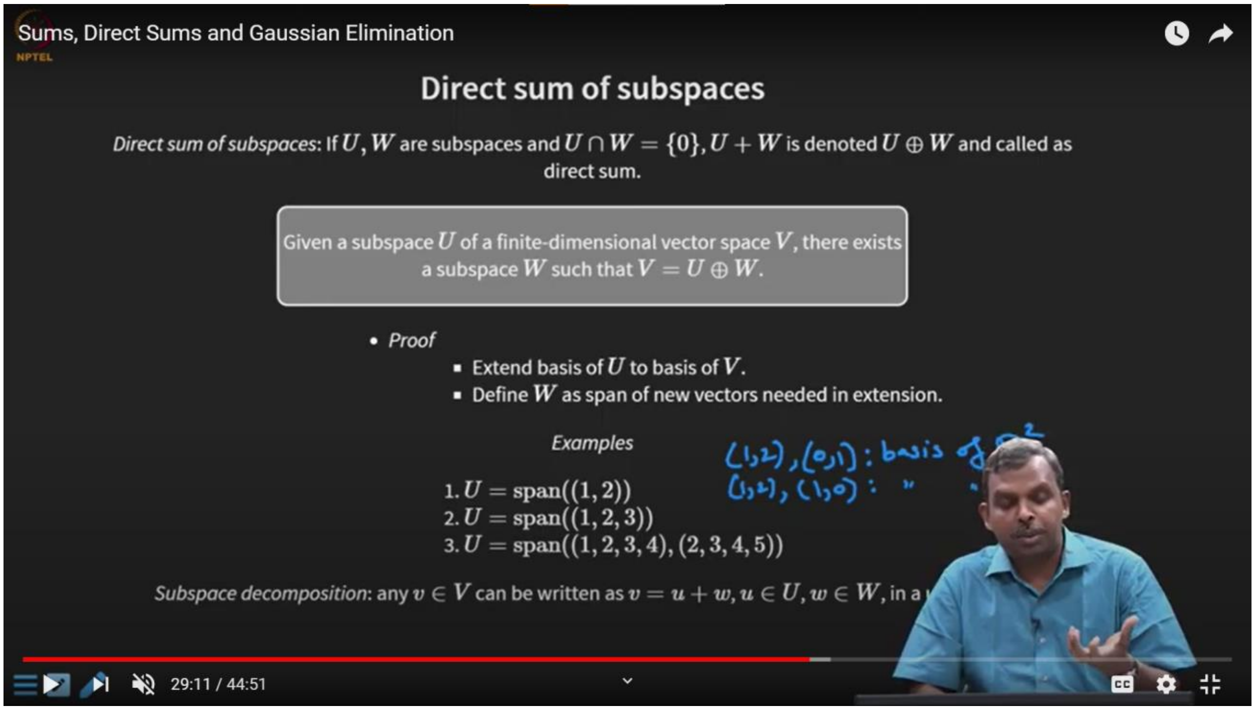

So the fact that V equals $U + W$ is quite easy to establish because, you know, $U + W$ gives you the entire basis. I told you how to find span of those two, it's okay. Maybe slightly tricky is to say that U and W have no intersection, okay? That's also very easy to establish. If U and W have a common intersection, non-zero intersection, what should happen? There should be a linear combination of U. u_1 through u_k which should be equal to another linear combination of w_1 from w_k, okay? It should be non-trivial, which means what there is a linear combination of u_1 through u_k and w_{k+1} to w_n which gives you 0. Which means those are linearly dependent. But that is not possible because they are a basis, okay? So it is easy to show that these two are true, okay?

Okay, so this is sort of implied when I say direct sum, okay? The moment I write direct sum, this implies this, right? I am also claiming that that is true, okay? So it's V equals $U \oplus W$, meaning I am claiming $U \cap W$ is zero, and when you sum them, I get V, okay? So it's not very difficult to prove. I am not going to write down the mechanics of it. I gave you the argument of how it worked. If U and W have a non-trivial intersection, have a non-zero vector in common, then a linear combination of a basis should be 0 and that is not possible, right? Because basis has linearly independent vectors, okay? So that is a quick proof of this nice result.

Now it turns out a lot of interesting things one can do with this. Here is, here are a few examples, right? I've written down a few examples. So one can think of various ways to, you know, think of using the subspace decomposition and understand what it means. So let us take the first one. Maybe I will write a few things here to illustrate. So this is U. Now how can I extend $(1,2)$ to a basis of $\mathbb{R}^2$? So it turns out there are infinitely many ways to do it, isn't it? So one could do it as, say, $(0,1)$ is a basis of $\mathbb{R}^2$. But so is $(1,2)$, say, $(1,0)$. This is also a basis of $\mathbb{R}^2$, isn't it? Okay, so while we know that there is at least one subspace W for which $U \oplus W$ is V, it is not unique by any stretch of the imagination, okay? You can find any number of subspaces which give you the entire vector space V for a, given a U okay? So that is something important to understand.

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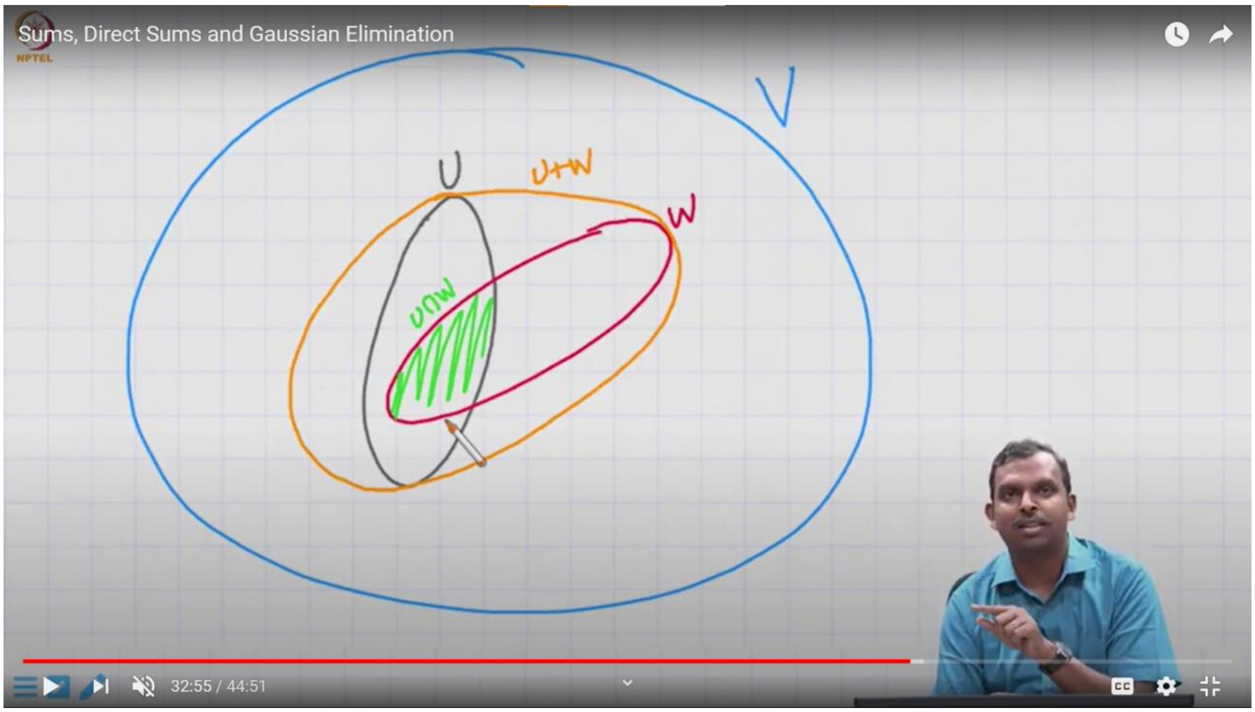

So you can sort of draw a picture here and convince yourself that that is true, okay? So supposing I take this $(1,2)$ line as U, okay? Any line through the origin which does not coincide with U will be a W, okay? Any line through origin that is not U, okay? So every such line can be taken as a W and then your $\mathbb{R}^2$ will become $U + W$, okay? So remember that. That is something important to remember. The W is not unique. You can have any number of W. So the decompositions can be very interesting. The second example and the third example I will leave as exercises to you. You can think about how you can think of extending. So look at the third example for instance. In $\mathbb{R}^4$ I have given you two vectors. $(1,2,3,4), (2,3,4,5)$. How do you do the extension, okay? How do you do the extension? That's also not clear, right? So we haven't seen any numerical method for quickly doing extensions of bases. Once again Gaussian Elimination will help you there. We, I will talk about it in the next few slides, okay?

So here is another interesting result. Turns out, because of this direct sum decomposition $V = U + W$, any vector v in the vector space can be written as $u + w$, right? Obviously where $u \in U$, the subspace U, and $w \in W$. And that decomposition is unique, okay? Because it's a direct sum, it ends up being unique also. So how do you prove it's unique? Think about how you would prove something is unique. So always, the technique for showing something is unique is - you assume it's not unique and show that it's, it'll end up being unique anyway, okay? So if we assume it's not unique, you can do $u_1 + w_1$, and $u_2 + w_2$ and $u_1 \neq u_2, w_1 \neq w_2$. Then what you can do is, you know that $u_1 + w_1 = u_2 + w_2$. You push u_2 to this side. You see $u_1 - u_2$ equals $w_2 - w_1$. And that gives you a non-trivial intersection between U and W, right? $u_1 - u_2 \in U$. $w_2 - w_1 \in W$. And if they are non-zero then they have to have intersection. And that is not allowed, right? $U \cap W$ is only zero, okay?

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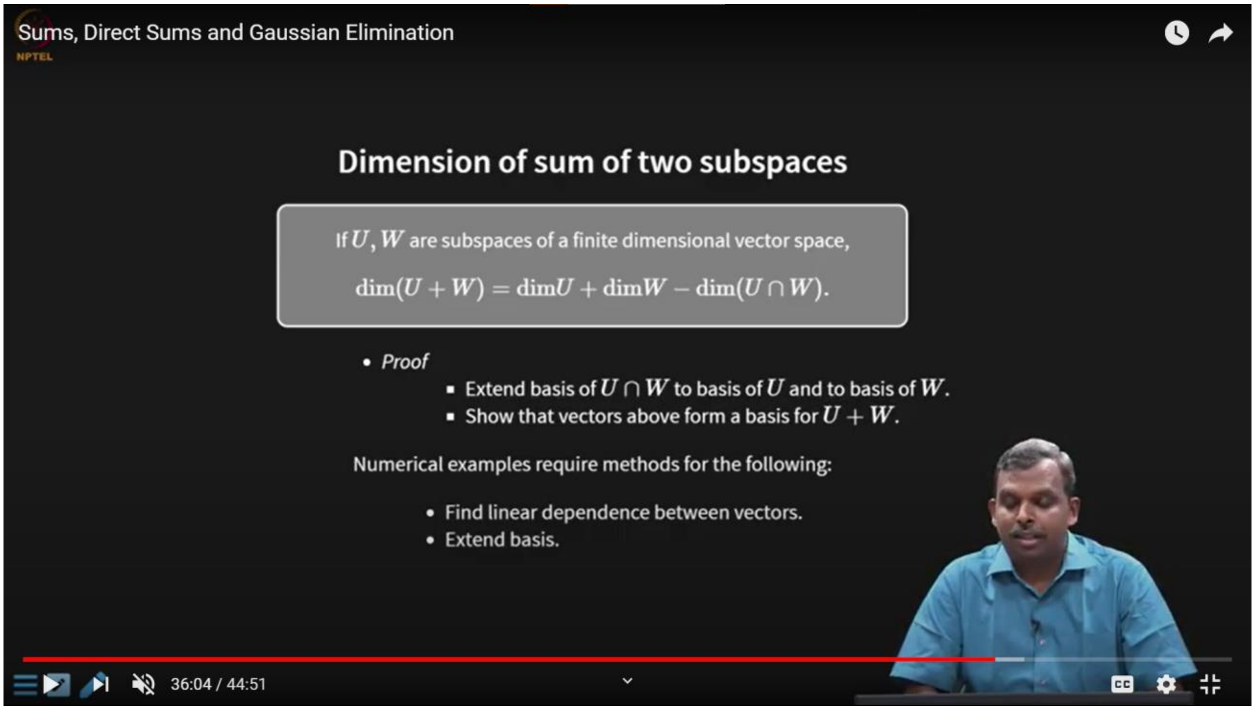

So that shows you that this decomposition is special in some sense. The moment you identify a subspace U and a subspace W such that $U \oplus W = V$, any vector v can be written as $u + w$ and that $u + w$ decomposition where $u \in U, w \in W$ is unique, okay? So that's also nice to know, okay? So these kind of simple things in decomposition are very, very powerful, and they help, will help us later on, okay? Once again I want to emphasize the numerical methods here, right? Given a basis, how to extend it. Given a basis for a subspace, given a set of linearly independent vectors, how do you extend it to a basis, okay? It will be good to have a method for that. We don't yet have a method. Let's see if we can think of, I mean, I'll describe next a method called Gaussian Elimination which will give you ideas on how to do these things, okay? Let's…

Okay, so before we, I mean... I have been promising this Gaussian Elimination a lot, but there's this other wonderful relationship between all these things we have spoken about. We spoke about the sum of $U + W$, we spoke about the intersection of U and W, right? And then we spoke about dimension of all these spaces, right? $U + W$ is a subspace. U is a subspace, W is a subspace. $U \cap W$ is a subspace. It turns out there is a very nice numerical relation between all these dimensions, okay? And that is given there for you. U and W are subspaces of a finite dimensional vector space. Then the dim $U + W =$ dim $U +$ dim $W -$ dim $U \cap W$, okay? So that is the idea. So maybe if you want, we can do a picture. Here, a picture is interesting. So let us say this. So I'll just draw some sort of a, you know, some sort of venn diagram sort of picture.

Let's say this is V. So quite often when you see a result in mathematics, even abstract mathematics, it's good to have a picture in your mind, okay? So what is the picture, mental picture that you draw to understand or recollect this? So for this subspace thing, maybe this is a good picture. I mean, you can have a different picture also, I am just pointing out one way of thinking about it. So let us say this is U, okay? Let us say this is W, okay? This is the intersection, okay? That's $U \cap W$. And then you will have the summation, okay? So maybe that would be something like that, okay? That's $U + W$, okay? So now you can sort of imagine why this is true. So this is, this $U \cap W$ will show up both in U and in W. And when you do $U + W$, there are these, you know, multiple ways in which these things can add together, okay? So when you count dimensions of $U + W$, you should count the dim U plus the dim W and then subtract from it one $U \cap W$. So the $U \cap W$ you should not count twice, okay? That's the idea in this whole $U + W$ thing.

So this dimension is almost like, you know... How do I say it? It's a number that quantifies the size or spread of your vector space, right? So a subspace... So when I say a subspace U, and when I say it has a certain dimension, I am... That is sort of like a quantification of how big or the extent of that subspace in some sense, right? And then you have W which is again another subspace. And the dimension of W tells you the quantification of it. And then you have to look at the $U \cap W$. Now that is the quantification of what's common to both. Now when you do $U + W$, it's almost like a union, right? So you take all of those things together. So you can count all the dimensions in U, count all the dimensions in W. But you would have double counted the dimension in $U \cap W$, so you have to subtract that once to get the actual count of the dimension of $U + W$, okay?

So this is just an intuitive way to think about it. We'll of course prove this result. Maybe I won't give you a full version of the proof, just sort of illustrate what the idea is, you know? This basis extension will play an important role in all these proofs, okay? So what you do is - you start with the basis of $U \cap W$, okay? The smallest subspace that you have, start with the basis of that. Now this, that basis can be extended to a basis of U, okay? Think of U as the whole world, and then $U \cap W$ is a subspace of U, right? So you can extend that basis to a basis of U, you can also extend that same basis to a basis of W, okay? So now you have a basis for $U \cap W$ which is extended to a basis for U, which has also been extended to basis for W, okay? You take all these vectors, the common bases for $U \cap W$ you take only once. Then you take the remaining vectors, you add it to make a basis of U. Then you take the remaining vectors, you add it to get a basis for W. Put all these things together, these will form a basis for $U + W$, okay? So you should show that. And they are linearly independent, they span $U + W$. This is something that you can try to write down and prove sort of intuitively. You can argue also, it will work out okay?

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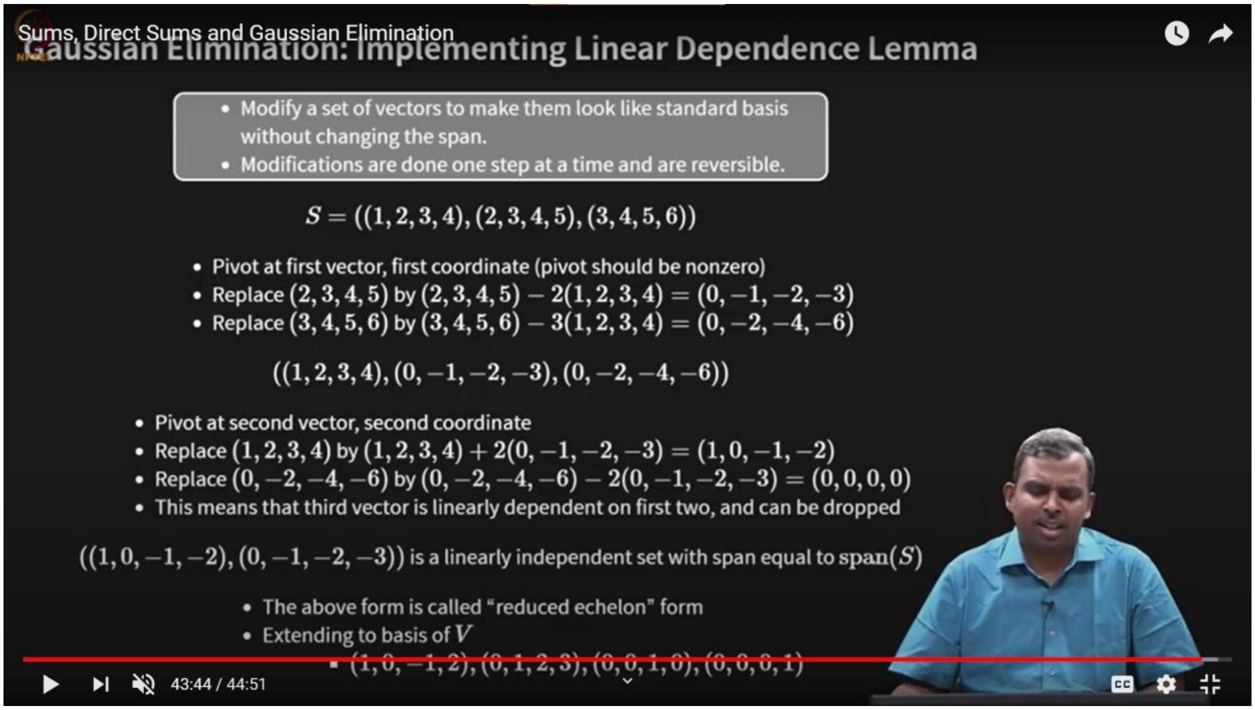

But this result is very useful. So you know, I told you that you have to numerically find $U + W$, $U \cap W$, a lot of interesting things will come up in problems particularly using these kind of ideas, okay? So this is very important to understand. So you will see even in the quizzes and exercises I have some problems for you which use this relationship to rule out some things, right? For instance... Let me give you a very simple thing to start with. Supposing I am in a 6-dimensional space, or let's say... Let's take a big example. Let's say I'm in a 100-dimensional space, okay? So pretty big space. I have two subspaces of size of dimension 80 each, okay? U is a subspace of dimension 80. W is a subspace of dimension 80 again, okay? Now should they intersect? Can they have a trivial intersection alone? Should $U \cap W$ be something? What should be the dimension of that? If one were to ask that, it seems that maybe you can't say much.

But then you look at this equation here. You know the dimension of U is 80. 80 + 80 is 160. Now I'm in only 100-dimensional space which means $U + W$ can at most be 100, okay? So I have 80 + 80, but then $U + W$ can at most be only 100. So 160 means the dimension of $U \cap W$ better be at least 60 so that the 60 gets subtracted and I get only 100, okay? So here is the result for you. Now if you have two 80 dimensional subspaces in a 100-dimensional space, their intersection will at least have dimension 60, okay? Isn't that interesting? Okay, so these kinds of results are possible, no? I didn't tell you what the subspace was, I didn't tell you anything about it. It's not a numerical result, but ahead of time you know that their intersection should at least have dimension 60. So these are the kind of things one can do with these kind of results, okay?

Now the final point here, and the last part of this lecture is that quite often when we looked at numerical examples, when we wanted to compute the actual basis of the span or compute the actual basis of the sum or intersection or things like that, we needed to find linear dependence between vectors and extend the basis. We wanted quick ways of doing it and so far I have not talked about any numerical methods in this class. We won't spend too much time emphasizing numerical methods in this class by the way. It'll mostly be the abstract notions but nevertheless this this one little idea of Gaussian Elimination is very important. So like I said, Gaussian Elimination is a very important, simple numerical method that is often used in linear algebra to get results, okay?

So what is it that we wanted? Given a set of vectors, we wanted to find out if they are linearly dependent. You know reduce it, etc., extend it to a basis, all these things we wanted to do, okay? One thing we saw already in some of the examples is - if things looked like the standard basis, if the given vectors look like the standard bases, as in $(1, 0, 0), (0, 1, 0)$. So if you had zeroes and ones like that. Then things like linear dependence, independence are very easy to figure out, right? So you can look at it and you know whether it's linearly independent or not, and you can also extend very easily, okay? Wherever you do not have the zeroes, ones and zeroes, you can keep extending, okay?

Now the problem is - in general when somebody gives you a set of vectors, it may not look like the standard basis, okay? So you want methods to modify the vectors to make them look like the standard basis, but you don't want to change the span, right? You don't want to go out of the span of those vectors, you want to be within the span of those vectors, given a set of vectors. But can I find vectors in that span with a lot of zeroes, okay? Lot of zeroes, ones, so that I can quickly identify whether the linear dependence is there, independence is there, and figure out ways of doing extension, okay? So essentially Gaussian Elimination is a method like that, and you can see it gives you ways to implement, find linear dependent vectors and then implement extensions and things like that.

So I am going to do it with the illustrations. I will also assume that you have some basic familiarity with this already, you have seen it somewhere before. But nevertheless I will do an illustration with one set of example, one example. There are some corner cases, some changes here and there that you might have to make, but it's usually intuitive, it's not very hard, okay? So let's start with this example, okay? So I am giving you three vectors $(1, 2, 3, 4)$, $(2, 3, 4, 5)$, $(3, 4, 5, 6)$, okay? And you have to find out if they are linearly dependent. If they are linearly dependent, which one is dependent on the others, right? So you remember the linear dependence lemma. Okay, so how do you execute that? Okay, so Gaussian Elimination starts with this notion of a pivot, okay? Pivot is one coordinate and one vector which you are going to make non-zero, okay? Once you know it's non-zero, you can make that coordinate and all other vectors in your list also zero, okay? So remember that. You shouldn't say also zero, non… equal to zero, okay? So you fix the pivot, and then make sure that all other vectors in that coordinate are actually equal to 0, and you are still in the span, okay?

So this is actually possible using linear combination. So that's what I'm illustrating here. So let's say the first vector first coordinate is where you are pivoting, okay? So that's 1, and then you can replace $(2, 3, 4, 5)$ by $(2, 3, 4, 5) - 2(1, 2, 3, 4)$. So you can see what I've done there. I've made the first coordinate of the second vector 0. Likewise, you can make the first coordinate of the third vector 0 just by linear combination. So I'm doing simple linear combination. So I'm not going out of the span, it's also a reversible linear combination. What do I mean by reversible linear combination? This we'll come back to later on, but remember that. So when I say $(2, 3, 4, 5) - 2(1, 2, 3, 4)$ to get $(0, -1, -2, -3)$, I can do a linear combination of $(1, 2, 3, 4)$ and $(0, -1, -2, -3)$ to give me back $(2, 3, 4, 5)$. So as in $(2, 3, 4, 5)$ I can still get back, I have not lost it, right? I've replaced $(2, 3, 4, 5)$ by something else. But $(2, 3, 4, 5)$ I can still get back. So in that sense it is reversible. I have not lost $(2, 3, 4, 5)$ in that bargain, okay?

So that is what I do. So once I do this, I have $(1, 2, 3, 4), (0, -1, -2, -3)$ and $(0, -2, -4, -6)$, okay? Those are the 3 vectors I have. The span of this set and the span of the original set are exactly the same, okay? Except that I have this whole bunch of zeroes, so now I know that the first vector and second vector are linearly independent, first vector and third vector are linearly independent, right? Why is that? Because it has a one in the first position, zero in this position. What about second and third? I do not know already, but you can maybe look at the numbers and already guess what it will be.

But for even that, systematically one can do the same pivoting procedure. What do I do? I pivot on the second vector second coordinate which is actually -1, right? So once I fix on the second coordinate of the second vector, I can make, using that I can make the second coordinate of all other vectors 0, ok? So how do I do that? Replace $(1, 2, 3, 4)$ by $(1, 2, 3, 4)$ plus 2 into this, so that linear combination makes the second coordinate of $(1, 2, 3, 4)$ zero. I get another vector, and once again notice the reversibility here, okay? I have not killed the reversibility, and also $(0, -2, -4, -6)$ you can replace by something, and there it becomes interesting. What do I get? $(0, 0, 0, 0)$, okay? So I end up getting $(0, 0, 0, 0)$ when I try to make the second coordinate of the third vector 0, okay?

So now this means, at this point you have linear dependency. Anytime you get $(0, 0, 0, 0)$, you know that this vector that you have is actually linearly dependent on the other vectors, so you can drop that from your list, okay? So in this way I have found that this $(3, 4, 5, 6)$, okay, is actually linearly dependent on $(1, 2, 3, 4)$ and $(2, 3, 4, 5)$, okay? Or in some other form I found it, but it is the same thing, right? So you can go back, you can reverse it to get to that result, okay? So that is the thing. So you can drop $(0, 0, 0, 0)$, okay? So you get that the span of S, original set S, which had 3 vectors. It had a certain span. I have found a basis for that subspace. What is the basis? $(1, 0, -1, -2), (0, -1, -2, -3)$. Not only it's a basis... I mean, of course it's linearly independent and all that, but it also has this very simple form. If you look at the first two coordinates, it's 1, 0 and then the second one is 0, -1. If you want you can make the 0, -1 to 0, +1 by multiplying by -1 and all that but it's okay.

So this is good enough, so this is this reduced echelon form, as they call it, okay? So you have that 1, 0 and 0, 1 showing up, and in this form it's easy to extend to a basis of V, okay? So you have 1, 0 and 0, 1 and simply you need to extend by doing $(0, 0, 0, 1)$ and then $(0, 0, 1, 0)$, okay? So it's easy to extend once you have the echelon form, okay? So this process, which is called Gaussian Elimination, it goes back to Gauss. Of course Gauss did so many things, and this is also one of the very interesting things that he found of how to, you know, bring pivot in one coordinate, make everything else zero, don't affect the span. It's a reversible way, and then finally you end up finding linearly dependent vectors in a list. You can also have methods for extending to the basis, okay? So this can be used in many numerical examples and problems that I give you to help solve many problems.

Even intersection of subspaces and all these things, once you find a good enough echelon form for the basis, it's easy to work with and solve problems, okay? So that's a very basic introduction to the numerical methods. Once again I want to emphasize - this course is not about these kind of numerical methods, wherever we need it for problem solving and quiz questions and all that I will introduce the linear numerical methods that are needed, okay?

So I have a quiz for you. Once again this is a google form that you're welcome to fill. A lot of questions, simple questions just to get you thinking about what these things are. Please answer them. That will give me valuable feedback on how well the course has been understood by you, okay? Thank you very much. This is the end of week one, and we are done with most of the basics of vector spaces, finite dimensional vector spaces. From next week onwards we will start studying linear maps or linear transformations. And linear maps and transformations are very, very important parts of linear algebra. Without linear maps, linear algebra would be very boring, okay? So one needs to really study linear maps very well and understand them very well. Hope you've had a good week and let's look forward to week two and linear maps. Thank you.