Applied Linear Algebra
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Week 02
Linear Maps and Matrices

Hello. Welcome to this lecture. The topic we'll be studying today is easily the most important reason why anybody studies vector spaces, which is linear maps and the matrices associated with them. So vector spaces by themselves are not as interesting. Linear maps are really what makes everything very, very vital. In fact for the rest of the course, from now on, we'll primarily be studying properties of linear maps, okay? So let's get into it.

So here's a quick recap of where we are. All that we studied last week and just a quick summary. I think it's good to have a quick summary as well. We studied, we looked at the definition of a vector space over a field $\mathbb{F}$ and we primarily said we are going to look at the real or the complex field. And we looked at linear combinations. And linear combinations are once again the most essential operation that one needs to perform in a vector space. We define the notion of span in which given a very small vector, list of vectors, how you can make all sorts of linear combinations to get a much larger set.

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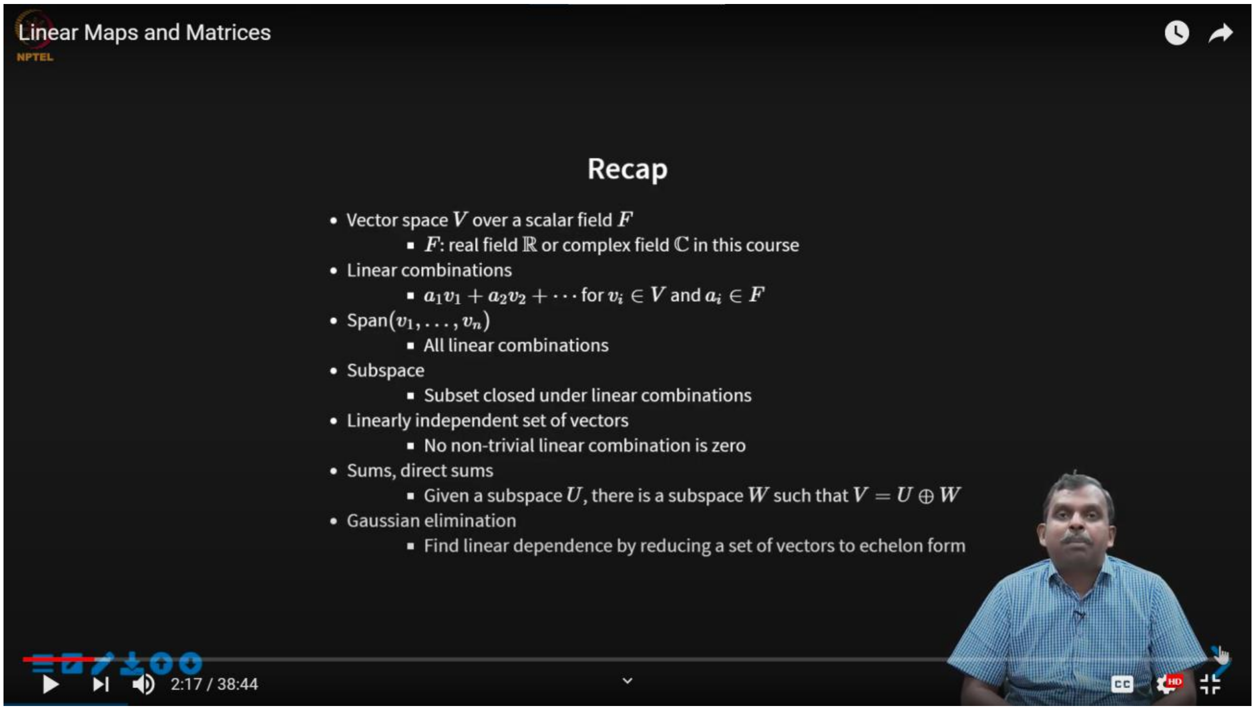

We described notion of a subspace which is very crucial again. Breaking up vector spaces into smaller subspaces. And then the important notion of linearly independent set and what that implies. We looked at bases, you know, all those definitions. Dimension, all very important definitions, And then this notion of sums and direct sums. This is also very important and this lets us do this divide and conquer of vector spaces with subspaces, looking at them by, you know, assembling subspaces together. And finally one of the numerical things we looked at to study actual examples and do some calculations. Gaussian Elimination is very useful, it gives you a very useful form for the basis, this echelon form for the basis which lets you conclude a lot of things like, you know, finding linear dependence, extending vectors to a basis. All of that is made possible by Gaussian Elimination, okay? So this is a quick recap.

Let's jump into linear maps in which I said... This, like I said, is the most important thing in linear algebra, okay? So before that, let's just look at what are these functions or maps, okay? So why is it that these play an important role in mathematics in general. And what you have studied before, I think most of this will be familiar to most of you but maybe you have not thought about it in this fashion, okay? So usually when you want to model real life or want to look at, you know, the world around you and make some statements about it, make some mathematical statements about it, physical statements about it, it's always about relating one quantity to another, okay? So you might have studied this from school. You would have said - okay, there’s pressure, temperature, how do I relate pressure to temperature. If you look at electrical circuits for instance, there’s voltage, current, how do I relate voltage to current. So there's always multiple quantities, and you want to say when one quantity increases in a certain fashion or decreases in a certain fashion, another quantity has the corresponding behavior. And I have a function or map which tells me how one quantity varies when another quantity varies, okay?

So this is a very typical thing in mathematics. Anytime you define a new object, you want to really figure out functions or maps from one set to the other which has these objects, okay? So there are various types of these functions, you must have a lot of experience studying these functions. You must have studied so many functions of one scalar to another scalar, right? So from real numbers to real numbers or complex numbers to complex numbers. You must have studied so many functions so far. The most popular and simplest function you can do is what's called linear function, okay? So in linear functions, just one quantity increases in proportion with the other quantity. It's like a straight line, you know? When, irrespective of how large the quantity is, if it varies by a certain amount the other quantity also varies by a proportional amount, right? So that's the linear function. It's very popular. In fact most models use linear functions to start with. Only when linear doesn't work people look at, you know. maybe something non-linear which could be quadratic, polynomial. exponential, logarithmic, trigonometric. You might have seen all these functions and their properties and behaviors and all that, and this dominates, you know, modeling of real life, okay?

But the most important function of all this is the linear function, okay? There are lots of reasons why that is true. In fact any nonlinear function in a short enough interval, linear function is a good approximation for it. So this piecewise linearity is very popular again. And so linear functions are very, very important and that's what one focuses on always. You first understand linear functions. If you cannot understand linear functions, then you can't, you probably can never understand any other function as well. So first start with linear functions and see what happens, okay?

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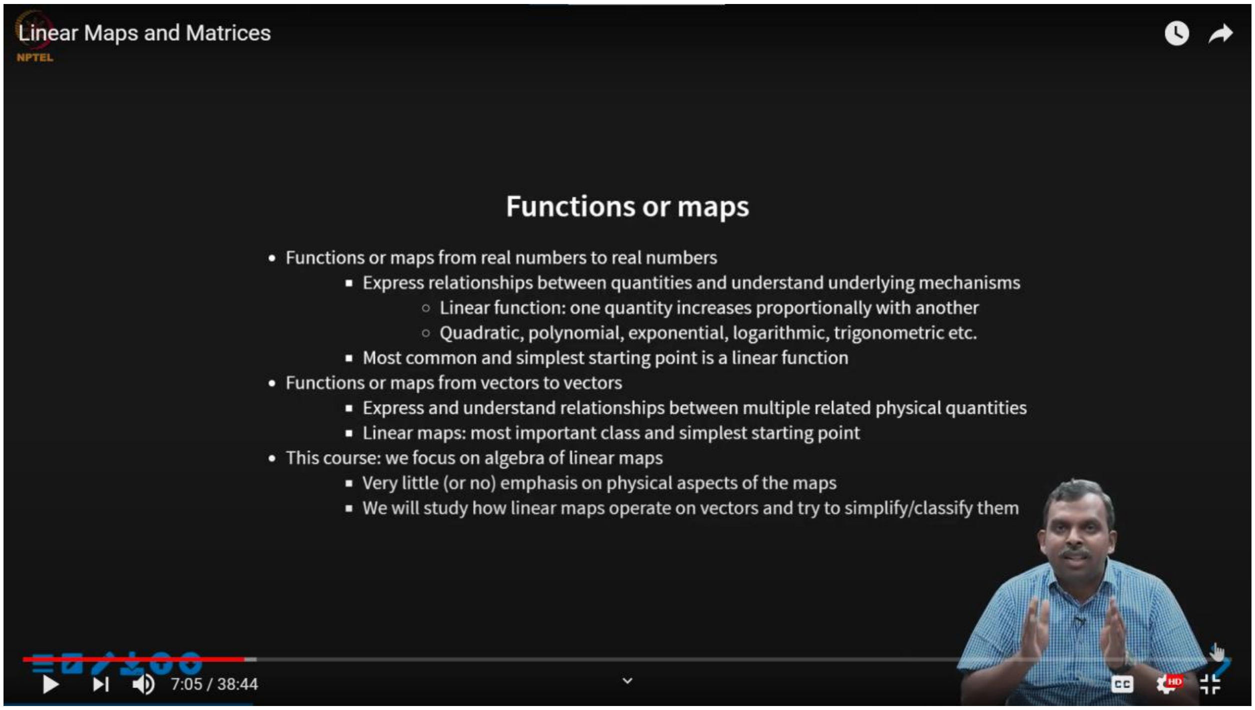

So we've been studying vectors, right? So we've been studying vector spaces, you know subspaces etc. So maybe there are multiple vector spaces and we are interested in functions or maps from one vector space to another vector space, okay? So this in real life, I'll give you a very simple real life example towards the end of this lecture. But there are, in real life this models situations where there are multiple variables and mapping to multiple variables, right? So multiple outputs, right? So and then you want to express some relationships between them, understand, you know, complicated interactions between these variables and things like that. So in that case, this is very useful and linear maps are the most fundamental and simplest and important of starting points, okay? So what we'll do in this class - how we'll look at linear maps. Once again this is a sort of a mathematical introduction, so we will focus very little on the physical aspects. Except when I give you some explicit examples, we will really not worry about the physical aspects of it. Except we will study the linear maps in sort of an abstract way, define what it is and think of an abstract vector space to another abstract vector space - an arbitrary linear map. How should it be, you know, what should be the definition, what are its various properties, how to study it.

We'll really go into depth and understand everything about these linear maps, okay? So that's going to be our goal. We'll do that and once you do that, in real life when you actually apply it to a situation and you realize something is a linear map, then all this will come to help you, okay? So every single thing that you proved about linear maps here will help you in a real situation be it, you know, solving electrical circuits, doing finite element methods or whatever. And then there is a whole bunch of applications - networks, graphs... I mean there's countless applications for these kind of things, okay? So we will study the mathematical aspect. So hopefully that gave you like a high level introduction to why these things are important. As I go further I may not have that much time to keep repeating it, but remember this little point. We are ultimately trying to find a relationship between quantities, express them, you know, succinctly. And this linear map is a very good model for many actual phenomena in real life, okay? So let us go on.

Okay, so here is the definition for a linear map. We need two vector spaces, right? So anytime you have a map, it's from one set to another. So in this case we will look at both sets being vector spaces, okay? There is a vector space V and the vector space W, and my linear map or map is going to go from V to W. Input is from the vector space V, output will be from the vector space W, okay? That's the way to think about these things, okay? So here's the definition - a linear map from V to W. Usually we'll denote it with T. $T : V \to W$ means T is a linear map which takes you from the set V or the vector space V to the vector space W, okay? And it has to satisfy two important properties, okay? Anything linear, when you say linear, what you mean is these two things - additivity and homogeneity, okay?

What is additivity? You give me two vectors u and v, okay? I know I can add them right? Where they are, I can add them, right? Two vectors u and v, I can add them. Now this linear map, the so-called linear map T would have taken u to $T(u)$, right? So u under the linear map T goes to $T(u)$. So let me write that down maybe a little bit more clearly for you, okay? So when I say T goes from V to W, and if I give you a v in V under the transform T, under the map T, $T(v)$ belongs to W. So this will be my notation, okay? So if you want a pictorial representation, this is a good picture to keep in mind. You have a vector space V, you have a vector space W. All these little ellipses are basically a, you know, pictorial representation of all the vectors in the vector space V, all the vectors in the vector space W. And my map T is going to take v to $T(v)$, okay?

Okay, this is all notation. This is how we will denote it. Now remember the v, small v, actually belongs to the vector space capital V. But this $T(v)$, right, will belong to the vector space W, okay? This is what a map does, okay? Now what is this linear map supposed to do? I know that u will go to $T(u)$, v will go to $T(v)$ and I can do the addition $u + v$. I will get another vector right? $u + v$ is another vector in V. Now if this T operates on $u + v$, what should happen? The vector that I get should actually be the sum of $T(u)$ and $T(v)$ in W, okay?

So maybe I should illustrate that for you. I have a u which under T goes to $T(u)$, okay? And I will also have a $u + v$, right? So maybe $u + v$ is here, I do not know where it is, right? So this could be $u + v$, or in general let me write it as $au + bv$, okay? Or maybe let me not do that... So $u + v$, okay? Now this $u + v$ will get transformed to something here under T. And what should that something be for it to be a linear map? It needs to be this $T(u) + T(v)$, okay? So that is the condition. So that sort of condition tells you the linear map. When is a map linear? It's supposed to be linear if it satisfies this kind of condition, okay? It should take u to something, of course. It will take v to something, that is also fine. But then I can add $u + v$ here. In the domain, I can add $u + v$. And in the range on the other side where it takes me, I can add $T(u) + T(v)$, right? Both of these have to be identical. Where, as in, when I map $u + v$ under T to something, that should be exactly the same as $T(u) + T(v)$, okay?

So that is a restriction. We will see how that affects the definition of linear map, how it restricts the linear maps or something like that later, okay? So this is the sort of thing, this is a sort of picture to keep in your mind when you see these kind of definitions, okay? The next thing is homogeneity and that also has a very similar picture, right? So I have v going to $T(v)$ and then I can do this whole bunch of $\lambda(v)$, right? So this whole, you know, the line so to speak passing through v, right? That entire line should go to the line passing through $T(v)$, okay? So that is how linear maps work. Now if you have a line, it gets mapped to another line, okay? That's how it works, right? That's a good picture to have in mind. So that's additivity and homogeneity for you, okay?

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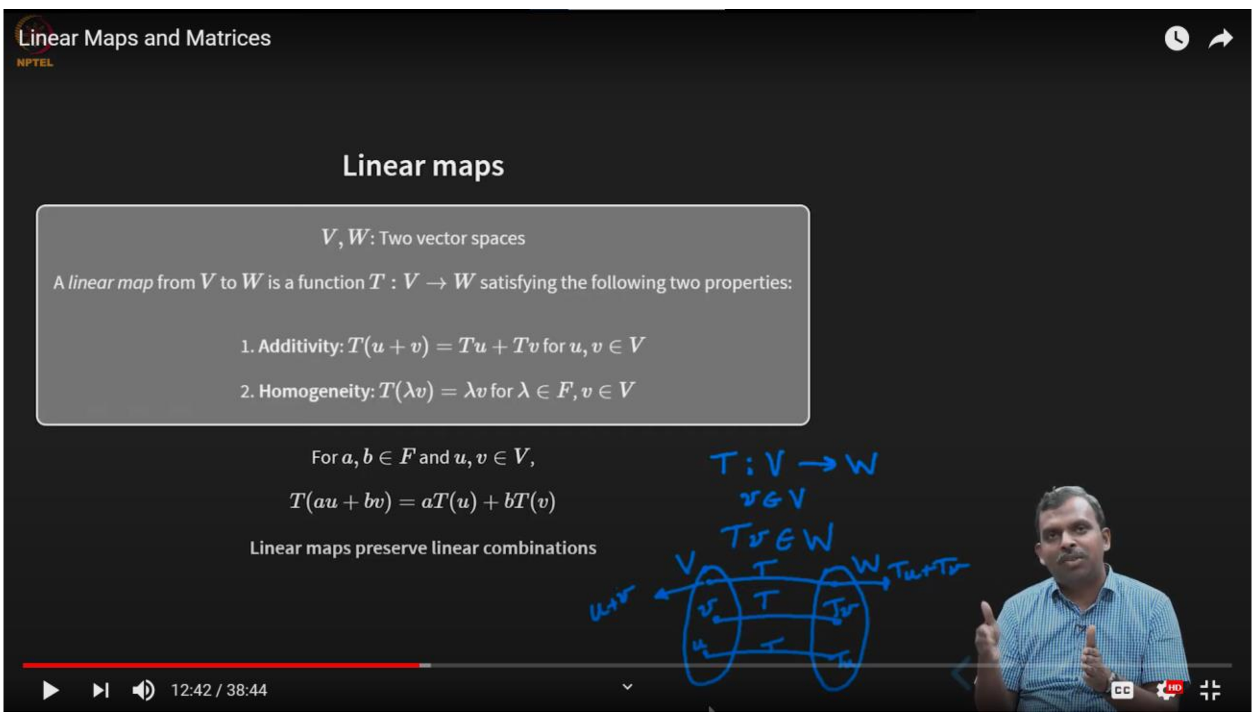

So you can put additivity and homogeneity together and in short say that linear maps are those maps that preserve linear combinations, okay? So if you have $au + bv$ here, and then you hit it with the linear map T, It should be the same as $aT(u) + bT(v)$, okay? So you hit u with the linear map T, hit v with the linear map T and then do the same linear combination, you should get the same thing on W, okay? So you have all these nice… I mean these two conditions maybe sound a bit too restrictive to you. Maybe you think there's not too many linear maps... But it's not true. So you have quite a few interesting linear maps and this set of linear maps is also very rich. There's lots of linear maps out there, and we'll keep studying them, okay? Hopefully this definition is clear. Quite often when you read these definitions maybe the entire significance doesn't come to you immediately. We will study more and more aspects. And as you study more and more, you will see how this linearity plays an important role, okay?

A good picture is this picture that I drew here. You can think of this set sort of picture V to W and T keeps mapping, and there are these connections on, all these restrictions on what it has to map, okay? So straight lines have to go straight lines and all these things have to be true, okay? All right, so that's linear maps for you. So we've seen the definition. Let's see a whole bunch of examples. So most of these lectures and even other lectures on linear maps I'll primarily give a lot of examples to motivate. But we will also see some general results as we go along. But right now let's start with some examples, okay? Are there many linear maps, right? You may ask. So here are some very simple examples.

The first and the most… The first two examples we will see are sort of trivial in some sense, okay? They're very simple definitions in some sense, but they are also important, they'll play an important role as we'll see later on. The first linear map is the zero linear map. What is the zero linear map? I simply take any vector and map it to 0, okay? So you will see that many things we denote it as the same in this course, right? So that might be one of your complaints. For instance 0 stands for the real number 0, stands for the complex number 0, stands for the 0 vector in any vector space, you know, in $\mathbb{F}^2$, in $\mathbb{F}^3$, in $\mathbb{F}^5$ whatever, it stands for the zero vector. Now zero will additionally stand for the zero linear map, okay? All zero linear map, any vector I map to zero I'll again denote it by 0, okay? Hopefully this is not confusing to you. Depending on the context it will be abundantly clear what 0 is. But, you know, keep that at the back of your mind. Sometimes we'll write, you know, some equation equals 0 and then you have to sort of interpret it based on that equation what zero it should be, okay? So that's something to keep in mind.

Anyway, so that's the zero for you. Definition is very simple. You can again draw a picture okay. So these pictures are good to draw. So when I think of a zero, what am I doing? I have a V, I have a W. There is the zero in the W. What does my T do? It takes my entire vector space V, and it maps it to 0, okay? So you can check sort of trivially that this is a, this is the 0... T equals 0 maps everything to 0, right? You can check that this is a linear map. See, remember, anytime I give you a definition, you have to go back. Any time I claim something is a linear map, you have to go back to the definition and make sure that those definitions are not violated, okay? So if I do this, you can check that both the definitions are not, both the conditions are not violated. Additivity holds, homogeneity also holds, okay? Any linear map, whatever you may do, finally you always get zero, right? So it doesn't matter what you do, that condition is satisfied, okay?

Another linear map which is again trivial but very important is when V and W are the same, okay? So your W, which is another vector space is the same vector space of V. So you are mapping vectors from V to itself, okay? Such linear maps are called operators, okay? So later on we'll study much more about operators. Operators are much much more interesting to study. But once you say V to V, you have this very interesting linear map which is the identity linear map, which will be denoted as 1, okay? Once again, 1 stands for many things in this class. So in the world of operators, 1 will be the identity operator, okay? What does it do? As the name specifies, any vector you give to it, it will simply spit out the same vector, okay? Identity linear map, you give it v it will give you v again, okay? So that's the identity map.

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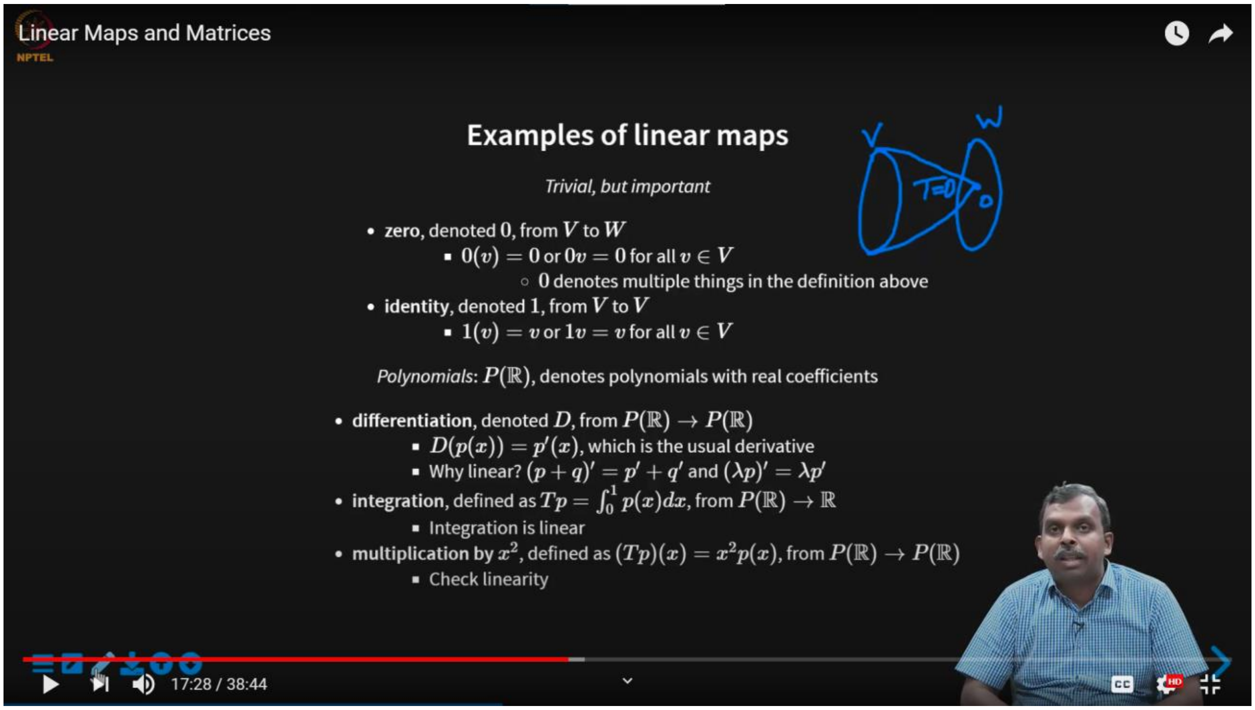

A few more examples here which are in the polynomials. Just to give you, just to emphasize that not just finite dimensional vector spaces. Any vector space in fact has linear maps. Many interesting linear maps are there out there, here are a few examples. If you look at the vector space of polynomials with real coefficients... Remember this is infinite dimensional, right? I am not putting any degree restrictions, any degree is okay. On that space, on that vector space, I've given you here three examples - one is differentiation, another is integration, another is multiplication by x^2. You can think of other examples. So multiplication by x^3, multiplication by something else, you can think of so many nice interesting examples. All of those are linear maps and you have to go back and check your conditions. And these come from just the basic definitions that you have out there, and all sorts of interesting linear maps you can have. Linear maps from polynomials to polynomials, polynomials to reals, all sorts of interesting things you can do there as well, okay? So I don't want to go into too much detail here, but just just to show you there is a rich set of linear maps out there, even in the other sorts of vector spaces that we've been studying, okay?

So that's simple examples. Now it turns out the most interesting linear maps are these kind of maps, maps that take you from $\mathbb{F}^n$ to $\mathbb{F}^m$, okay? The n-dimensional vector space over the field $\mathbb{F}$ to the m-dimensional vector space over the field $\mathbb{F}$, there are many linear maps that do this. And most of the time in this class we'll study only these linear maps, okay? These linear maps from finite dimensional vector spaces to another finite dimensional vector space, that's what is most interesting to us, okay? So this is very important, and in this slide I have a whole bunch of examples for you. And these examples are very important. Just think about them. This is the first time if you're looking at it, look at them for a while, think about them and then figure out if each of these are linear maps or not.

By the way one more thing I want to mention. Quite often people will say linear transformation instead of linear map. I might also do that a lot. I might say linear transformation later on. But both of them are exactly the same, okay? So linear transformation, linear map, both of them mean the exact same thing, okay? So let us go on and look at each of these examples here. There are a whole bunch of examples. And next to each of them I am going to write either linear or not, okay? So if I say linear, then it is a linear map. You can check those conditions, it'll be true. Now if I say not, then it's not a linear map, okay? Some condition is violated, okay? So let's see that.

The first one is the most trivial example. It's just $T(x)$ equals $4x$. $\mathbb{F} \rightarrow \mathbb{F}$, you can check both the conditions will be true. It's a very, very easy thing to check. This is linear, okay? The next one is sort of agonizingly close to this $4x$. This is $T(x)$ equals $4x$, this is just $4x + 3$. Okay? So all of us think of this as also linear. I mean, quite often this is thought of as linear but this will not satisfy the conditions that you have, okay? You can do a quick check, you know? You put x_1, x_2. x_1 goes to $4x_1 + 3$, x_2 goes to $4x_2 + 3$. $x_1 + x_2$ will go to what? $4x_1 + 4x_2 + 3$. But what is $T(x_1) + T(x_2)$? $4x_1 + 4x_2 + 6$, okay? So there is a violation there, okay? Because you are adding this constant to every transformation, that constant will double and you have no way of capturing that doubling in your definition, okay? So this is not linear, okay? Anytime you have a constant, addition it's not linear. A lot of people study this under what is called affine maps and affine maps are quite close to linear. They're easy to study as well but in this class we'll primarily focus on linear maps, okay? So for us $4x + 3$ is not a linear map, okay?

So $3x + 4y$ is from $\mathbb{F}^2 \rightarrow \mathbb{F}$, okay? $\mathbb{R}^2 \rightarrow \mathbb{R}$, like, you know, plane to the real line. And that you can quickly check is a linear map, okay? So you can check. Both of those conditions you can put, you know, $(x1, y1)$, and then look at $(x2, y2)$. Then look at $(ax1, ay1), (bx2, by2)$. And then plug in the formula. You add, you will see you will get the exact same thing, okay? So you can see it is $3x + 4y$. You can see where the linearity comes in also. The variables are involved only with linear terms, you cannot have constant term, you cannot have cross terms, you cannot have higher powers, okay? That's the way to think of linear, right? So only x and y should come, okay?

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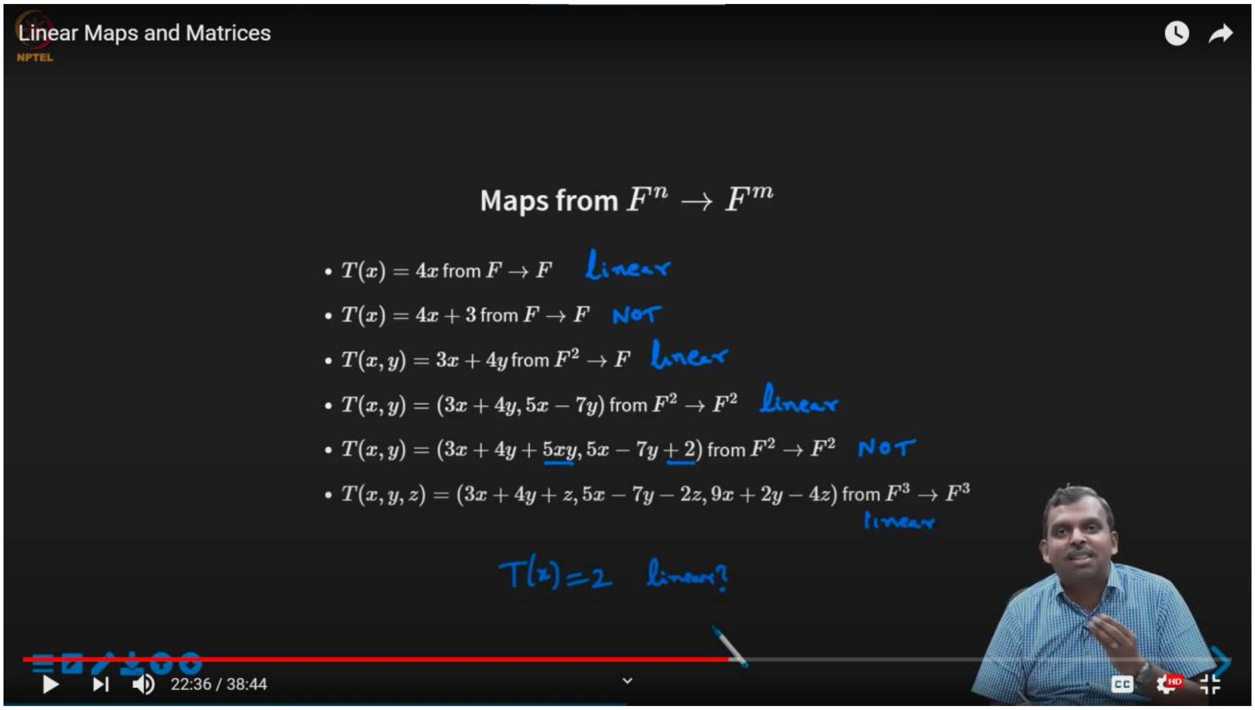

So here's another example $\mathbb{F}^2 \to \mathbb{F}^2$. There are two values now $(3x + 4y, 5x - 7y)$. Again, it's all linear. You can check with the actual examples, this will end up being linear, okay? So here's another example. It's $(3x + 4y + 5xy)$ then $(5x - 7y + 2)$. And it turns out these two make it non-linear, okay? So it's not linear, all right? And the last one if you check it out you see it's all a bunch of linear terms. So this is linear, okay? So this is sort of, gives you a feel for how a linear transform will look from $\mathbb{F}^n \to \mathbb{F}^m$. And like I said, that's the most important example. You have a bunch of coordinates, n coordinates at the input, and every coordinate at the output should just be a linear function of that. So it can involve x, y, z but nothing more, right? It cannot have x^2, it cannot have, you know, xy, cannot have even a constant, okay? If it has a constant also, there is a problem, okay? So for instance if you want a simple exercise, what about this one? $T(x) = 2$. Is this linear, okay? Okay, think about it, okay? This is a simple little exercise for you. The answer is quite easy but still think about whether or not this is linear, okay? So it will give you an interesting notion of what linearity really tightly means, okay?

So hopefully this gives you a sense of what it is. And like I said we will primarily study these linear maps in most of the rest of this course. Okay now... So we have this notion of linear maps. We have defined it. So how do you specify a general linear map? So how do you think of a linear map? You know, what should we do to fully specify a linear map, right? So what is the meaning of fully specify? So anytime you want to specify a map, what do I do? I want to define a function

$f(x)$. What do I do, okay? You have to sort of tell - given an x, how do you find the $f(x)$, right? So supposing you take $f(x)$ equals $\sqrt{x}$, for instance, right? So all of us know that given x, you know what $\sqrt{x}$ means. You know, I mean you can go into a calculator, put it in, it will give you the answer. There is a very well-established method to compute that function. So likewise in a linear map I should be able to tell you, when I define a linear map, right? A clear and precise method to compute that output given an input, right? So how do you do that, okay? And for doing that, for vector spaces you have to use the basis. Basis is a very simple way of doing it. Of course, there may be other ways of doing it but specifying a linear map through the basis or specifying how to compute the output of a linear map given the input through a basis is the most direct and simple way to do, okay?

So here's the method. I put it inside a box for you. You have a linear map T from V to W, okay? V is some vector space, W is another vector space and T is a linear map that goes from V to W. You start with a basis of V, okay? Any basis. You know that there are infinite bases. So many different bases are there. You pick your favorite one, maybe the standard basis, right? So when you're confused, you don't know what basis to pick, you pick the standard basis, isn't it? So you take some standard basis or some sort of a basis like that in a finite dimensional vector space. $v_1, v_2, \dots, v_n$, okay? So it turns out it's enough if you specify $T(v_1), T(v_2), \dots, T(v_n)$, okay? So once again think about it. I have this big vector space, and it looks like for every vector I have to specify what $T(v)$ will be, right? So that's how you are thinking you have to specify a vector space, But it's enough if you specify, for a linear map it's enough if you specify, given a basis, where each of those basis vectors go, okay? I have a basis with n vectors $v_1, \dots, v_n$, all I have to do is say what v_1 gets mapped to, what v_2 gets mapped to so on till what vn gets mapped to. Once I give you that, it turns out for any other vector v I can find out where that will map to okay? So it is not too difficult to imagine. Linearity plus specifying it in the basis is enough for you to specify the map for the entire vector space.

The proof is so silly. Here you can, you see what I have done here. If you take any vector in the vector space, I know since we this $v_1, \dots, v_n$ is a basis, I have a linear combination of these v_1 to v_n, which give me V, right? So that much I know. So I make that linear combination. I get v. So now how do I find $T(v)$? So you know it's sort of... The linearity directly helps you, right? So T of the linear combination a_1v_1 to a_nv_n, I know all of that can be taken out. So I get simply $a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n)$. And I know $T(v_1)$ to $T(v_n)$, okay? So once I know that, I am done, right?

So again the picture is important to keep in mind. So what is the picture you want to keep in mind? I have my V here, I have my W here, okay? All I need to worry about is my basis. So maybe I'll take the basis in red, okay? My v_1, v_2 so on till v_n, right? I have to say what each of these things will map to, right? This maps to something here, this v_2 maps to something here, this v_n maps to something. What is that? So let's say, you know, I can put down some w_1, w_2, so on till w_n. See, I am showing w_1 and w_n as all being different, but that is just, you know, just to cover the general case. And w_1 and w_2 can be the same, right? So I can pick anything I like, okay? Whatever I want here, I can pick for w_1 to w_n, okay? And I simply specify v_1 maps under T, you know... Under T, v_1 gets mapped to w_1, v_2 gets mapped to w_2, like that. I can pick my favorite basis v_1 to v_n, I can pick my favorite set of vectors. In fact there can be repetitions here. All the w's can be 0, right? Anything I want I can pick here. Any set of n vectors I will always have a linear map which takes me from v_1 to v_n to those vectors, okay? So that's what this result means. So linear maps get fully specified by specifying them on one basis of your choice to a set of vectors of your choice. Once you do that, you have a linear map, okay?

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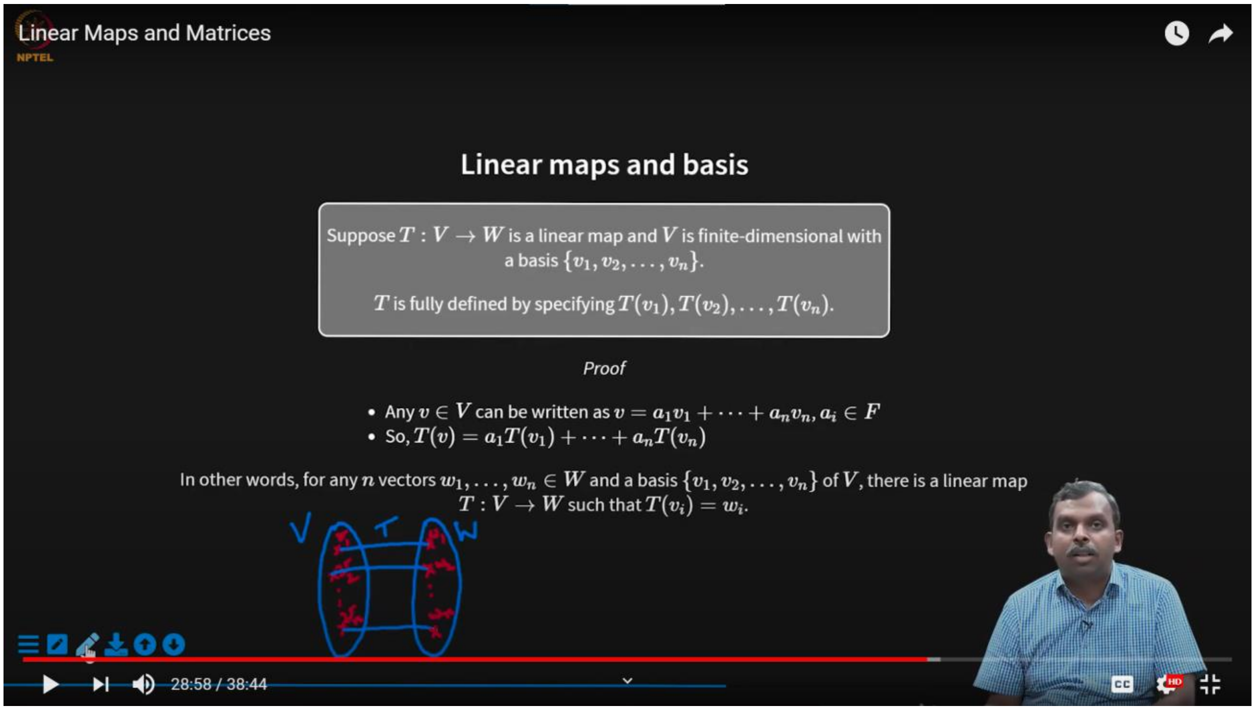

So this also tells you that there is a rich class of linear maps out there, right? So we know so many bases are there, okay? You just pick a basis and then there are, if supposing there are so many vectors in W, then you can just keep mapping, right? So as and when you keep mapping, you get linear maps. Will you get different linear maps? How do you know when you get different linear maps? How do you know that the two linear maps specified using two different bases are the same? All those are interesting questions. We will address all of them as we go along but for now at least this much is clear. Given a basis, given a set of vectors, I can specify a linear map for you and that completely specifies it. You can go ahead and use it to compute the linear map for any vector in your vector space, okay? So that is specifying linear maps through bases. This is very, very important. This is how we will specify linear maps, right? So give a basis, specify the output for the basis vectors then you can figure out every single output for any vector that you want, okay?

Okay, so now next comes the next most important thing, right? So now that you know... See I am doing a linear map from a finite dimensional vector space to another finite dimensional vector space. So it's like $\mathbb{F}^n$ to $\mathbb{F}^m$, right? So I know, once I know the vector is, you know, finite dimensional, I can pick say the standard basis or some basis, some form of a basis. Maybe not standard basis, some basis I can pick with n vectors in it. So every vector can be represented as n coordinates, I know that is possible. Once I pick a basis. Same thing happens with W also, right? So once I pick a basis with m elements, I know every vector in W is represented simply by m coordinates. So I have that, you know, comma comma comma picture that I draw for vectors, I know I can do that, okay? So now how am I specifying the linear map itself? I am saying $T(v_j)$, I am specifying it as a linear combination of w, right? Once I have a basis, I have to specify $T(v_1), T(v_2), \dots, T(v_n)$ and each of these $T(v)$'s is actually a vector from W, which is a linear combination of the basis w_1 to w_m, okay? So I will have a picture like this, okay?

So in general every output I specify will look like this, isn't it? Right? So the, every basis v_1 to v_n I have to specify what $T(v_1)$ is, $T(v_2)$ is. Now each of these $T(v)$ is actually a vector in W. So it has to be a linear combination of the small w's. And look at the coefficients. Those coefficients I can pick, you know, some coefficients will come there, okay? Those will be what v_j gets mapped to, right? So that is how I specify. So these coefficients as it turns out end up specifying the entire linear map, okay? So the scalars a_{ij} are crucial. Once I specify the scalars a_{ij} fully, right? For all values of i, i goes from 1 to m, j goes from 1 to n. Once I specify that the entire linear map is specified from V to W under this chosen basis, okay?

So these values a_{ij} you conveniently capture into a matrix form, okay? So that is how the matrix gets formed, and that is the connection between matrix and linear map, okay? So when you have a $m \times n$ matrix, you have to think of it like that, okay? So where did it come from? There is some linear map, no? And once you fix the basis, the result of a linear map acting on each basis element will give you a vector in W and that can be represented in another basis in W, and those coefficients give you the elements of this matrix, okay? Specifically think about this. This guy is what? This column is $T(v_1)$, okay? This column is $T(v_2)$, so on till this column is $T(v_n)$, okay? So every column represents the output corresponding to the basis vector at the input, okay? So the first column is $T(v_1)$, okay? What do we mean by saying first column is $T(v_1)$? The coordinates there, the coefficients there multiply the basis w_1 to w_m to give you the vector in W, right? So that represents the output $T(v_1)$. The second column is the output $T(v_2)$. The last column is the output $T(v_n)$, okay? So this is how you find a matrix corresponding to a linear map.

How do I find a matrix corresponding to a linear map? You have to fix the basis. Only when you fix the basis, you have a matrix. If you don't fix a basis you will never have a matrix, right? So you fix a basis for V, fix a basis for W. v_1 to v_n, okay? And you take and evaluate your map T on each of the input basis vectors $T(v_1), T(v_2), \dots, T(v_n)$. And then what do you do? You express $T(v_j)$ in terms of the basis of W, find those coefficients, put them in the j^{th} column, okay? $T(v_j)$'s coefficients put in the j^{th} column, you get a matrix. You get a $m \times n$ matrix with elements from $\mathbb{F}$. And that is your matrix representing the linear map. Is it okay? So this is how you go from a linear map to a matrix.

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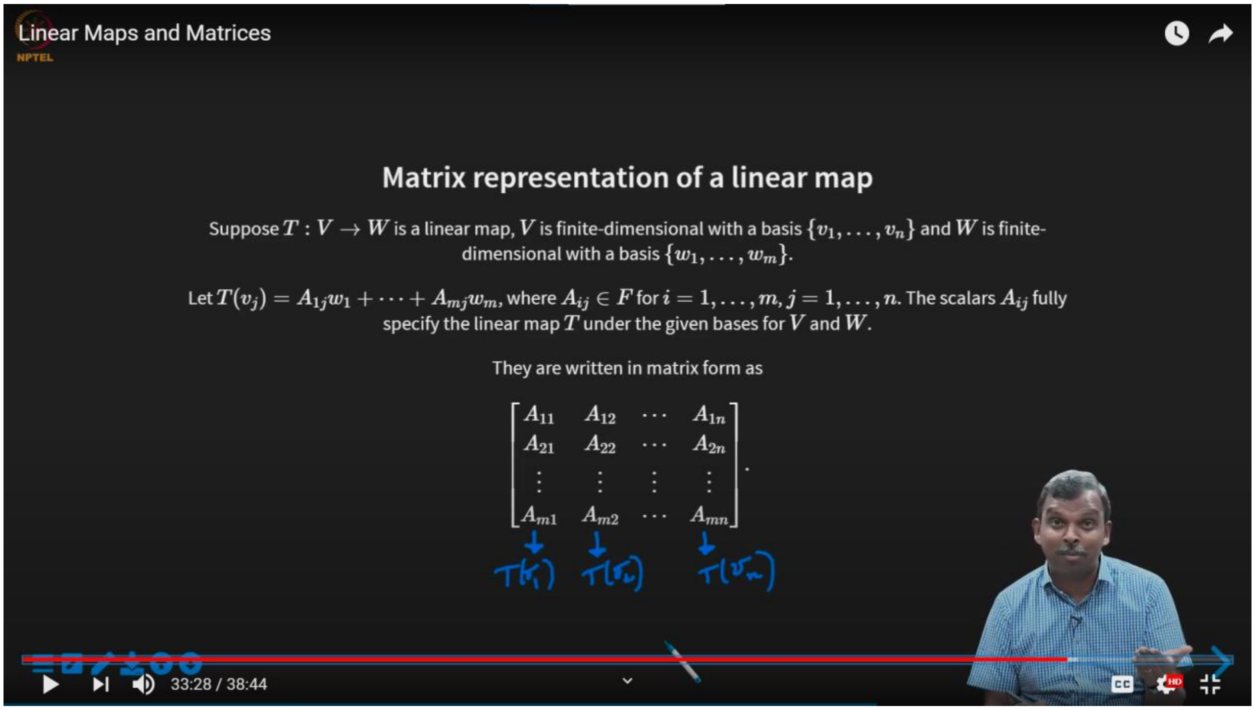

You can also quickly see how you can go from a matrix back to a linear map. We'll see that later on in more interesting examples. So this is how you have to think of a matrix. A matrix usually represents a linear map, okay? And that is very important to know, okay? So a whole bunch of examples here. I will pick the example to go from $\mathbb{F}^n$ to $\mathbb{F}^m$, and I will say it is a standard basis, okay? So that there's no confusion about choice of bases etc., okay? You know what the standard basis is, right? (1, 0, 0), (0, 1, 0), (0, 0, 1), like that, okay? So we pick standard basis and $\mathbb{F}^n$ to $\mathbb{F}^m$, and let's see a whole bunch of examples of this linear transformation and the matrix corresponding to it and all that, okay?

So that's your transformation. The first example I took $T(x) = 4x$, right? So what is my basis? It's just $\mathbb{F}$. So basis is just 1, okay? There's only one element here, there's nothing much to go. So the matrix is just, you know, 4, right? There's nothing much to worry about there, right? So I took the basis, found the T corresponding to the basis, right? The basis was just 1. I took $T(1)$ which is just 4, so the matrix is just 4, okay? So it's a 1×1 matrix, there's nothing much interesting going on here.

So the next example you can take is $3x + 4y$. Now I have a transform from $\mathbb{F}^2$ to $\mathbb{F}$, okay? So my basis has 2 different elements, (1, 0) and (0, 1). How do I find the matrix corresponding to this transformation? I evaluate $T((1,0))$, I get 3. And then I evaluate T((0, 1)) I get 4. Put T((1, 0)) in the first column. $T((0,1))$ in the second column, I get a matrix (3 4), okay? So that's the matrix corresponding to this. So let's look at the next one $(3x + 4y, 5x - 7y)$. I have 2 basis vectors again. $T((1,0))$ now gives me (3, 5). $T((0,1))$ gives me (4, -7). And now I have to put it in columns, right? So (3, 5) goes into the first column. (4, -7) goes in the second column. That's my matrix.

Same thing you can do again, now you know what to do, right? So once you have a $T(x,y,z)$, I have $\mathbb{F}^3$ to $\mathbb{F}^3$, now how does it matter? So I need to find $T(1,0,0), T(0,1,0), T(0,0,1)$. I will have a vector with three coordinates now, right? So I am going to $\mathbb{F}^3$, so (3, 5, 9), (4, -7, 2), (1, -2, -4), correct? So I have to put them in corresponding columns, I will get my matrix, right? So (3, 5, 9), (4, -7, 2), (1, -2, -4), sort of child's play, right? Just looking at the thing and figuring out what it is. Believe me, we will study more complicated properties of the linear maps, but for now it's just simple things with what these are, okay?

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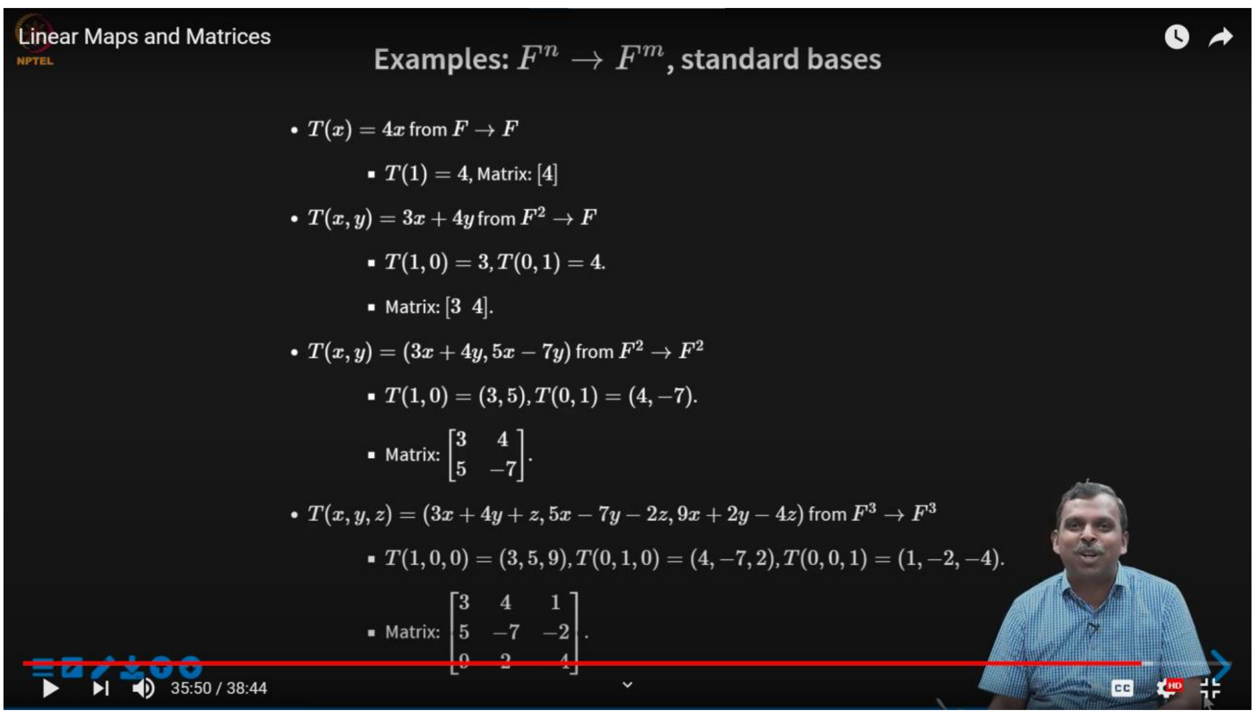

All right, so let me finish this lecture with a very practical sort of example. Maybe it's very relevant today given the context, but this is sort of an example - how people do modeling. So I was talking in the beginning about physical aspects of these maps, how they capture some quantities and their relationships in real life. So let's say here's the situation. I have this (x_1, y_1) (x_2, y_2). What is x_1? x_1 is the number of people who got flu, some sort of flu, let's say, you know. Given the situation it's an apt topic to talk about. Who got flu in 2020, how many people got flu in 2020. y_1 is the number of people who did not get flu, okay? So I have x_1 who got flu, y_1 did not get flu. And (x_2, y_2) is the corresponding number for the next year, okay? Number of people who will get flu, let's say in 2021 and y_2 is number of people who will not get flu in 2021. So these numbers you can imagine are very important right? For planning and all that. So a lot of modeling happens like this, believe me, real life works a lot with models and models are like this. So what's my model? I'm going to have a model which predicts that people who already got flu in the last year, their incidence for flu will be 10 percent, okay? I'm just cooking up a number here. I have no basis to ground it in any reality. Please take this as an abstract model. Just in my own head, I've just put those numbers, okay? The next number is incidence of flu in people who have not gotten flu in the previous year, okay? That I'm going to take is 30%, okay? And let's say these population changes equal number of people means number of people who die are number of people who are born, and we're not going to look at those kind of differences and just sort of keep it simple, okay? So if you want to do that with this model you can start writing out equations, okay? So you see x_2 is going to be $0.1x_1 + 0.3y_1$. Then y_2 is going to be $0.9x_1 + 0.7y_1$, okay? So there you go, that's the linear map. And $T(x, y)$ is that linear map and there is a matrix associated with this linear map, okay?

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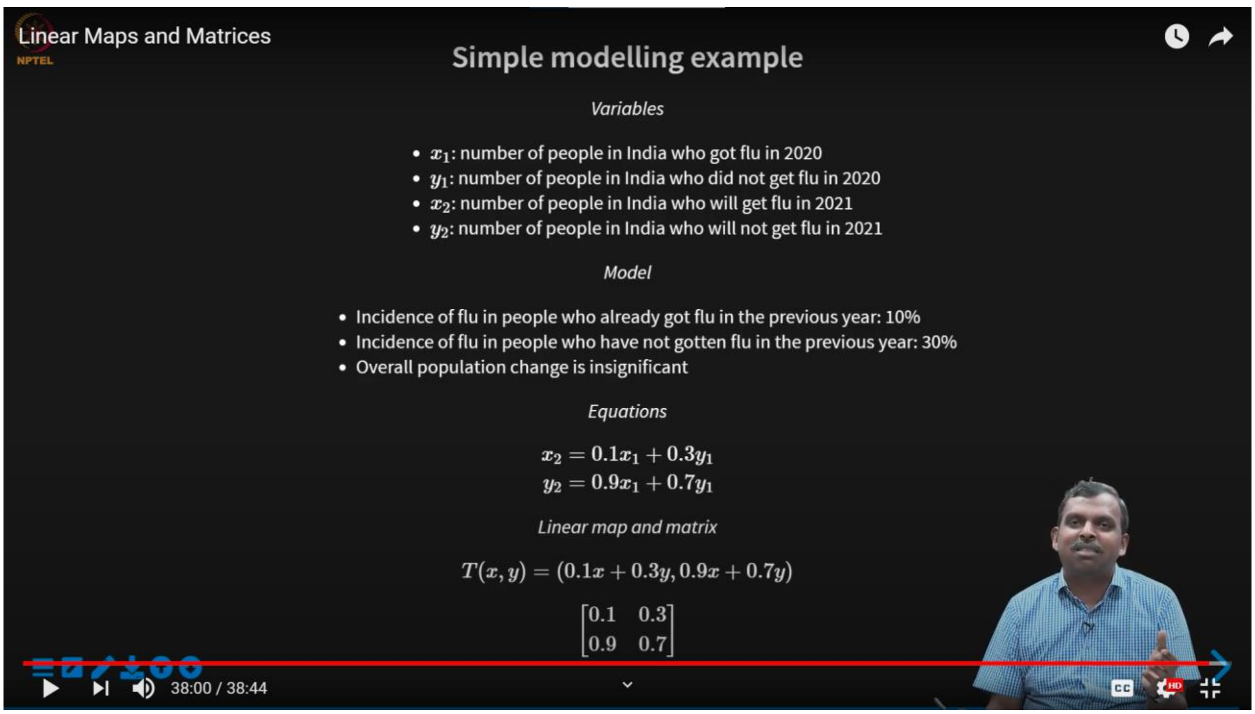

So a simple example. I mean, of course you can think of more complicated examples. Like for instance you might have seen electrical circuits, a bunch of voltages in some nodes, currents through some components. You can write a linear relationship between the currents, those components as a function of those node voltages and you know how to write those, how to solve those etc. etc., right? So more complicated relevant models are there, but I just wanted to throw a very simple model at you to end this lecture, okay? So at the end of this lecture once again there is a quiz here. I will urge you to please fill this quiz with what you know, it gives me very good feedback on what concepts have gone through, what have not gone through. I can pick up on these and clarify during our interaction sessions, okay? Thank you very much.

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