Applied Linear Algebra
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Week 02
Null space, Range, Fundamental theorem of linear maps

Hello and welcome to this lecture. In the previous lecture we saw linear maps' basic definition and quite a few examples, some motivation for why these are interesting, why we should study etc. This lecture we're going to start continuing to study linear maps, okay? We'll start taking a deeper look at them. How does the linear map really look? What are the important objects associated with the linear map that we should study. In particular what is this null space, what is the range and more importantly fundamental theorem of linear maps, right? So sounds like a very big result. Nobody would call it a fundamental theorem if it was not a very big result. It is a big result. It's also a simple enough result that we can present almost immediately after we study linear maps, okay? So it is... Beginning to study linear maps... We'll keep on doing this more and more and more and by the end I hope you will have a really, really good understanding of what linear maps are and how they work in a, you know, vector space from one to the other, how to think of it, how to work with it and etc. etc.

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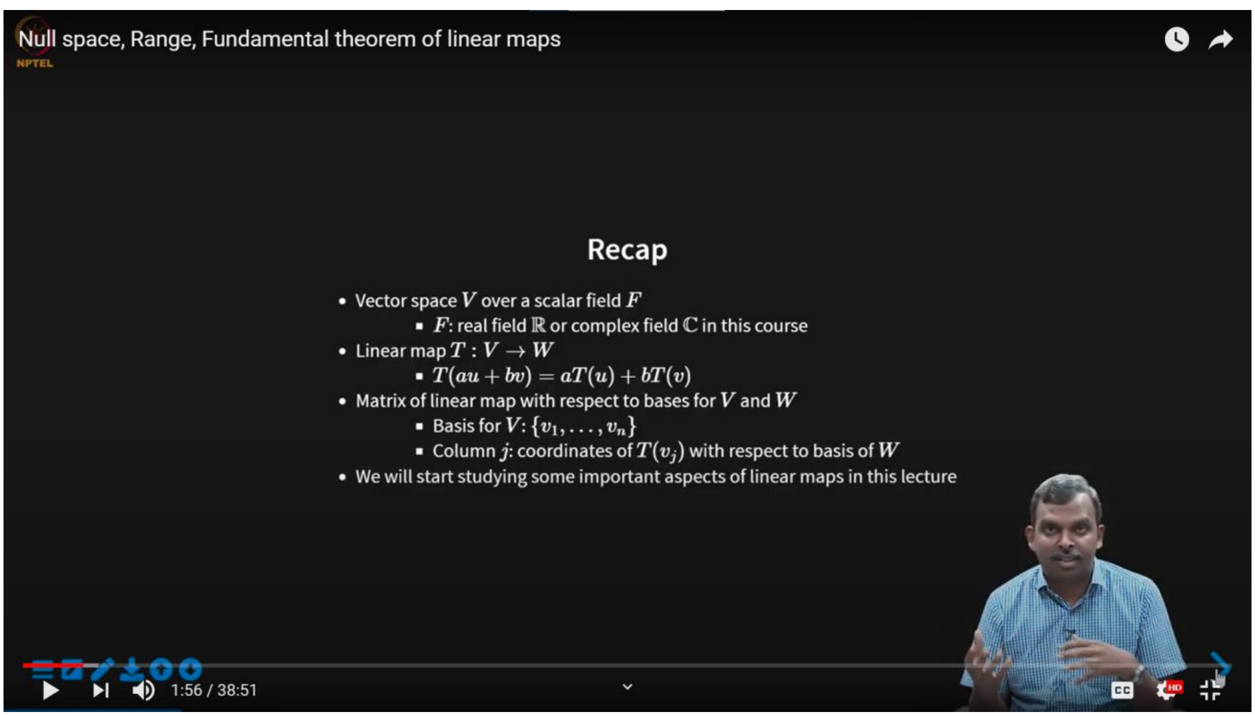

Okay, so let us go on. So quick recap. We saw that a linear map is a map from one vector space to another which preserves linear combination. So two linear combination of the input, and after the map also if you do the same linear combination there is a correspondence between the input and output, okay? We saw this important way of finding a matrix of a linear map, right? How to specify a linear map? It's enough if you specify it on a basis. And once you pick two bases for V and W you can always go to a matrix, right? The jth column of the matrix is simply has elements from F which represents the output $T(v_j)$ right? So that is a very simple way of associating a matrix with a linear map and that is very important. So from now on, when you see a matrix you have to think linear map, right? So do not just say collection of numbers put in some square array or something. Yes that is true, but for us in this class it will represent a linear map from one vector space to another, okay? That is what it is, okay?

Okay. So the first object that we will study associated with the linear map is something called null space, okay? So once again, just to back up a little bit, you have a linear map. You have that picture, right? So you have that one ellipse mapping to another ellipse, right? So I want to study what the linear map does in more detail, right? So does it do… Study a part of it. I mean you know what it does to some things or... Does it do anything special or how do I, you know, break it up and sort of study it in a systematic way? That's what we are doing slowly. So the first thing that we want to study is... Right? If you have a linear map what are the vectors that the linear map pushes to zero at the output, okay? So that set of vectors is called null space. It's denoted $null(T)$ okay? And that's the definition. Set of all inputs which get mapped to zero by the linear map, okay? So this looks like an interesting object. And it turns out it is a fundamental, interesting fundamental object to understand the linear map. To understand the linear map you have to know its null space, okay? So to understand its properties we have to know its null space and that's the first definition, okay? And it's sort of also intuitive why this would be important, right? If there's a bunch of vectors that the linear map is pushing to 0, you can imagine any linear combination within that will also get pushed to 0. So something interesting is going on here. So if you can understand this $null(T)$, then you made a big first step in understanding what your linear map is doing, right? So this is the starting point.

So let us look at a few examples before we go on, okay? First example we saw was the zero linear map, right? So we saw the zero linear map. What did the zero linear map do? It simply took any vector and put it out to zero. Now the null space for this linear map is very easy. It's the entire vector space V, right? Then we also saw the identity linear map right? So identity linear map, remember it goes from V to V itself every vector is mapped to the same thing, right? So for this, you see that the null is only the zero, right? It's the trivial zero set, right? So no other vector will get mapped to 0 because every vector is getting mapped to itself, okay? That's easy.

What about differentiation, okay? So we saw polynomial differentiation is one of the examples of linear map. What about differentiation? When you differentiate, what are the polynomials which gets mapped to 0? Those are the constant polynomials, right? So the set of all constant polynomials, that gets mapped to 0. What about multiplication by x^2? We saw this example. So when you multiply polynomials by x^2, which polynomial will get mapped to 0? Okay, that's again 0, right? So there is no non-trivial polynomial which when multiplied by x^2 will suddenly give you 0, it should have been 0 in the first place, right? So that we know, just by product.

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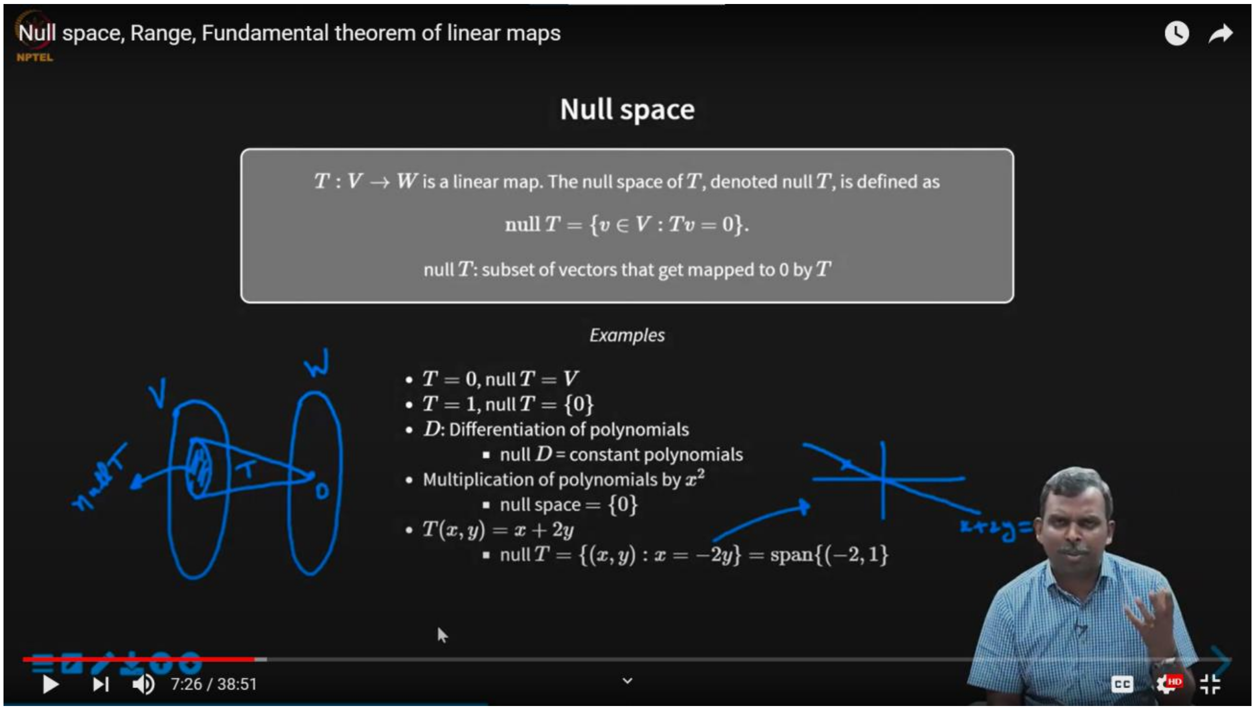

Here's an interesting example, right? Like I said this, $\mathbb{F}^n$ to $\mathbb{F}^m$ type example is what's most interesting for us and that's what will give you nice things to work with. So here is a simple one. $T(x, y)$ is $x + 2y$. It goes from $\mathbb{F}^2$ to $\mathbb{F}$, right? So that is what... I mean, if you want to do $null(T)$, I want to look at set of all inputs which give me $x + 2y = 0$ or x should be $-2y$, or if you want to write it differently $y = -x/2$. What object is that? That is a straight line passing through the origin. $y = -x/2$. And when x is -2, y becomes 1, right, so that is the span of $(-2, 1)$ okay? So hopefully you can see that. So this example if you want me to annotate it out for you a little bit... If you look at this guy $x = -2y$, that's the same as $(-2, 1)$, the line passing through $(-2, 1)$, right? So that's the line, okay? So that's this line. So you can see $x + 2y$ will be 0, okay? That's the line, right? So $x + 2y = 0$, isn't it? So that's, that gives you the null space for T. So simple examples like this. As you keep complicating things, you will get more and more complicated, maybe, linear maps. How to find the null space etc. we will see as we go along but this is how you think of a null space, okay? Given a linear map, what are all the vectors which get mapped to 0, okay? Some books also call it kernel.

But let me also draw another picture for you to keep in mind. So in case you want to visualize what this is, what is going on here. So you have V, you have W, you have 0 here, right? And then inside V there is this set which we call $null(T)$ and under T everything in $null(T)$ gets mapped to zero, okay? So that sort of a picture if you want you can keep in mind, okay? These pictures are important. They do not really convey too much but some people find it easier to remember pictures. So the null or the kernel of a linear map simply is the set of all the vectors that get mapped to zero, okay? So it is important enough to understand, okay?

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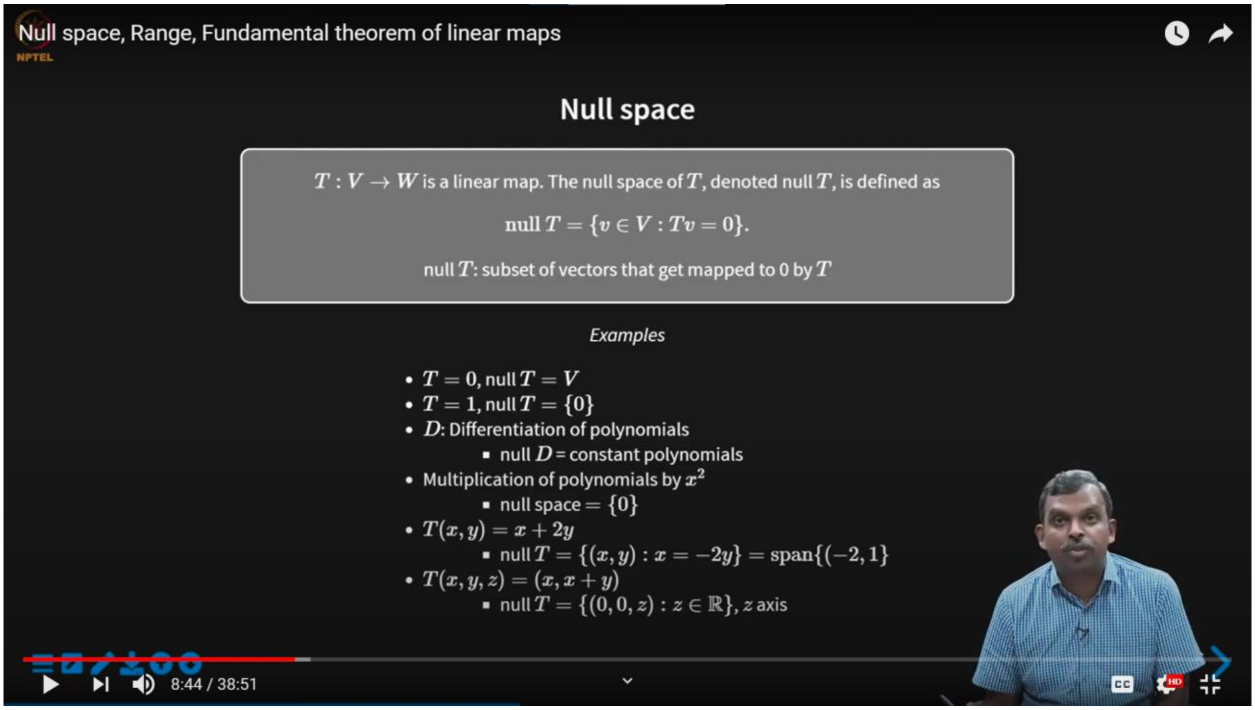

Here is another example. $T(x, y, z)$ is $(x, x + y)$, okay? So looks a little bit more tricky. So I want x to be equal to 0. I also want $x + y$ to be equal to 0, okay? So when will that happen? That will happen whenever both x and y are 0, right? So if x and y are not 0, that is not going to happen. So the null of this is simply $(0, 0, z)$ where z is any number. So z becomes, this becomes the z axis, okay? So you can imagine if I have (x, y, z) going to $(x, x + y)$ and if I want x to be 0, $x + y$ to be 0, clearly x should be 0 and y should be 0 and z could be anything. I don't care what z is, okay? So the null space of this transformation is simply the z axis itself. Any point on the z axis gets mapped to 0, okay? So you see, I mean, it's maybe looking to be a little bit interesting. There is this... Given a linear transformation it looks like the object which, you know, maps, the subset which gets mapped to 0 is something interesting. And yes it is and we will study more properties of it in the next slide, okay?

So the first property that we'll study. This slide says null space and injectivity. I will come to injectivity soon enough. Why injectivity is important, we will look at it. But first let us look at a very simple and elegant property of a null space. It turns out if T is a linear map, okay, $null(T)$ has to be a subspace of V, okay? It's not just a subset. In the previous slide I defined it as a subset and it turns out $null(T)$ for a linear map is actually a subspace. In the example you can go back and see if you want. All the examples for subspaces, it turns out that's a general fact, okay? Any linear map, then its null space is a subspace of the domain, subspace of V, okay? The proof is really actually very simple. If you have two vectors in the null space, u and v okay? You take any linear combination $au + bv$ and hit it with T, what will happen? You get $aT(u) + bT(v)$. $T(u)$ is zero, $T(v)$ is zero. Clearly that's also zero, it will belong to the null space, okay? So any linear combination of vectors in the null space also has to be in null space okay?

So this gives you this very simple proof that any linear map has to map zero to zero, okay? If it does not map zero to zero, it will not be a linear map, okay? And this is proof for that, it's sort of, I mean there are maybe multiple ways to prove it but this is a one of the corollaries of this fact, okay? So null space has to be a subspace, okay? So this picture that we drew before. okay, we had this V. And there is this thing that is was going to zero. That little set, the null space of T has to be a subspace. By itself it will have a basis, you can describe all that, right? So it has to be a subspace. Keep that in mind, ok?

Here is the next interesting property of a map, okay? So we said - okay, we are studying linear maps. The first property we'll study is what are all the vectors that get mapped to 0, okay? That was the null space. We studied that. The next property of a linear map I might be interested in is - is it injective. Injective, another word for injective, is one-to-one, okay? What is one-to-one? Any map you want to study, one of the properties you want to study - is it one-to-one. One-to-one meaning that... The definition is given here. When is T from V to W injective or one-to-one? Whenever you have $T(u)$ equals $T(v)$ then u has to be equal to v, okay? So this is sort of a technical sounding description, but one-to-one is a good picture to have in mind, right? You have a u, it goes to $T(u)$. There is no other vector which will also take you to the same $T(u)$, okay? So any other vector, you go to some other point only. There is no question of two vectors going to the same vector, okay? That's what is one-to-one. That is injectivity property. Is that okay? Hopefully you understand this. So you have you have a linear transformation from V to W. I am worried about, I want to define and find out when is it going to be injective, okay? When is it going to be one-to-one. That's an interesting property for a map, right? So you want to know whether it's one to one. Can there be more, two, more than one vector which gives you the same vector at the output, right?

So these are interesting things to ask, okay? So a bunch of examples are given here. You can see. Identity is one-to-one. Multiplication by x^2 is one-to-one but the zero map is not one-to-one. The differentiation is also not one-to-one, right? So differentiation is not one-to-one. You add any constant to it you get the same output, okay? So it's not one-to-one. So there's nice, interesting examples, okay? So it turns out null space and injectivity are very very closely connected, okay? One - the null space sort of controls the injectivity of the entire map, and that relationship is given next, okay? So it's a wonderful and simple relationship. A linear map $T: V \to W$ is injective or one-to-one if and only if...

So I keep using this iff. When I say iff, it means if and only if null of the T, null of the linear map is only the zero subspace, okay? So null space has to be trivial zero subspace, okay? If and only if that is true you have T being injective. So what is this if and only if? It means the result goes in both directions, okay? A implies B and B implies A, okay? Both are true means then I will say if and only if the statement and its converse are true, okay? So what is, what is the meaning of this? How do you read it? If T is one-to-one then its null space has to be only zero. If I find that a linear map has its null space being only 0, then it is one-to-one also. So it goes both ways. That's the thing about if and only if, okay?

So how do you prove this statement? Again it's a very easy statement. Just follows from the definitions. But let us write it down. See, whenever I have to prove an if and only if, I have to prove both statements. I have to say ... if and only if means A implies B, B implies A, okay? So let me first do A implies B. So there I'll assume T is injective, okay? So if T is one-to-one, supposing somebody tells me there is a vector v in the null space, okay? v is in null space so what does that mean? $T(v) = 0$, right? Now I know there is one more element, 0, which gives me 0, right? $T(0)$ is also 0. So $T(v)$ equals $T(0)$. Now T is injective which means v is 0, okay? So that's the proof to show that if T is injective any vector that is said to be in the null space has to be zero, okay? Only zero goes to zero, nothing else will go to zero if T is injective, okay? Just that property makes that happen.

Now on the other hand if you say my linear map T... I don't know if it's injective or not I'm trying to prove the converse now, right? Somebody tells me my linear map T has only 0 in its null space, nothing else is in the null space, okay? Now what happens? I have to show T is injective, right? So for that what do I do? Let's say there is u and v such that $T(u)$ equals $T(v)$. Somebody tells me that's true, then what happens? I push this to this side, I get $T(u - v) = 0$, which means what $u - v$ has to be in the null space, right? Now that, now I know the only vector in the null space is zero, so $u - v$ is zero and u becomes equal to v, okay? So this little proof tells you that if T is injective then null space has to be only 0. If null space is only 0, T has to be injective, okay? So think about this proof and that will tell you the power of what this result is.

Notice what this result is claiming. If you have a linear map from V to W and you want to know if this T is one-to-one or injective, right? Is it one-to-one, all I have to do is to find its null space, okay? So if it's one-to-one the null space will only be from zero to zero, okay? So one-to-one null space is only this, okay? Very powerful and elegant and simple result which for a linear map fully characterizes the property of injectivity, okay? Null space has only zero means it’s injective otherwise it is not, okay? So that's a powerful result to have.

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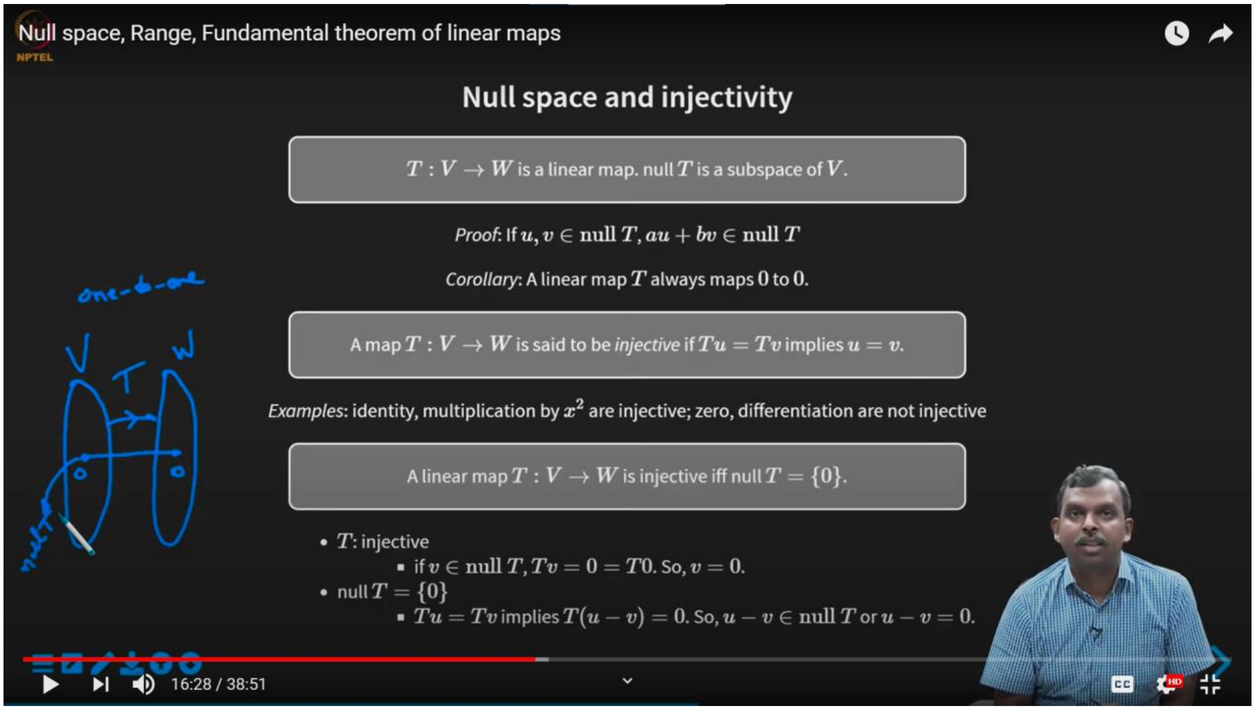

Okay. So the next object that one studies in maps in general and particularly for linear maps is this notion of a range, okay? So suppose T from V to W is a map. The range of T is basically the set of all its outputs, okay? You collect all the output you can possibly get from T, okay? You hit, you take v, you take every vector in your capital V, you hit it with T, okay? You'll get a set of vectors there. Collect all of them together, that will give you a range, okay? So a whole bunch of examples we'll see. So remember when you hit the linear transform on every vector in V, it is not true that you need to get everything in W, right? You can get something smaller than W. For instance the zero is an example. So there are lots of things where the range becomes important. So we need to know where all I can go through T, okay? So for that the range is very important, okay?

So here's an example. The first example is - if you look at the zero linear map, then the range is just zero, right? So it's a zero linear map, it doesn't do anything, right? Everything goes to zero, the range is only zero. You can't do anything else with it. If you look at one, then the range is the entire V, okay? The identity is easy, if you look at differentiation of polynomials, then the range is all polynomials, okay? So that's not too difficult to see. For every polynomial I can find some other polynomial which when differentiated will give you that, right? So that's possible. Multiplication by x^2 is interesting. What is the range? The range is polynomials which have zero constant and zero coefficient for x, right? So once you multiply by x^2, you can never have a... Either polynomial should either be 0. If it's 0 then it's fully 0. if it is non-zero then once you multiply by x^2, the smallest coefficient you will have is only x^2. So you cannot have constant coefficient, you cannot have x coefficient, both of them will go off to zero, okay? So that is what multiplication by x^2 means. You only have multiples of x^2 showing up.

Here is another example $(x + 2y)$. It goes $\mathbb{R}^2 \to \mathbb{R}$ and you can see it is the entire real line, right? So given (x, y), $(x + 2y)$ can be made to be any point on the real line. So you will get the entire real line on the range. Here is another example I showed before (x, y, z) being $(x, x + y)$. In this case the range is $\mathbb{R}^2$, the entire x-y plane you will get in your range, okay? So some simple examples we saw for range in any given linear map. Hopefully we can find the range and the range will tell us a lot of interesting information, okay? So now this range is connected to another property called surjectivity for the linear map. I will tell you what surjectivity is in a little while.

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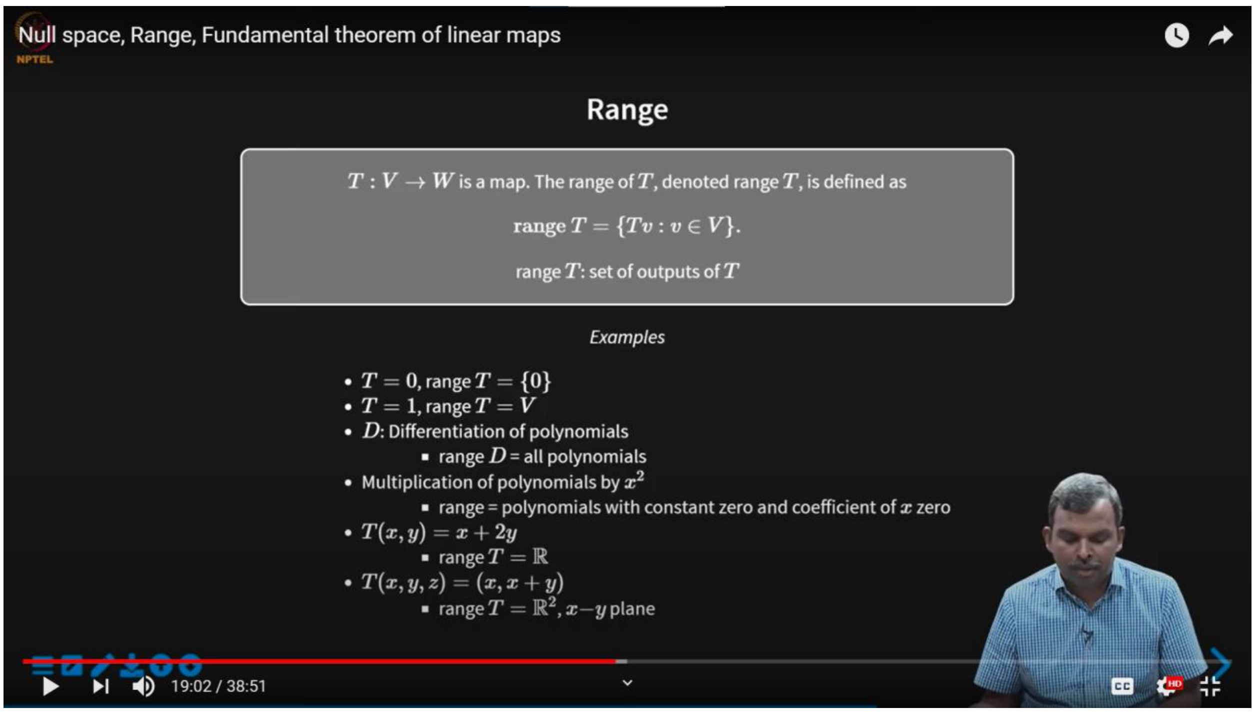

But before that we have this. And again a very simple relationship here. If T from V to W is a linear map, it turns out range of T is a subspace of W, okay? So previously we saw this null space which is the subset which maps everything to zero, that became a subspace of V, okay? So there is V, inside V there is a subspace which is the null of any transform, okay? That's one subspace. The range of that transform will be a subspace in W, okay? Which is the target of the linear transform. Proof once again is very easy. You take w_1, w_2 in the range of T which means what? There was a v_1 which gave me w_1, right? Only then it can become, be in the range, otherwise it's not in the range. Likewise there is a v_2 which gave me w_2. So $T(v_1)$ equals w_1, $T(v_2)$ equals w_2. Now if I make any linear combination of w_1 and w_2, I can make the same linear combination for v_1 and v_2 and I would have found a vector which under T will give me the linear combination here, okay? So this is a very simple proof you can write down. So that tells you that this linear combination also works, okay? So that's range. So range becomes a subspace as well, okay?

The next definition is surjectivity. For a map T, it is said to be surjective or there is another term for it, it's called onto also... A lot of people call it an onto map. A map is said to be onto if the range of T is equal to the entire W, okay? So I have, I'm taking from, going from V to W. If my range under T is the entire W, I can go to any vector in W under the T then that map is said to be surjective. Surjective is also called onto in some math literature, okay? Here are some examples. Identity is surjective, differentiation is surjective. 0 and multiplication by x^2 are not surjective. $x + 2y$ and $(x, x + y)$ are surjective, okay? So within the range they go everywhere, okay?

So let us draw this picture once again. This picture I keep drawing again and again. It is a good picture to sort of understand linear maps we are studying. We are studying the same object and studying more and more detail here, okay? We started with some arbitrary linear map, now we know there are all sorts of interesting results here. There is this null which takes me to 0 here. And there is this range which is again a subspace, let me put 0 here, again a subspace and that's where the entire V comes, okay? So this is a linear transformation T. This is $null(T)$. This is $range(T)$, okay? Hopefully that completes the picture of a linear transform in your mind. We will draw many, much more interesting pictures of how linear transforms will look eventually. You know. there are so many ways to view it, so many interesting properties.

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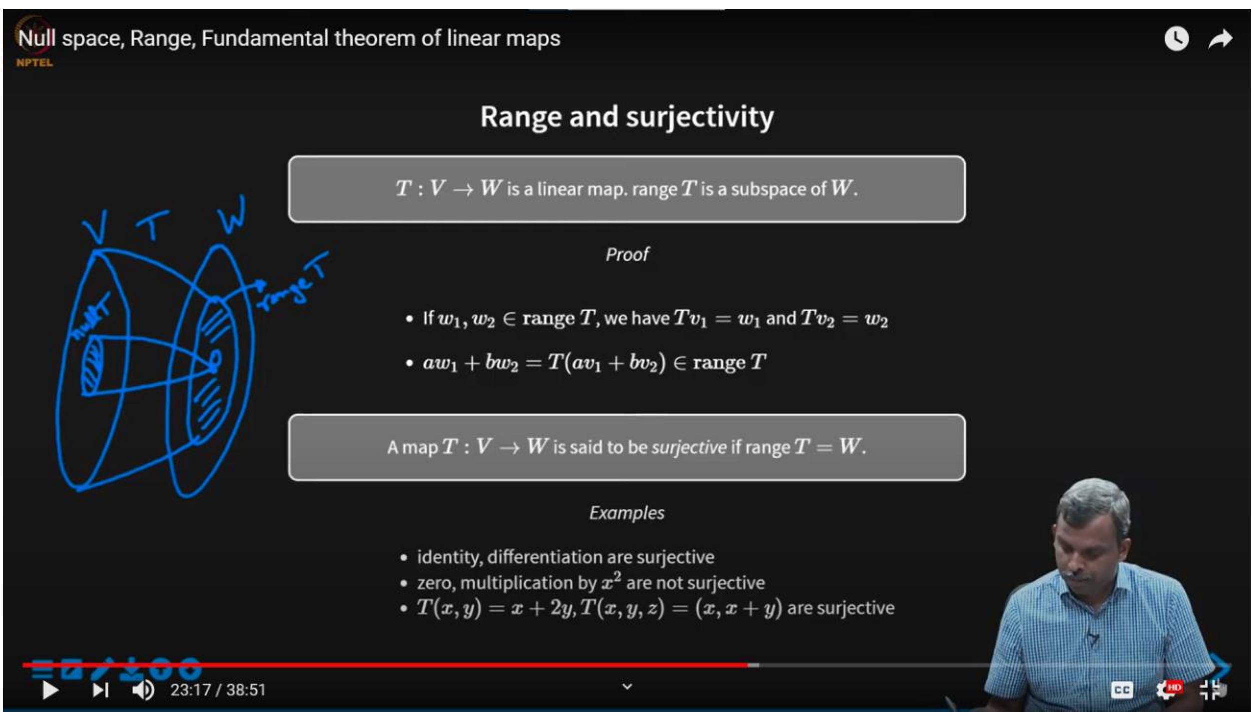

But as it is we already have very nice properties, right? So any linear transformation will look like this. There is one subspace inside V which maps everything to zero. And on W if you hit the entire V with the T you get another subspace which is inside W, okay? So this is the thing. And what is injective and surjective? The map is one-to-one if and only if the null space is the, is just zero, trivial. And the map is surjective if and only if the entire range is equal to W, okay? So this is surjective, injective and null space and range, okay? So that's about range.

Now here is the fundamental theorem of linear maps, okay? So it's a wonderful result which ties up all that we studied together and gives a very nice way of understanding what a linear map is, okay? So it gives you a very, I mean it's quite interesting, it's very, it's got a lot of nice applications. People use it quite often that's why people call it a fundamental theorem, okay? So I don't know if you've studied other fundamental theorems or maybe you didn't remember the names. There are two other fundamental theorems that you must have studied by now. One is fundamental theorem of algebra, look it up, try to find out what it is. Another is fundamental theorem of calculus, try and look it up you might have studied. I'm sure you've studied both of those. If you read them you'll know what it is. They are very important theorems and these fundamental theorems really have huge applications all across mathematics in general.

Likewise in linear algebra this is a fundamental theorem of linear maps and it has very big applications all across, okay? It is very easy to derive also, you see it's not very hard, okay? So let us say V is finite dimensional and T from V to W is a linear map. So technically here we do not need W to be finite dimensional. Usually we will assume it like that but we do not need W to be finite dimensional for this result to hold, okay? So then it turns out the range of T has to be finite dimensional, okay? That's the number one thing, okay? First result is the range of T is finite dimensional and moreover the dimension of V equals dimension of $null(T)$ plus dimension of $range(T)$, okay? So that's the nice result that we have. Remember once again, go back to the previous picture, right? What did the previous picture tell you? So you have this vector space V, inside it there is a null space T which is a subspace, it will have a dimension. My vector space V will have a dimension, then the whole V gets mapped to a range. That also is a subspace, that will have a dimension. So the dimension of V equals dimension of $null\ space(T)$ as a subspace of V, plus the dimension of $range(T)$ as a subspace of W. These two if you add, you get the dimension of V, okay? So that is the result. And that is the fundamental theorem of linear maps. Sort of a grand name but you will see it's very easy to write it down, okay?

So the proof actually is quite easy in some sense. It's very intuitive and nice to think about and will give you a nice picture of what a linear map will look like. So how do you start the proof? You start the proof with a basis for the null space, okay? I know V is finite dimensional, I know null space of T is a subspace of V which means null space of T is also finite dimensional, which means null space of T will have a basis, right? So I start with a basis $u_1, u_2, \ldots, u_k$. Let's say it's a basis for null T. So now anytime I have a basis for a subspace, I know I can extend it to a basis for the whole space. I will do that next, okay? What is my extension? $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_l$ let's say, okay? So there is totally $k + l$ vectors now which all together form a basis for V, okay? The first u_1 to u_k is the basis for null T and u_1 to u_k and v_1 to v_l, if you put them all together, I get a basis for V. So immediately what is the result that you know? From here you can see very quickly that dimension of V equals $k + l$, right? So that is immediate from here, and dimension of null T equals k, okay? So from here you can quickly see for this theorem to be true I have to really show the dimension of range of T equals l, right? Then I am done.

And in fact you can show a little bit more. You can show something very interesting. You can show that $T(v_1)$ to $T(v_l)$ is a basis for range of T, okay? So we will see this proof. I will quickly write down a proof for this, it's not very hard. But once I show this, what does this mean? Dimension of range of T equals l and the whole proof is done, right? So this is, this is all we have to show. The only thing we need to show now is this $T(v_1)$ to $T(v_l)$ is in fact the basis for range of T, okay? So this is a very interesting proof and you will see how this is done. So how do you show something is a basis, okay? So you have this range T. How do you show something is a basis? To show some set is a basis, I have to show two things. What are the two things I have to show? First thing I have to show that it is a spanning set, right? Any vector in the space I can obtain by a linear combination of the basis vectors, right? So that is the first thing I have to show. The next thing I have to show is linear independence, right? That these vectors are linearly independent, there is no non-trivial linear combination of these vectors which will give me zero, right? So both of, both the properties I have to show.

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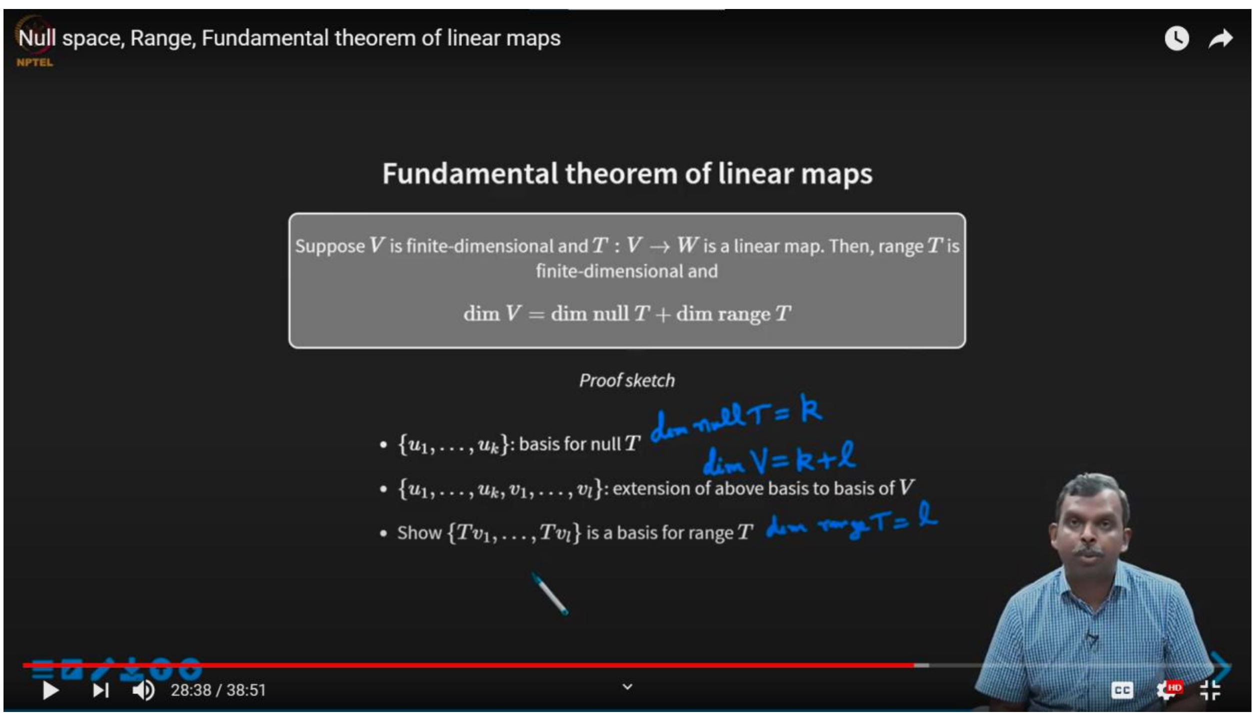

So let me go to this white board and prove both those properties for you, okay? So what do I know now? $u_1, u_2, \dots, u_k, v_1, \dots, v_l$ is the basis for V. Which means what? Any $v \in V$ is, okay, for any $v \in V$, this can be written as a linear combination of these vectors. Do you agree? Okay? That is easy enough to see, right? That is the basis, right? So its basis for V, any v can be written like this. So what will be $T(v)$, right? It is going to be $a_1T(u_1) + \dots + a_kT(u_k)$. But what do we know about u_1 to u_k? These guys are a basis for null T. And what does $null(T)$ do? $null(T)$ makes any vector zero, right? Any vector in the null it will go off to zero. So $T(u_1)$, this whole thing becomes 0, plus $b_1T(v_1)$ plus so on till $b_lT(v_l)$ okay? So that implies $T(v)$ is in the span of $T(v_1), \dots, T(v_l)$, okay? So I have shown the first property that I want, right? When I want to show that $T(v_1)$ to $T(v_l)$ is a basis for range T, I have to show that any $T(v)$, right, for any vector v in my original domain V, $T(v)$ should be in the span of $T(v_1)$ to $T(v_l)$. I have already shown that, right? So that comes from a simple property.

The next property is what? I have to show linearly independent property for $T(v_1)$ to $T(v_l)$, okay? Now suppose $c_1T(v_1) + \dots + c_lT(v_l) = 0$, okay? Then what does that mean? Push the c's inside $\dots c_lv_l = 0$. What does that mean? That implies $c_1v_1 + \cdots + c_lv_l$ belongs to $null(T)$, right? So any linear combination with zeros and the T times this vector $c_1v_1, \dots, c_lv_l$ is actually zero. Which means that vector is in the null space(T). Now what do I know about null space of T? u_1 through u_k is a basis which means $c_1v_1 + \cdots + c_lv_l$ is in the span of $u_1, \dots, u_k$, okay? Which means this must be equal to some $d_1u_1 + d_ku_k$. Now what do I know about $v_1, \dots, v_l, u_1, \dots, u_k$? These are linearly independent, okay? So that implies $c_i = 0$ and $d_i = 0$, okay? All right, so that means that this set $T(v_1)$ to $T(v_l)$ are linearly independent, okay? Go through this proof once again. So supposing there is a linear combination of $T(v_1)$ through $T(v_l)$ equals zero. Then I can rewrite it as T times the linear combination of v itself being 0, which means that particular linear combination of v should be in the null, which means that particular linear combination is in the span of u. But unfortunately or fortunately all these guys are linearly independent, so which means all these coefficients are 0, which means the original thing was linearly independent, okay?

A simple enough proof. Like I said, it's quite easy to write down this proof. But you get a very, very powerful result which is this fundamental theorem of linear maps, okay? The dimension of V, the domain, equals the dimension of the null space plus dimension of range of T. The fact that range of T is finite dimensional sort of came through in the proof, I did not try to write down a very detailed proof for you, but it's quite easy to see how once they show a spanning set, clearly it's also finite dimensional, okay? So that's something that came out of that point in the proof, okay? All right, so this is fundamental theorem. So fundamental theorem has to be a little bit more powerful, right? So let's see what light it sheds on the linear map, okay?

So let's see. Once you know the fundamental theorem, can I understand the linear map better? In fact the proof of the fundamental theorem is a very very strong illustration of how a linear map works, okay? You go to the null space, find its basis, everything there gets mapped to zero. Now if you extend that basis to V, you have a wonderful way of understanding the linear map, okay?

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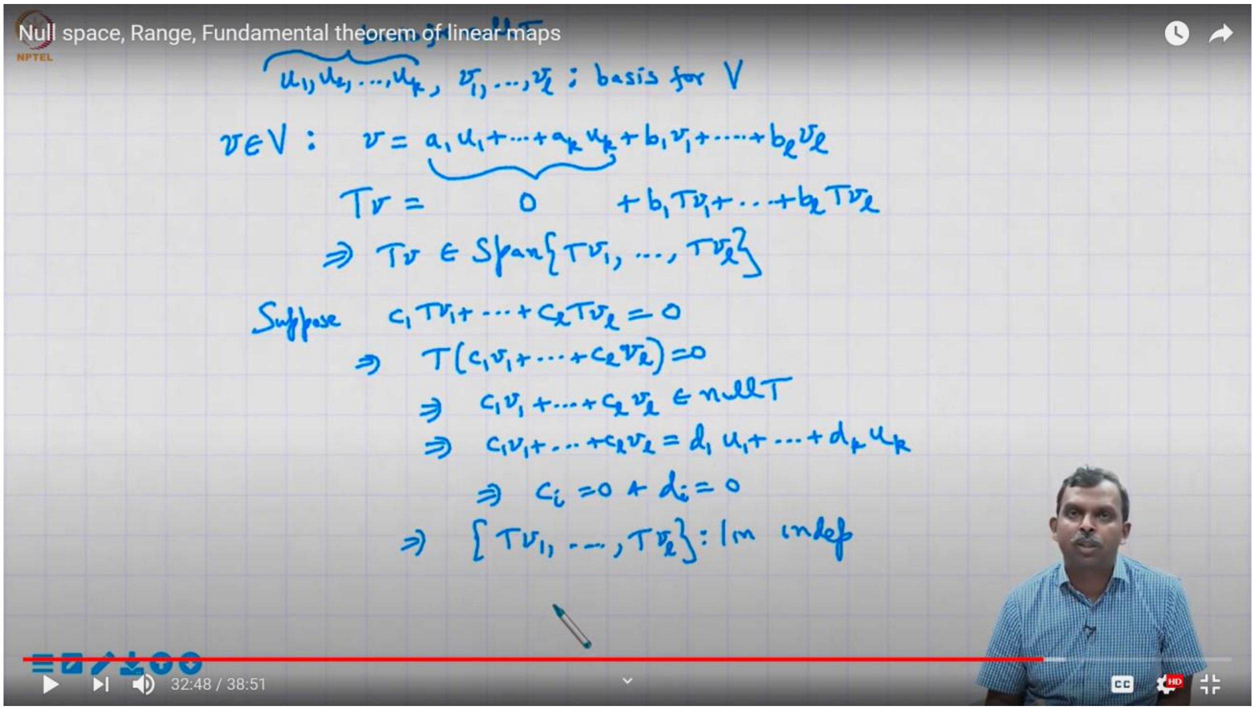

So let me elaborate that picture for you in the next slide, okay? So how do all linear maps work, okay? So the proof of the fundamental theorem in my opinion gives you a nice window to look at the linear map and understand what it does, okay? So I'll draw a little picture.

But before that let's also do this, okay? So you have the null space of T which gets mapped to zero. You have a basis for the null space. You extend it to get the extension, which is the basis for the whole vector space V, okay? So let us do that. So we have this vector space V and this vector space W and inside that I have my u's, okay? u_1… I will draw them all together here up to u_k, and then I have my… Maybe I should draw them in another color. Then I have my v_1 through v_l, okay? I am drawing them slightly differently. It’s okay, it’s not a very badly representative ... v_l, okay? Now what happens to all these guys? All these fellows get mapped to zero, right? They are in the null space, okay? $u_1, \ldots, u_k$ is a null space, okay? Now you have the null space, and then what are the vectors in V which are outside the null space, right? So that's how you think of extension, right?

So think about extension. How do you think of extension? I think of the null space first and then I think of vectors outside the null space. I try to find one linearly independent vector v_1, okay? So that's there. And then you look at vectors of the form av_1 plus vectors from the null space, okay? So all of those guys get mapped to this $T(v_1)$, right? Do you agree? So you have the v_1 getting mapped to $T(v_1)$. If you take linear combination of v_1 with any other vector in the null space also you will get the same $T(v_1)$, right? So keep that picture in mind. So you have this null space. Then you have something outside of the null space which is linearly independent of everything in the null space, and that goes to one $T(v_1)$, okay? And if you add that with anything in the null space also, you will get $T(v_1)$ only, okay? The same way you can sort of picture all these $T(v_1), \dots, T(v_l)$, okay? Right? And these are linearly independent vectors, this v_1 to v_l is linearly independent. $T(v_1)$ to $T(v_l)$ is also linearly independent, right? And together all of them form the range, okay... So these guys. I should just draw a blue picture… So this was the null, right? This part is the null, this part gives you the range, okay? So that is a picture, a very powerful picture which, you know… Any linear map looks like this. And you should have that in your mind, okay?

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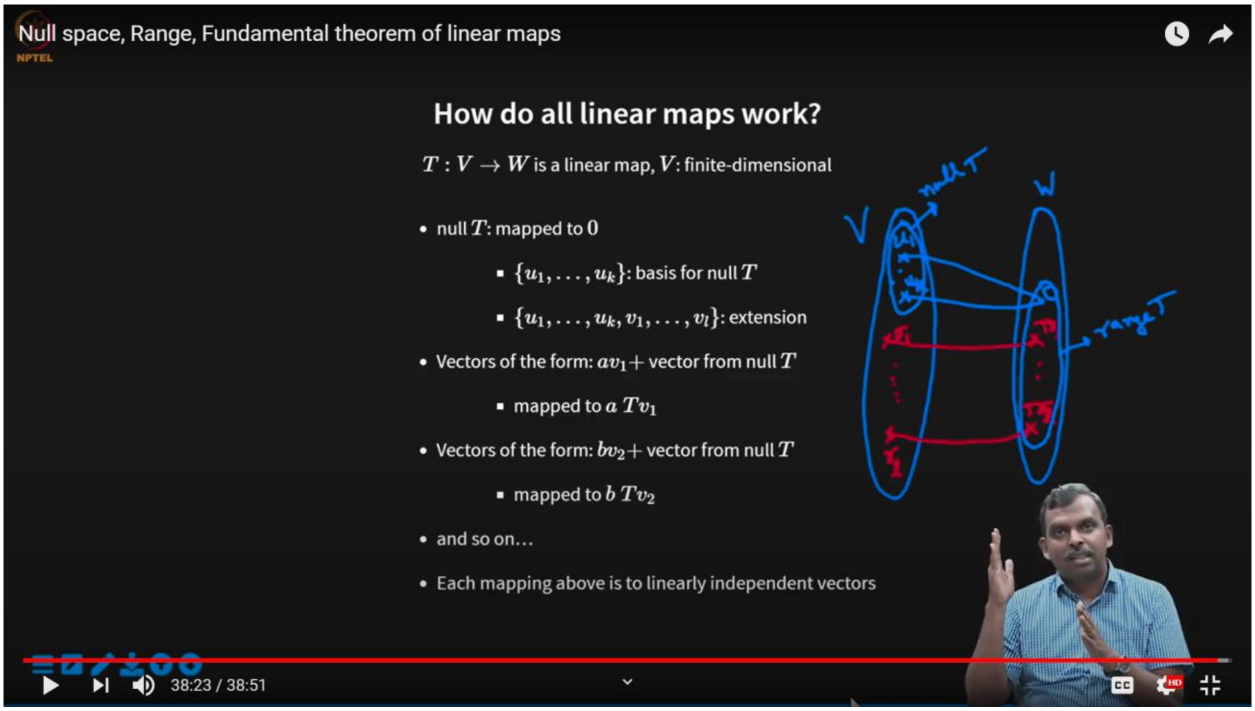

So we've started studying linear maps, right? So we've already done quite a, quite deep inroads into it. We studied null space, we studied range space. We know when it's injective. We know when it's surjective. Not only that, we know this fundamental theorem which is giving you this wonderful view of the linear map, okay? You take a vector space V, there is a null space, I mean... Remember there are so many bases inside the null space itself you can do. So many different bases. But you pick any basis you like, okay? And then you extend. Even extension there are so many ways to extend, there's no unique extension. You can extend in so many ways. You take your favorite extension, okay? And then how do you picture the whole linear map? Every vector in your extension takes you to linearly independent vectors in the range, and, you know, they're all different in some sense. Once you know it's linearly independent, they're like different dimensions, different directions, right? And then you take the span of all of those, you get the range. And on this side, the entire vector space gets split in that fashion, okay? So this picture is a very good picture to keep in mind when you think of a linear map, it works in this fashion. There is a null space and then there is this extension of the null space into the entire vector space, and then when you do the linear map you get linearly independent guys. And all of them span the range, okay? So that's a good picture to have in mind. We will start applying this picture in so many different ways and derive more interesting results with linear maps going forward. In the next lecture, we will see how to connect this matrix view to all this, okay? So what, how do all these results correspond to in matrix terms. How do you understand that. And that we’ll do in the next lecture. Thank you very much.