Stochastic Modeling and the Theory of Queues
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Module - 2
Lecture - 12
Merging of Poisson Processes - Part 2

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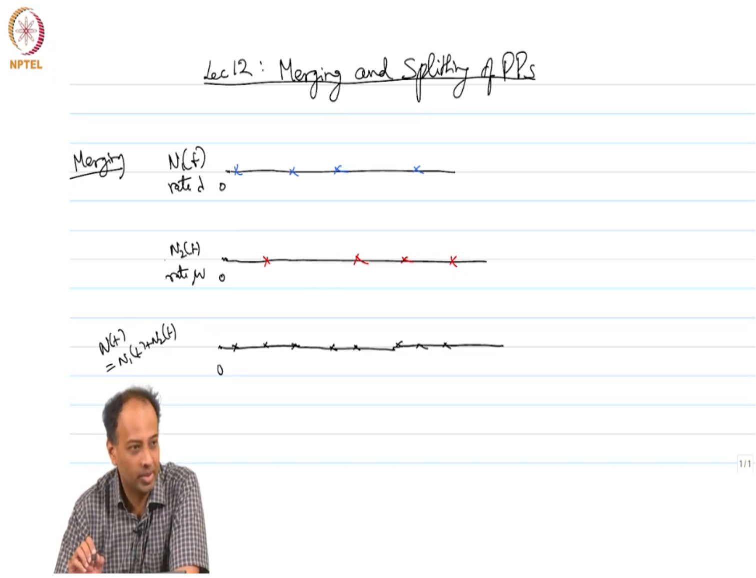

Good morning, welcome back. Today we will discuss merging and splitting of Poisson processes. We started discussing merging in the previous lecture, we will just complete what we have to say and also do splitting today. So, for merging, you have two Poisson processes. So, $N_1(t)$, which has rate, let us say, call this λ, and let us colour this blue. And there is another process $N_2(t)$ of rate μ, which is, let us say, a red process. This is 0.

Now, I am looking at the merged process, and I am not able to distinguish colours. I am just looking at whether there is an arrival or not. So, you see what I mean. There will be an arrival here, arrival here, arrival here, one here, one here and so on. So, this process is simply $N(t) = N_1(t)+N_2(t)$. So, this is a Poisson process of rate λ. This is a, $N_2(t)$ is a Poisson process of rate μ.

And $N_1(t)$ and $N_2(t)$ are independent, these two Poisson processes are assumed to be independent. Now, what we said in the last class is that, the merged process $N_1(t)+N_2(t)$ is also a Poisson process of rate $\lambda + \mu$.

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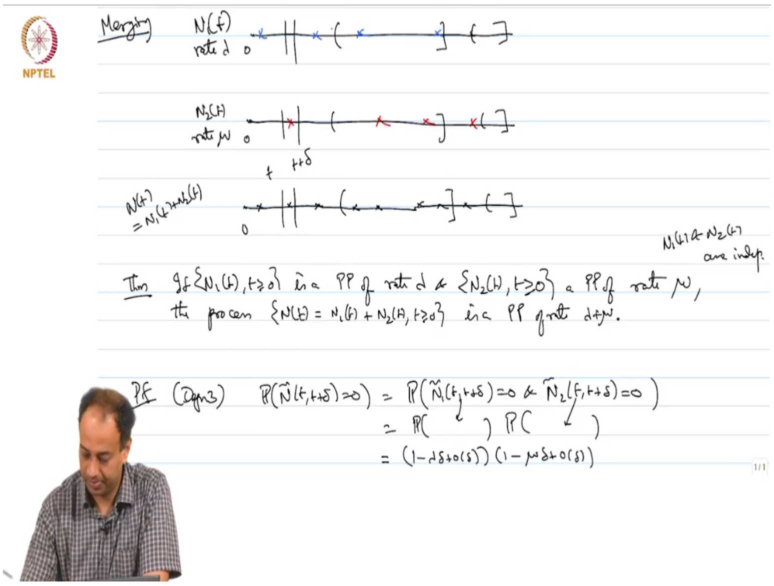

So, this is the thing. If $\{N_1(t), t \geq 0\}$ is a Poisson process of rate λ and $\{N_2(t), t \geq 0\}$ a Poisson process of rate μ, the process $\{N(t)= N_1(t)+N_2(t),\ t \geq 0\}$ is a Poisson process of rate $\lambda + \mu$. I forgot to say that $N_1(t)$ and $N_2(t)$ are independent; please write that. $N_1(t)$ and $N_2(t)$ are independent, otherwise this is not true. So, we looked at this using this δ micro slot picture, if you remember. So, you can look at some δ slot here, let us say you look at some δ slot here, some δ slot $(t, t + \delta]$.

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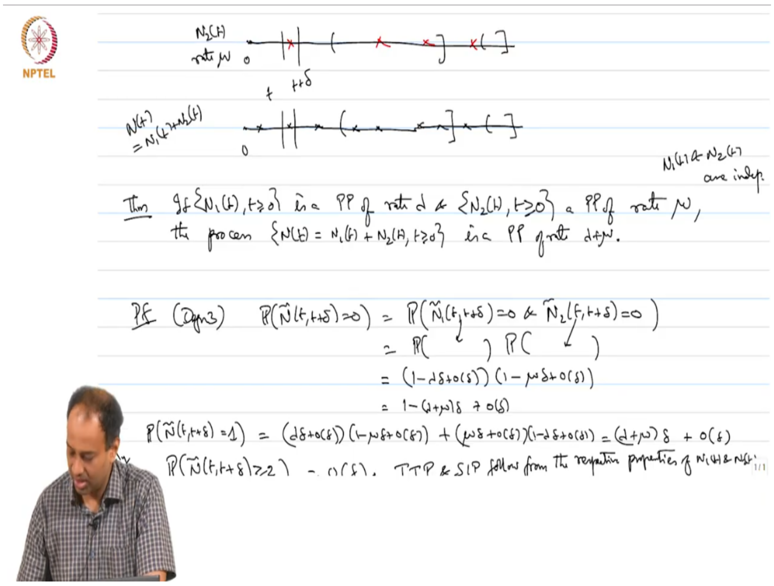

So, here, definition 3 you can use. What is definition 3? The probability of arrival in any micro slot is $\lambda\delta + o(\delta)$, and IIP and SIP holds. So, you can have

$$P(\widetilde{N}(t, t + \delta) = 0) = P(\widetilde{N}_1(t, t + \delta) = 0 \,\&\, \widetilde{N}_2(t, t + \delta) = 0)$$

And this $N_1(t)$ process and $N_2(t)$ process are independent.

So, this becomes a product $P(\widetilde{N}_1(t, t + \delta) = 0) . P(\widetilde{N}_2(t, t + \delta) = 0)$. Now, this we know to be $(1 - \lambda\delta + o(\delta))(1 - \mu\delta + o(\delta))$ which is equal to $1 - (\lambda + \mu)\delta + o(\delta)$. Then you will get $\lambda\mu\delta^2$, $\lambda\delta o(\delta)$, all that can be absorbed into another $o(\delta)$ term. Similarly, you can write what is the $P(\widetilde{N}(t, t + \delta) = 1)$.

This is possible if there is one arrival in the first process and zero arrival in the second process, or the other way around; and they are independent of course. So, this will just turn out to be $(\lambda\delta + o(\delta))(1 - \mu\delta + o(\delta))$. So, this corresponds to one arrival in the first process, no arrival in the second process; or you can have it the other way, and these are disjoint.

You can have $(\mu\delta + o(\delta))(1 - \lambda\delta + o(\delta))$. There is no other possibility. This, if you simplify, you will get something like; so, you will get $(\lambda + \mu)\delta$. So, that corresponds to this

$\lambda\delta$ multiplying this 1, and this $\mu\delta$ multiplying this 1. And then, you will get a whole bunch of δ^2 and $o(\delta)$ terms. So, essentially what remains is all one $o(\delta)$.

And similarly, you can argue that $P(\widetilde{N}(t, t + \delta) \geq 2)$, you can argue, is equal to $o(\delta)$. So, we have just gotten the right incremental distribution; we still have to prove SIP and IIP. So, in particular, I have to prove that the $N(t)$ random variable has both IIP and SIP. So, to prove, it is not difficult to argue at all. So, let us say, you take some interval like that, and some other interval like that.

You want to prove that, in the merged process, you want to prove that the number of arrivals here is independent of the number of arrivals here. So, the number of arrivals here is nothing but the sum of arrivals in the corresponding interval in the two processes. Now, for the first process, of course, IIP holds. So, in the corresponding interval, if you look at it here, you can; this is not at all difficult to argue, as you can imagine.

So, you look at the corresponding intervals here. And this process satisfies IIP. So, this is independent of that number of arrivals. Number of arrivals here is independent of the number of arrivals here; and of course, the number of arrivals here is independent of the number of arrivals here. Why? The number of arrivals in that interval is independent of the number of arrivals in that interval, why? Because the processes are independent.

So, using IIP for the individual processes and the fact that the processes are independent, you can get IIP for the composite process or the merged process. Now, we have to argue SIP. Sorry, maybe this should be here. This should be here. So, for SIP, what do you do? So, you have to prove that the number of arrivals in any interval is only a function of the width of the interval, but the total number of arrivals in this width is nothing but the total number of arrivals in the two processes put together; and for that, SIP holds, for the separate processes.

From this you can argue SIP for the merged process. So, I will just say IIP and SIP follow from the respective properties of $N_1(t)$ and $N_2(t)$, and the fact that they are independent. Now, I happened to use definition 3, it turned out to be quite natural, in the sense that I am just looking at the increments of the merged process. Now, you can also look at other definitions, definition 2 or definition 1.

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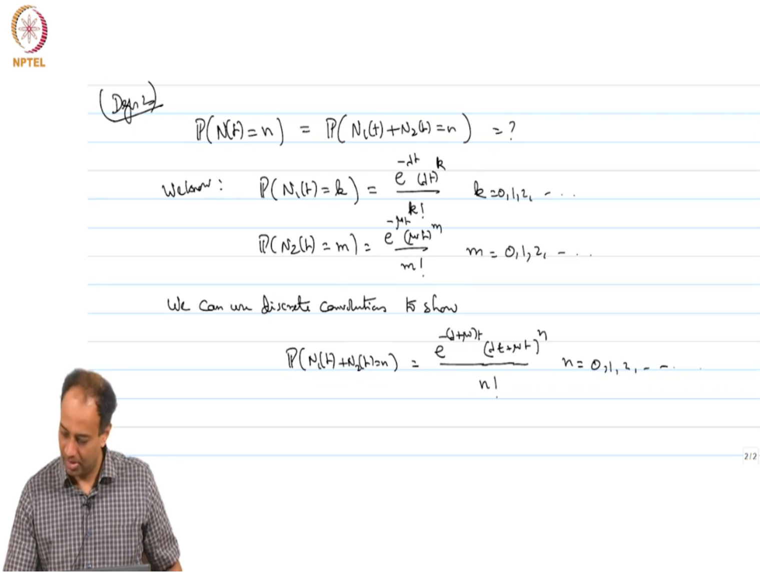

For example, if you want to do the second definition; proof using definition 2. What is definition 2 by anyway? Definition 2 says that $N(t)$ is Poisson PMF for every t; $N(t)$ is Poisson with parameter λt, whatever, $\mu_1 t$, whatever the parameter is; and of course, SIP and IIP. See, that SIP and IIP hold, I have already shown; using the previous argument still holds. I just have to prove that; so, if I look at this, $P(N(t) = n)$.

If this turns out to be Poisson PMF with parameter $(\lambda + \mu)t$, then I am done. So, this is nothing but the probability that; so, I want the PMF; so, for any t, I want the PMF of $N_1(t)$ and $N_2(t)$. So, what is this, is the question. But I already know that; we know this. What is the PMF of $N_1(t)$? $P(N_1(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$, $k = 0, 1, 2,....$ And similarly, this also I know; $P(N_2(t) = m) = \frac{e^{-\mu t}(\mu t)^m}{m!}$, $m = 0, 1, 2,....$

So, I know that $N_1(t)$ is Poisson with parameter λt; $N_2(t)$ is Poisson with parameter μt. And I want to find the PMF of $N_1(t)$+$N_2(t)$. And this $N_1(t)$ and $N_2(t)$ are of course independent. So, you are looking at the sum of 2 independent Poisson random variables. And you may already know that the sum of 2 independent Poisson random variables with parameter, let us say λ_1 and λ_2 is Poisson with parameter with $\lambda_1+\lambda_2$.

So, we can either use; so, how do you prove that? So, we can use discrete convolution. See, you are allowed to use convolution because $N_1(t)$ and $N_2(t)$ are independent, otherwise you cannot use convolution. You can use discrete convolution to show that this is also a Poisson with parameter $(\lambda + \mu)t$.

$$P(N_1(t) + N_2(t) = n) = \frac{e^{-(\lambda+\mu)t}((\lambda+\mu)t)^n}{n!}, n = 0, 1, 2,...$$

You know how to convolve, right? You can do convolution.

But, so, if you were doing an undergraduate probability course, you will probably sit and do the convolution. Now, we already know something about the Poisson process. So, we can; there is actually a cleverer way to do this. See, you just consider one Poisson process. If you have one Poisson process, you look at intervals $(0, t_1]$, let us say, and intervals, in another interval $(t_1, t_2]$; sorry $(0, t_1]$.

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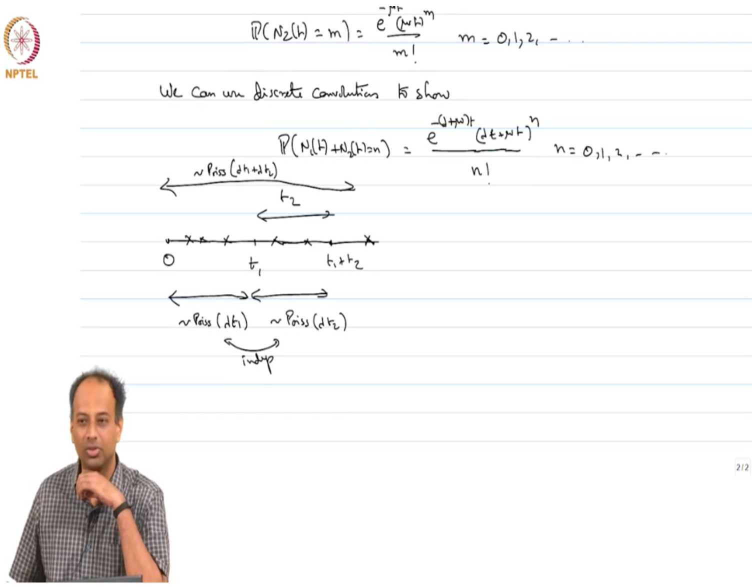

So, this is a separate consideration, just to see that the sum of two independent Poissons is a Poisson. So, say this is 0; let us say t_1; and this is t_2. So, this is t_1+t_2. And you have some Poisson process running. So, you know that this, the number of arrivals in $(0, t_1]$ is Poisson distributed with parameter; what? λt_1. And the number of arrivals in this $(t_1, t_1 + t_2]$ is Poisson with parameter λt_2. Why?

SIP, because the width is t_2. And of course, the number of arrivals in this width, in $(0, t_1]$ is independent of the number of arrivals in $(t_1, t_1 + t_2]$; so, these two are independent. Now, the total number of arrivals in this entire width; you look at this width; we know this to be Poisson with parameter $\lambda t_1 + \lambda t_2$. That also we know. So, what have we established?

So, I know that the sum of two Poissons with parameters λt_1 and λt_2, independent Poissons is a Poisson with parameter $\lambda t_1 + \lambda t_2$; the parameters are getting added up. See, that is all that I want. In the other case, when I am merging two processes, again the parameters are getting added up. The only difference there is that the rates are different, the t is the same. Here, the parameter λ is the same and the t's are different.

But that does not matter, right? You are just looking at two Poisson PMFs, this has nothing to do with stochastic processes. So, call λt_1 to be ξ_1 and λt_2 to be ξ_2, you are getting a Poisson PMF with parameters $\xi_1 + \xi_2$; and that is all that is happening even in the merging case. So, this is just a clever way of showing that the sum of two independent Poissons is a Poisson with the parameters added up. Is this argument clear?

So, you do not have to; if you are bored of convolution, you do not have to convolve; this is a clever way to see this. So, this is also a nice proof. And of course, IIP and SIP, you already did, right? You have to show IIP, SIP also. So, definition 2 also is a valid way to approach this result. Do you know how to approach this using the first definition?

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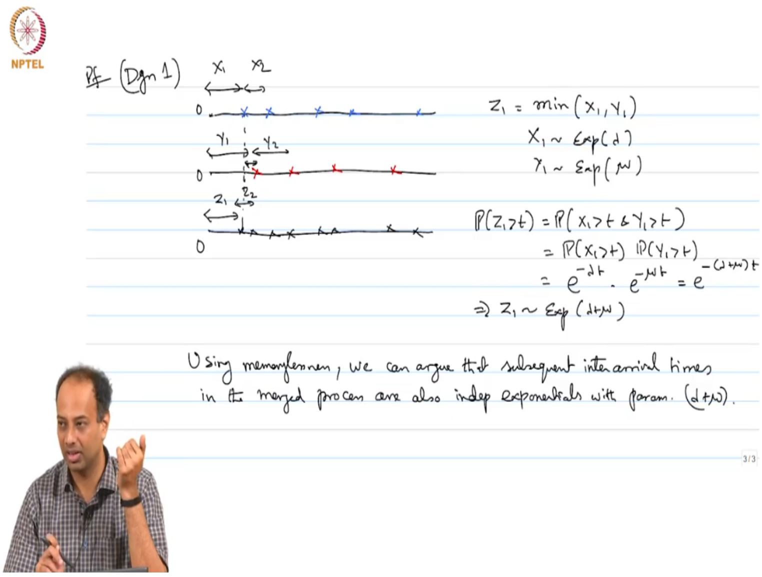

So, proof from definition 1. Definition 1 is what? Independent exponential interarrival times. So, just again, let us look at the blue process and the red process. See, I am doing this so that; for a splitting case for example, I will not do the proof using all 3 definitions. I am just doing this once, so that you get the hang of it. So, this is your blue process and that is your red process.

So, I want to call this guy X_1, X_2, et cetera; and let us say, I call this guy Y_1, Y_2, et cetera; of course, the merged process will have arrivals from both. Now, here, if you look at the first interarrival time, let us say, call this Z_1, Z_2, et cetera. What is Z_1 in terms of the X's and Y's? So, the first arrival in the merged process will simply be at; now, X_1 is exponential with parameter λ, Y_1 is exponential with parameter μ; that is what I am given; and of course, X_1 and Y_1 are independent.

Now, you probably know this already; but you can also derive it from first principle, it is not at all difficult; that the minimum of two independent exponential random variables is; yeah, he is right. What he is saying is that, the complementary CDF of the minimum is the product of the two complementary CDFs. So, essentially, you will get that Z_1 is also exponential with parameter $\lambda + \mu$.

So, if you look at this, $Z_1 > t$, this is equal to the probability that $P(X_1 > t, Y_1 > t)$, because the minimum is greater than t, then both are bigger than t. So, this is of course equal to the product; $P(X_1 > t)P(Y_1 > t)$; because of independence, the two processes are independent; so, X_1 and Y_1 are independent. And this is nothing but $e^{-\lambda t}e^{-\mu t}$, which is $e^{-(\lambda+\mu)t}$.

This means that Z_1 is exponential with parameter $\lambda + \mu$. So, what have we proven? So, we proved that, in the merged process, the first arrival time is exponential with parameter $\lambda + \mu$, which is correct, which is what you want. Are we done? No, we have to prove that the subsequent inter-arrival times are, they are all exponential and they are all independent. This basically follows from memorylessness.

So, the way I have drawn it, this blue process has created the first arrival; of course, it could also be that the red process created the first arrival. But nevertheless, see, the first process create an arrival here, let us say at some particular time, this is, let us say some time τ. For the red process, from the time τ to the next arrival is; what? Is also exponential. We proved that theorem, remember?

You fixed a time t and you looked at the time to the next subsequent arrival. So, you look at the first process. The second process does not know anything about the first process, it is just some time τ. At the time τ, the time to the subsequent arrival of the second process is an exponential with parameter μ. It is an exponential parameter μ; that is all that I am saying.

And of course, the next arrival in the blue process is of course an exponential with parameter λ, by definition. Now, the next arrival in the composite process is the minimum of this guy and the X_2. This time to the next arrival in the red process is exponential with parameter μ, from the theorem that we proved. And this is exponential with parameter λ. And these two are independent. Why? Processes were independent.

So, again, you are looking at a minimum of two independent exponentials. So, then you can argue; so, using memorylessness that subsequent interarrival times in the merged process, also independent exponentials with parameter $\lambda + \mu$. So, the trick is; so, you have gotten the first arrival in the merged process. In this picture, it comes from the blue process. You look at the time τ; the time to the subsequent arrival in the red process, this guy is exponential with parameter μ.

We know this from the theorem; and it is of course independent of everything that happened in the past. And likewise, this is also an exponential, the time to next arrival here. So, the time to next arrival in the merged process is a minimum of two exponentials which are independent, because the two processes are independent. So, again, it will be an exponential with parameter $\lambda + \mu$.

And to show that this should be independent of the previous exponential. That follows from, again, the theorem that said that this width, this exponential is independent of everything that happened in the past. And of course, this exponential is also independent of everything that happened in the past. So, these two together, you can get what you want. So, what have we shown? So, we have shown the same result in 3 different ways, using the 3 different equivalent definitions, which is not surprising.

I think all 3 are insightful. Perhaps, the most intuitively easy is the first one, you look at these micro slots and you get this straight away. In definition 2 and definition 3, you have to know something about the sum of Poissons being Poisson or minimum of exponentials being exponential. But there is no approximation involved. So, in the definition 2 proof and the definition 1 proof; but there is no approximation involved, that is the nice thing, it is exact.

But you have to argue somewhat carefully that these bits are independent of the past and all that. So, completing the proof requires some writing down a bunch of arguments. This, what I have said in this one sentence, you need to write it out properly, in terms of the theorem, that big theorem we proved, the memorylessness theorem that we proved. Good. So, this concludes the module on merging.