Stochastic Modeling and the Theory of Queues
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Module - 3
Lecture - 18
Non-Homogeneous Poisson Process

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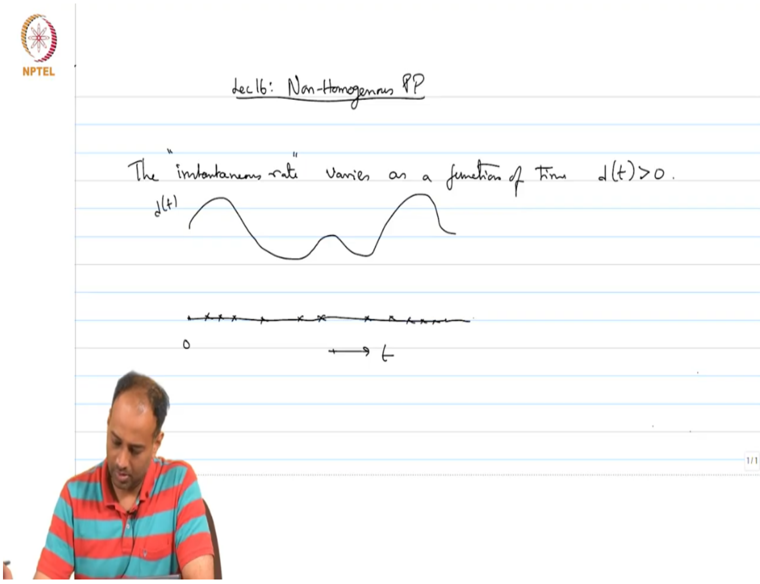

Welcome back. Today we will discuss the non-homogeneous Poisson process. So, this non-homogeneous Poisson process is like a generalisation of the Poisson process that we have been studying so far, which is the homogeneous Poisson process. If I do not say anything, it means homogeneous Poisson process. So, here, the rate λ varies as a function of time. So, basically, intuitively, the instantaneous rate of the process varies as a function of time, say some $\lambda(t)$.

The homogeneous case, this $\lambda(t) = \lambda$, because it is strictly positive. So, we assume that this $\lambda(t) > 0, \forall t$, bounded away from 0 $\forall\, t$. So, the way this works is; I will tell you how it is defined in a minute. So, let us say this is 0, this is time, and let us say my $\lambda(t)$ is doing something like that. Let us say this is $\lambda(t)$. Then, correspondingly, there will be more arrivals where $\lambda(t)$ is big and fewer arrivals when $\lambda(t)$ is small.

So, here, there will be lots of arrivals. So, larger the $\lambda(t)$, more arrivals there will be around that t. In a homogeneous process, of course, this is constant. So, this non-homogeneous Poisson process, apparently it has applications in optical communications, where you can control the intensity of the photons that you send out and so on. It is a process which has independent increments, but not stationary increments.

It is not even a renewal process really; so, the inter-arrival times are not independent and identically distributed; but it has the independent increment property, but not the stationary increment property. So, it is a counting process.

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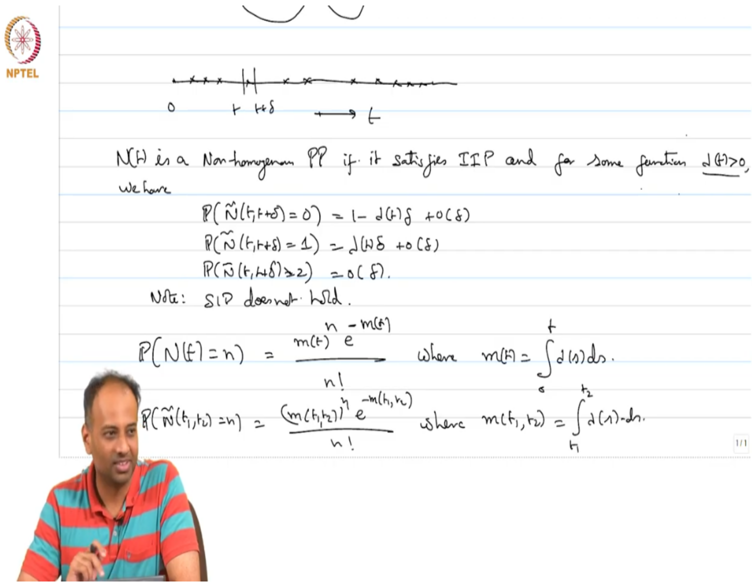

So, $N(t)$, this guy is a non-homogeneous Poisson process if it satisfies independent increment property, and for some function $\lambda(t)$ bounded away from 0, we have,

$$P(\widetilde{N}(t, t+\delta) = 0) = 1 - \lambda(t)\delta + o(\delta)$$

$$P(\widetilde{N}(t, t+\delta) = 1) = \lambda(t)\delta + o(\delta)$$

$$P(\widetilde{N}(t, t+\delta) \geq 2) = o(\delta)$$

So, if you take some $(t, t+\delta)$, the probability that you have an arrival in that $(t, t+\delta)$ is equal to $\lambda(t)\delta + o(\delta)$; and probability having no arrivals, $1 - \lambda(t)\delta + o(\delta)$; and more than 2 arrivals, 2 or more arrivals has probability $o(\delta)$, and it satisfies IIP. So, the arrivals in these disjoint intervals continue to be independent.

So, if I give you some interval here and some other interval here, the number of arrivals in those intervals are independent random variables. And of course, SIP is not satisfied, because the number of arrivals in any interval is now not just a function of the width of the interval. If you move it around, this $\lambda(t)$ will also change. So, if you move into a territory where this $\lambda(t)$ is big or small, we get more arrivals or less arrivals respectively.

So, SIP does not hold in general. So, it is an independent Bernoulli increment process, but it is not an identically distributed increment process. There are Bernoullis in these small time slots, but the probability of the Bernoulli showing up, I mean, the arrival showing up is not identically distributed across these intervals, it depends on what t you are looking at. Is the definition clear?

Now, this is like, you can argue similarly to the homogeneous case, and still should get a Poisson PMF for the distribution of $N(t)$. It is similar to the limiting argument that you made for the homogeneous case, except now you have to, this probability of having an arrival in any slot is $\lambda(t)\delta$. And you have to do the math a little more carefully.

It is done in your book; the bottom line is that it is not essentially too different from the derivation for the homogeneous process. So, what you get is that, if you look $P(N(t) = n)$, will turn out to be Poisson distributed with parameter $m(t)$. So, it will be $P(N(t) = n) = \frac{m(t)^n e^{-m(t)}}{n!}$, where $m(t) = \int_0^t \lambda(s)ds$.

In the homogeneous case, $m(t)$ is simply λt. And likewise you will get, this also you can get, $P(\tilde{N}(t_1, t_2) = n) = \frac{m(t_1,t_2)^n \ e^{-m(t_1,t_2)}}{n!}$, where $m(t_1, t_2) = \int_{t_1}^{t_2} \lambda(s)ds$. So, the way you prove this, as I said, it is easier to consider a non-uniform breaking of the time interval.

So, the way I have defined it, I have kept the δ to be constant, and the probability of an arrival is a function of $\lambda(t)$, is a function of time. Whereas, you can make this $\lambda(t)\delta$ to be some probability P. Call $\lambda(t)\delta$ to be some P, then, this δ will become, the width of the interval will simply become what? $\frac{P}{\lambda(t)}$, because, $\lambda(t)\delta$ is, I am calling it P.

So, the width of the interval δ will simply become $\frac{P}{\lambda(t)}$. And this is why I am assuming $\lambda(t)$ is bigger than 0. Now, you see why I am assuming $\lambda(t)$ is bigger than 0. Then, it is a sum of Bernoullis. Then you have a whole bunch of Bernoullis in which the width of the intervals are varying inversely as $\lambda(t)$. So, these intervals will be wider when $\lambda(t)$ is smaller, and the width of the interval will be smaller when $\lambda(t)$ is higher, so as to keep the probability of an arrival in any of these intervals the same.

Of course, the total number of arrivals will now be a sum of a bunch of Bernoullis, which is a binomial; so, you can write out a binomial formula and limit these intervals to 0 appropriately and you will get this Poisson distribution. I think it is not a greatly insightful exercise, so perhaps it is not so meaningful to spend the class time on it. I hope what I am talking about is clear. So, this generalises a homogeneous Poisson process, where in the homogeneous case, $\lambda(t)$ is simply equal to λ. Now, that is the small module on non-homogeneous Poisson processes.