Stochastic Modeling and the Theory of Queues
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Module - 3
Lecture - 24
Strong Law for Renewal Processes - Proof

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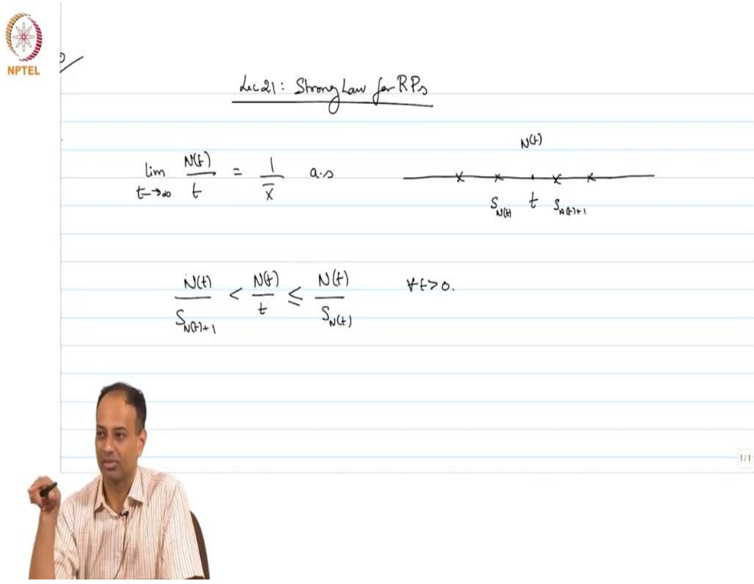

We will continue discussing the strong law for renewal processes. We will discuss the proof of the strong law. So, remember that the strong law says that limit t tending to infinity N t over t is equal to 1 over X bar almost surely. So, the intuitive reason that this works is as follows: You take some t; at its time, there have been N of t arrivals. So, the epoch of the arrival that occurred just before t is just S n t, and the epoch of the arrival that just occurred after the t is S N t plus 1.

So, we are looking at N t over t which can be sandwiched between N t over S N t, where this is less than or equal to, and N t over S N t plus 1, because numerators are all the same; denominators, when the denominator gets smaller, the ratio gets bigger, right? So, that is what I have used. So, this is true for all t. Now, if I send t to infinity, please notice that this N t over S N t is like some n over S n.

As t becomes large, this N t, the number of arrivals, you would expect that it goes, it increases 1 at a time, and it seems intuitively reasonable that N t should go to infinity as t goes to infinity. So, this ratio and this ratio, both behave like n over S n for large n. And S n over n by the strong law of large numbers goes to X bar almost surely. So, you want to argue that n over S n goes to 1 over X bar almost surely.

To formalise this argument, there are 2 ingredients that are required, as 2 lemmas. The first lemma is that N t goes to infinity almost surely, as t goes to infinity. And the second is that, N over S n goes to 1 over X bar almost surely. In fact, the second assertion is true for any continuous function. So, if you take any sequence of random variables X n going to alpha, where alpha is some constant almost surely, then for any continuous function f, f of X n will converge to f of alpha almost surely. So, if you have these 2 lemmas, will be able to prove this strong law. So, let us prove these 2 lemmas.

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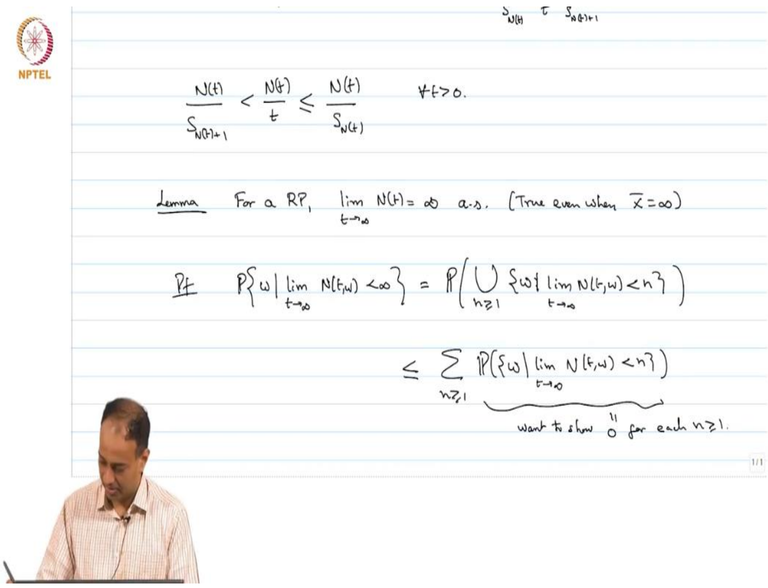

For a renewal process, limit t tending to infinity N of t is equal to infinity almost surely. This is true even when X bar is infinity. So, X bar can be finite or infinite. As t tends to infinity, N of t goes to infinity almost surely. Proof of this is follows: So, you look at those omegas for which limit t tending to infinity N t omega is less than infinity. You want to show that this probability is 0.

See, for those omegas for which limit t tending to infinity N t omega is less than infinity, there must exist an N for which limit t tending to infinity N t omega is less than n, because the limit is finite, we are saying. So, that limit must be something less than, some n; so, there must exist such an n. There exists translates to a union, so, that just becomes probability union n greater than or equal to 1, omega such that limit t tending to infinity N t omega less than n.

And this of course, by the union bound is less than or equal to sum over n greater than or equal to 1, probability of omega such that limit t tending to infinity N t omega less than n. So, now, I want to show that each of these things is 0. This is equal to 0 for each n greater than or equal to 1. So, if each of these terms is 0, then the sum is 0. Then I will be done. So, you fix an n and consider this probability.

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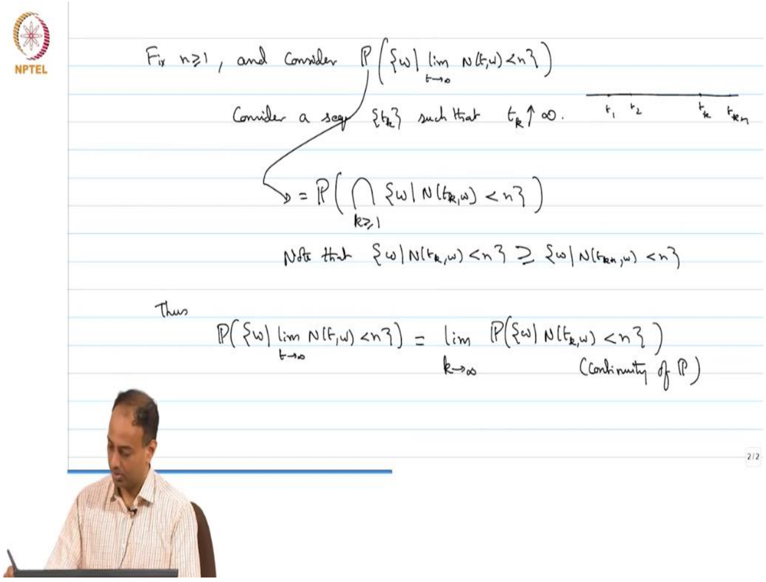

So, you fix n and consider this probability out here; omega such that limit t tending to infinity N t omega less than n. Now, this is a limit that is inside this probability. We would like to bring this out. And bringing the limit out, you carry out using a continuity of probability argument. So, the way you do this is, you consider a sequence t k such that t k is increasing and it increases to infinity.

You consider any sequence t k such that 0 is less than t 1 is less than t 2 and so on, and this t k, the sequence t k goes off to infinity, increases to infinity. So, I can write this probability as; so, this guy, let me just write it like this. So, this will just become intersection over k greater than or equal to 1 omega; so, this will be equal to this; N of t k, omega less than n. So, let me tell you what I have done.

So, I considered this sequence t 1, t 2 ..., some t k, which is going off to infinity. Now, I am telling; so, this is the event I want to consider, is the, those omegas for which limit N t omega is less than n, limit t tending to infinity N t omega is less than n, which means that for every t k, N t k omega must be less than n. See, if in the limit, N t omega is less than n, then N t k omega should be less than n; for if not, your limit will be bigger than n; that is, it cannot hold.

Conversely, if for every t k this is true, N t k omega is less than n, for every t k; and remember, this t k is going to infinity; then this limit will be less than n. So, you can easily show that this event is same as this intersection. You can show the equality between this event and what is inside here. It is this countable intersection. Now, it is clear that these kind of sets are nested decreasing sets.

So, what I mean is that; let me write it this way; note that omegas for which N t k omega less than n is, it contains the omegas for which N t k plus 1 omega is less than n. Because, if your omega is such that N t k plus 1, omega is less than n, at t k plus 1, you have less than n arrivals; then the number of arrivals at t k must be less than n. So, this event implies this event, which means that these sets are nested decreasing.

And you have a countable intersection over nested decreasing sets, and by continuity of probability, this is equal to the limit of the kth set, probability of the kth set. So, you can say, thus probability of; you go back to what you want; limit t tending to infinity N t omega less than n. So, you are fixing an n, like here; is equal to this guy, which is equal to limit k tending to infinity probability of those omegas for which N t k omega less than n; this is by continuity of probability. So, maybe I should recall that for you.

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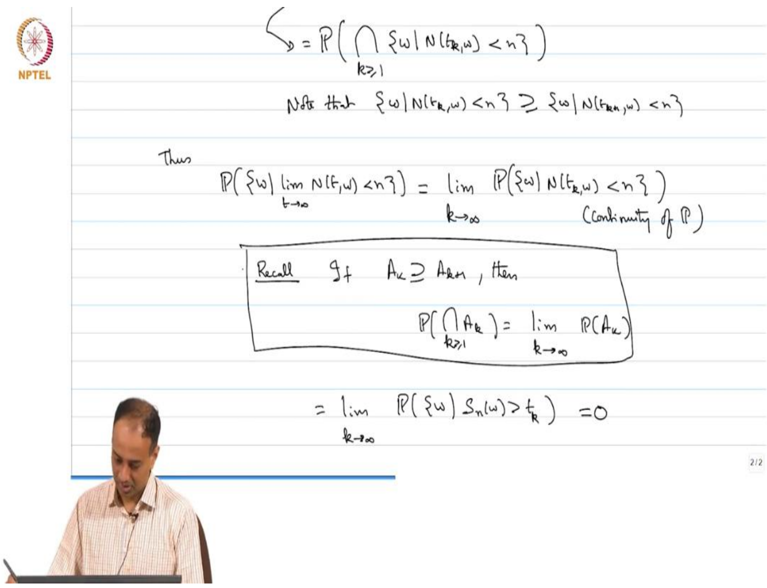

Recall: If some events A K are nested decreasing, then the probability of intersection k A k is equal to limit; did I write it correctly? This is correct, right? This is what I am using. This you must know; this you would have studied in probability course. This is the fundamental property of probability measures. So, you use that. So, this is equal to this. Now, we are in business.

So, if you look at this; so, this recall, I will put in a box, so that it does not break your proof. So, I will continue here. So, this is equal to limit k tending to infinity probability of omega such that S n of omega greater than t k. Why is that? N t k less than n is same as S n greater than N t k. This equivalence we proved, remember? What we proved was that N t greater than or equal to n is same as S n less than or equal to t.

We are just taking the complement of it. Now, this is equal to 0. Why? Because t k is going to infinity. Is the complimentary CDF. So, this t k is going to infinity and S n is some random variable. Probability S n greater than t k, as this t k goes to infinity, has to be 0, because this is a property of the complimentary CDF of any random variable. See, this S n is a legitimate random variable, so, it is finite with probability 1. So, its CCDF has to go to 0.

As k goes to infinity; if you want, you can write one more step; you can write limit t tending to infinity S n greater than t; and that will be 0. So, what have we shown? We have shown that this guy is 0, for each n. This is true for all n greater than or equal to 1. So, this guy is equal to 0 for all n greater than or equal to 1; therefore, these terms, each of these terms is 0. This is what I wanted to prove. So, I am done with the proof.

So, it requires a continuity of probability argument, which is missing in your book. You can just fill this in. The taking the limit out of the probability requires this nested construction; without that, it is not rigorously correct; I thought I will just point it out. With me? So, that is your first lemma that says that N t increases to infinity as t goes to infinity; of course, arrivals come one at a time, so, N t increases through all the positive integers, does not make any jumps between 2 integers, because no 2 arrivals can come at the same time.

So, it goes through all the positive integers and it goes off to infinity as t goes to infinity. So, essentially, if you look at this ratio, N t over t, it is sandwiched between N t over S N t and N t over S N t plus 1. And therefore, as t goes to infinity, this ratio goes like 1 over S 1; 2 over S

2; 3 over S 3; and so on. And, and this guy goes like 1 over S 2; 2 over S 3; and so on. So, there is just one more ingredient that is missing, which is sort of a continuity property.

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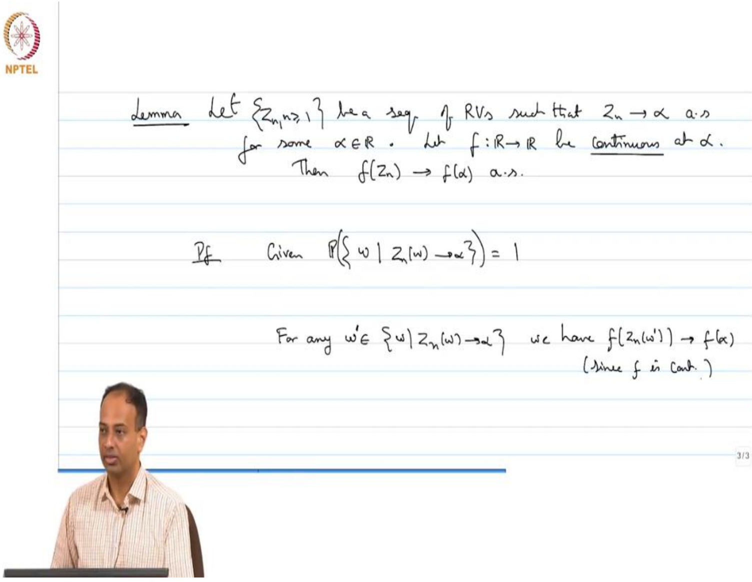

If Z n is a sequence of random variables such that Z n converges to alpha almost surely for some alpha in R. Let me say, let Z n be a sequence; then say, let f be continuous at alpha; then f of Z n converges to f of alpha almost surely. So, if you have a sequence of random variables converging to a number almost surely, then any continuous transformation of the sequence of random variables will converge to the corresponding f of alpha almost surely.

Key issue is here, f should be continuous. So, the key issue here is; so, the proof is fairly easy. So, you are given this, right? Given that the probability of those omegas for which X n of omega converges to alpha equal to 1. So, you know that X n converges to alpha almost surely. So, the probability of those omegas where X n of omega converges to alpha is equal to 1. This is by definition of almost sure convergence.

Now, for all omegas in this set, f X n of omega will converge to f of alpha. Why? Continuity. So, for any omega in this; **"Professor - student conversation starts"** Yes. This should be a Z n, right? That is all; I just put Z n because I did not want to, X n's are our renewal interrenewal times, right? There is no big deal otherwise. **"Professor - student conversation ends"** For any omega prime in this set, we have f of Z n of omega prime converges to f of alpha, f is continuous at alpha. So, what does this mean?

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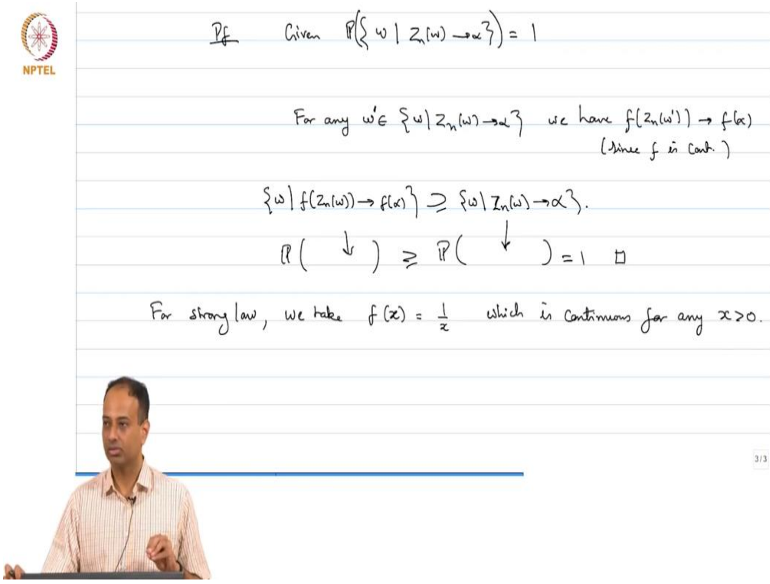

So, if you look at those omegas for which f of Z n of omega converge to f of alpha. If you look at those omegas for which f of Z n of omega converge to f of alpha, this will contain those omegas for which Z n of omega converges to alpha. I have just proven this, right? For every omega contained in this set, I have f Z n of omega prime will converge to f of alpha. So, every omega here is a member here.

Every omega for which Z n of omega converges to alpha, for those omegas, f of Z n of omega will converge to f of alpha. So, this is a bigger set than this. Have you written it in the correct direction? And so, we are saying that B is contained in A; so, probability of B has to be greater than or equal to probability of A. So, what can I say? So, I can write, so, probability of all that is greater than or equal to probability of all that.

But what is the probability of all that? 1, because Z n converges to alpha on a set of probability 1. So, probability of f of Z n of omega converging to f of alpha is greater than or equal to 1; but probability cannot be greater than 1, so, it has to be equal to 1. So, I will explain the containment once more. You take, see, this is the set of omegas for which Z n of omega converges to alpha. This is a set of probability 1; we are given that, right?

This is some subset of the sample space; it may not be all of the sample space, but it has all the probability. You fix any omega prime you want in this. Then Z n of omega prime is a sequence of real numbers converging to alpha. Now, f is continuous at alpha; therefore, by definition of continuous functions, you have, for that sequence f of Z n of omega prime, you will have convergence to f of alpha.

So, for every omega prime for which this convergence holds, this convergence necessary holds. So, if you look at this set, any omega here is a member of this set. Of course, this set could be bigger, it could have a few other omegas; so, we will get probability is greater than or equal to 1. Probability cannot be greater than 1, so, it is equal to 1; simple as that. So, in our case, in our setting, for strong law, we take f of X is equal to 1 by x, which is continuous for any X positive. So, now we can prove the strong law, but we can prove it for X bar being finite.

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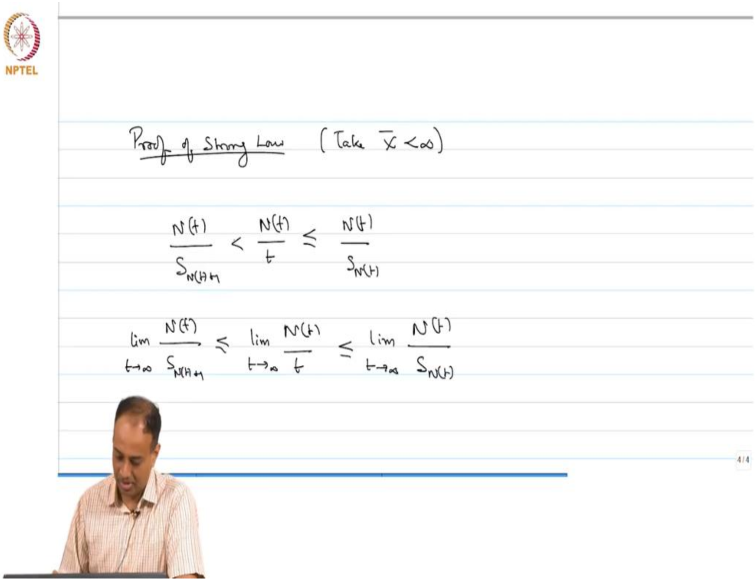

For finite X bar we already have this, right? N t over t is sandwiched between N t over S N t and N t over S N t plus 1. So, I can say that limit t tending to infinity, N t over t less than or equal to limit t tending to infinity N t over S N t. And here again write limit t tending to infinity N t over S N t plus 1. So, this less than will become less than or equal to when I take limit. It is like 1 over N is always greater than 0, but limit 1 over N is 0. So, this strict inequality will become a non-strict inequality in the limit. I hope you understand that. So, this much you will agree, right? This is just sandwiching property.

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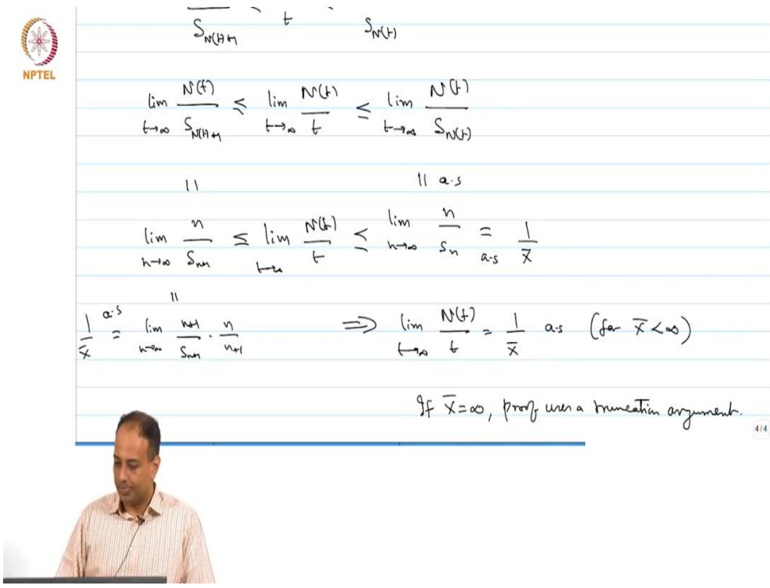

Now, as limit t tends to infinity, so, this is the same as, this guy is the same as limit n tends to infinity n over S n by the lemma that we proved. N t increases 1 at a time and it increases to infinity almost surely. So, this equality is true almost surely. And likewise, this will be limit N tending to infinity n over S n plus 1. Now, by the continuous mapping theorem, this guy is equal to, what is this equal to? 1 over X bar. Why?

It is a reciprocal mapping is continuous, and X bar I have assumed is, X bar is finite I have assumed. If X bar is not finite, then I cannot invoke continuity. So, if X bar is finite, this limit will be equal to, this guy will be equal to almost surely equal to 1 over X bar. This n over S n plus 1 also, you can write it as n plus 1 over S n plus 1 times n over n plus 1 and manipulate. So, this guy, you can write it as limit n tending to infinity n plus 1 over S n plus 1 times n plus 1 over N.

I think I made a mistake; Correct no? So, this will of course go to; same logic; it will go to 1 over X bar; this will go to 1. So, this will also be equal to 1 over X bar almost surely. So, what have I shown? So, I have shown this, limit t tending to infinity N t over t is equal to 1 over X bar almost surely. I have shown this for X bar finite; but the result I also claimed is true for X bar being infinite. So, that I will not go into in great detail.

I will just mention that, if X bar is infinity, proof uses a truncation argument. Have you seen truncation arguments before? So, you basically truncate the random variable. So, you are saying X bar is infinity, meaning that these X i's are random variables which are finite with probability 1, but the expected value is infinite. There exist such random variables; I suppose you know.

If you take a one-sided Cauchy distribution for example, it is finite probability 1, but expected value is infinite. In those cases, you define a new renewal process with interarrival times X i tilde equal to max of X i, B, for some B, very big B. So, you chop off the random variable at B, if it becomes too large. And because you are chopping off, this random variable will now have finite X bar; and for that, the strong law will hold, as we have proved it; and then, you have to take B to infinity.

You have to do this in the right order; you have to be a little careful, but it can be done. There are many theorems in probability which uses truncation argument. Even if you want to prove weak law for unbounded variance case; so, for bounded various case, you can prove weak law using Chebyshev's inequality; for unbounded variance case, you use a truncation argument for example. So, this is a standard trick in proving many results. But this I will not spend class time on. I think there is an exercise in your book to actually guide you through this steps of strong law.