Stochastic Modeling and the Theory of Queues
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Lecture –33
The Renewal Equation

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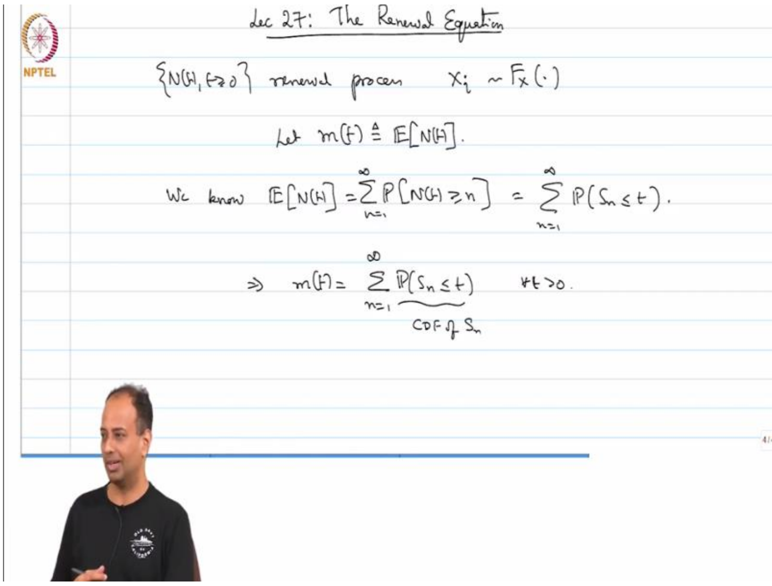

In this module, we will talk about the renewal equation which is an integral equation that relates the distribution of the inter arrival times to expectation of N t. So, as usual let N t be a renewal process with inter arrival distribution from F x CDF is given and let m of t = expectation of N of t. So, N t is a random variable for each t. Its expected value is sum function m of t.

So, I am going to relate m of t to the CDF and that will give us a integral equation called the renewal equation that is what we are going to direct. So, we know this N t is a non negative random variable. So, we know that expectation of N t is equal to sum over n probability of because N t is a non negative integer value random variable, but what is probability of N t greater than or equal to n?

It is equal to probability of S n less than or equal to t. So, this is true for all t therefore m of t is for all t greater than 0. So, in principle see I am given the CDF of x i's and what is this? This is the CDF of S n. So, given the CDF of x i's I can find the CDF of S n for each n by what convolution because they are iid random variables because it is just a n-fold convolution of the distribution of x with itself.

So, the CDF of S n is easy to find out and then you sum for each n n = 1 to infinity you take this sum and that should be your m t. So, in principle this is very easy, but in practice this is very cumbersome you have to do n-fold convolution and take this infinite sum for larger and larger n this is it can be a big headache. So, I turn out I can simplify this a little more.

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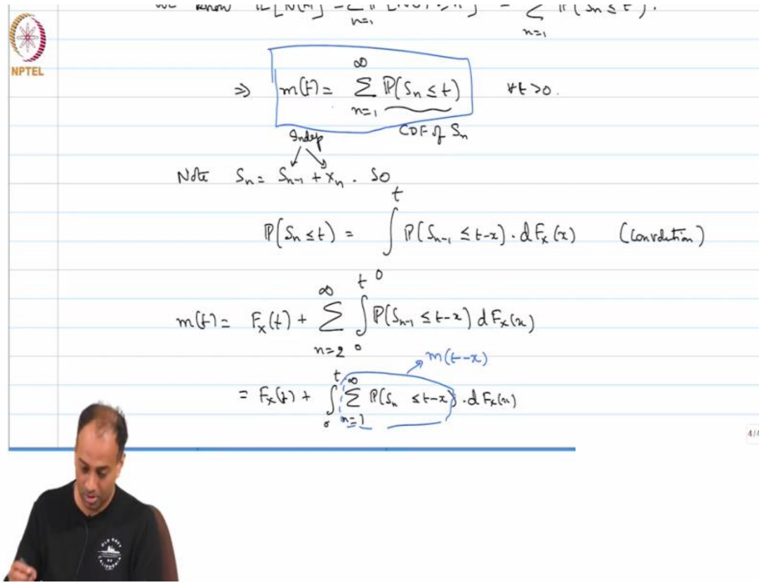

So, note that S n = S n – 1 + x n. So, I can write this probability that is basically the probability that S n – 1 is less than or equal to t – x and probability that x is less than or equal to x n is less than or equal to little x and I can run x over 0 to t. So, this is again because independence. Basically I am writing the convolution between S n – 1 and x n this is nothing, but the convolution except I am writing of the CDF.

Since these guys are independent x n and S n – 1 independent. So, I have if I go back to this substituting back I have m of t is equal to I have n = 1 I have probability that S 1 less than or equal to t which is just the CDF of x. So, I have F x of t + sum over n = 2 to infinity integral 0 to t probability S n – 1 less than or equal to t – x d F x of x. What have I done? I have just substituted my integral this convolution integral into that summation.

I pulled out the term n = 1 which is simply the CDF of x 1 which is F x. The first term is nothing, but probability S 1 less than or equal to t which is just F x of t then I have 2 to infinity of all this where I put in the summation or put in the integration. Now, I am going to interchange the order of this summation and integral and all that is legitimate because the inter (()) (06:55) is non negative.

So, what does that leave me with that is equal to plus integral 0 to t. I am going to pull the sum in n = 2 to infinity probability that S n – 1 is less than or equal to t – x d F x. What I will just interchange sum and integral and now note that I can make this sum you would not mind me doing this if I put n = 1 to infinity I can just write this as S n that you will not mind. I am just re-indexing.

Now, look at what this guy is what is this guy? This guy is just m of t – x because m of t is in this blue box. So, what do I get? So, I get.

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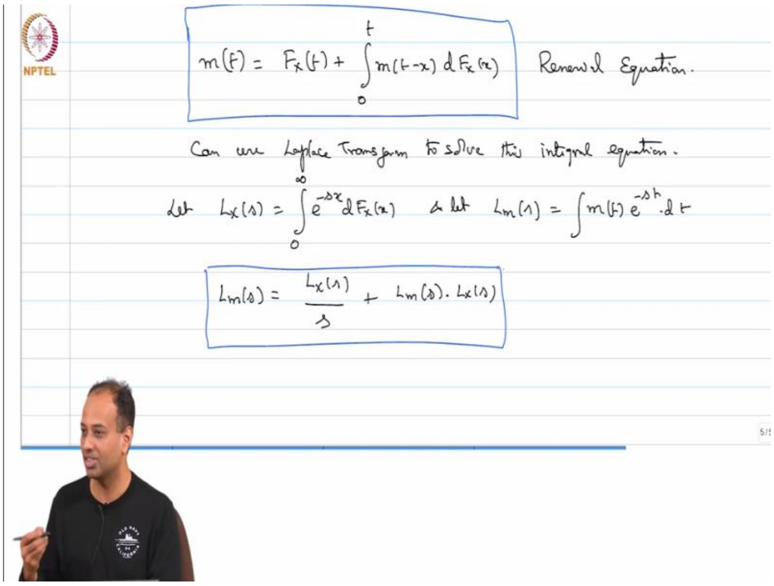

If I just write it again m of t = F x t + integral 0 to t m of t – x d F x of x. Now this equation is an integral equation known as the renewal equation. It is just an integral equation that relates F x the CDF x which is known to us to m of t which we want to find out. See in principle in this equation F x t is known. So in principle solve for m of t by solving this integral equation. So, when you have an integral equation like this.

So, when you have certain integral equations or differential equations which are linear what is the best way to solve it by taking some transform or the other. You typically take some moment generating function Laplace Transform whatever and so this is a linear integral equation. So, we can solve it using Laplace Transform or moment generating function which is all the same thing which is good approach to solve this equation.

So, solving in this t domain is a bit tricky because you are solving integral equation directly maybe a headache. So, you guys familiar with Laplace Transform right. Laplace Transform and moment generating function are basically the same thing except for a minus sign. Moment generating function has a minus sign Laplace Transform does not have a minus sign that is the only difference.

Let us just use Laplace Transform to solve this linear integral equation. So, let L x of s be the Laplace Transform of. So, you can just write the renewal equation as follows you will get L m of s is equal to you have the CDF here which is the integral of the PDF. So, when you integrate in the t domain what happens in Laplace domain the differentiating multiply by s so integrate you divide by s.

So, you get L x of s over s plus this guy is just a convolution. So, you should get what? In the Laplace domain convolution becomes a product. So, you will get L m of s times L x of s. So, this is you guys are experts in Laplace Transform signals and system so you get this. This is exactly the renewal equation in transform domain I have not said something new I have just taken transform.

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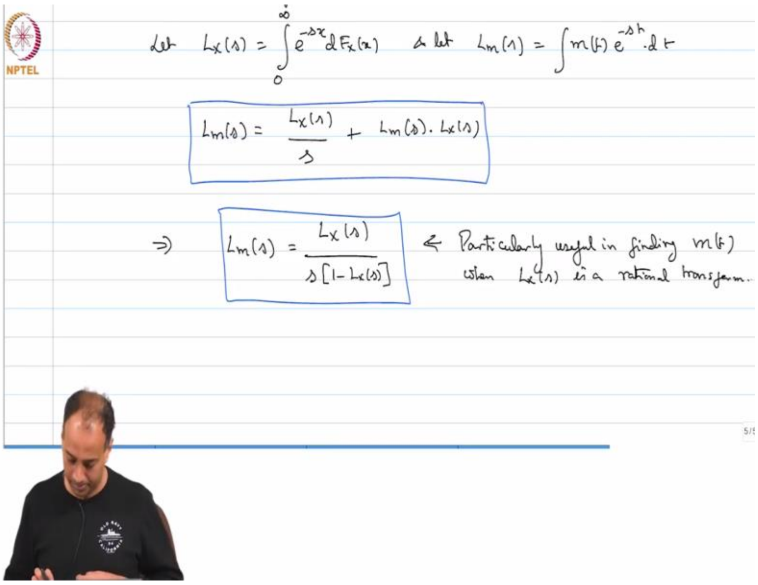

From this I can get which is the Laplace Transform of m t is nothing, but L x of s over s times 1 – L x of s. So, what have I done? So, I want to really solve for m t. I can solve for m t using the integral equation which is the renewal equation and since it is a linear equation I have taken Laplace Transforms and I have expressed the Laplace Transform of m t which is nothing, but L m of s in terms of L x of s is known to me because given the distribution of x i I can take its Laplace Transform and put L x in to this equation and get L m of s.

So, if I have give you that x is distributor like so and so you put the transform here. You will get the transform of L m of s and you invert back to get m of t. You guys are expert at this in fact with L x of s is rational transform if it is looking something like some polynomial over some other polynomial in s you can just this back you will get some other ratio of two polynomials which you can invert using all this partial fraction tricks that you are familiar with.

So, now you can do many number of exercises in this I will tell you the L x of x is so sorry x i are (()) (14:56) whatever. You put in here and calculate L m of s then invert back. So, this is always true, but it is particularly useful this guy is in finding m t as a function of t then L x of s is a rational transform. It is always true, but all I am saying is if L x of s is rational so with L m of s be rational.

So then you can invert back and get some explicit answer putting partial fraction find it out you will get some explicit answer for m of t. Actually it turns out that you can use this to prove elementary renewal theorem also. There is an approach to proving m of t over t goes to 1 over x bar even using this the transform. We use the truncation argument even through this route you can prove elementary renewal theorem. Okay I stop here.