Stochastic Modeling and the Theory of Queues
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Lecture –34
The Renewal Equation (Contd.)

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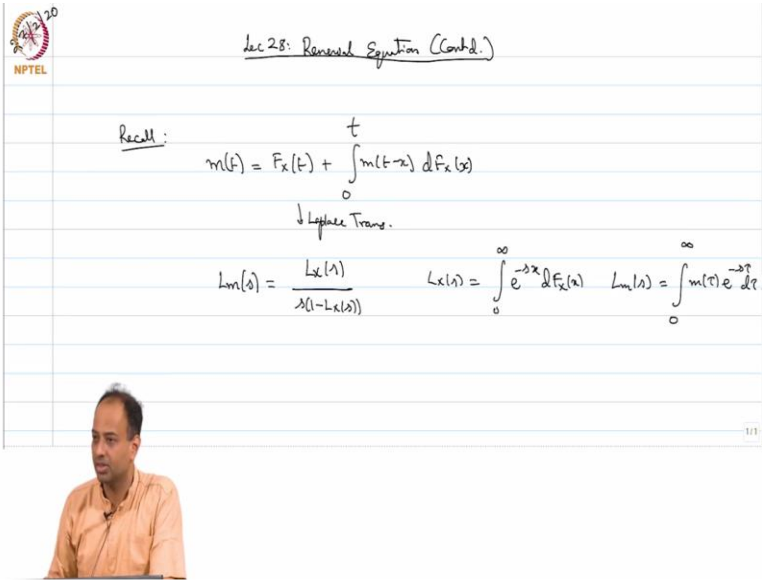

Welcome back. Yesterday we proof the elementary renewal theorem which involves characterizing m of t which is the expectation of N of t. We wrote down the renewal equation which is this m of t = F x of t + integral 0 to t m of t – x d F x of x this is called the renewal equation and you take Laplace Transform you get L m of s = L x of s over s times 1 – L x of s.

So, here L x of s is just the Laplace Transform of the underlying distribution x so this is e power s x d F x of s and L m is the Laplace Transform of m of tau. L x of s is known to you because the renewal process inter arrival distribution is given. So, you can calculate L x of s and you can put L x of s into this equation to get L m of s and you can invert back to get m of t.

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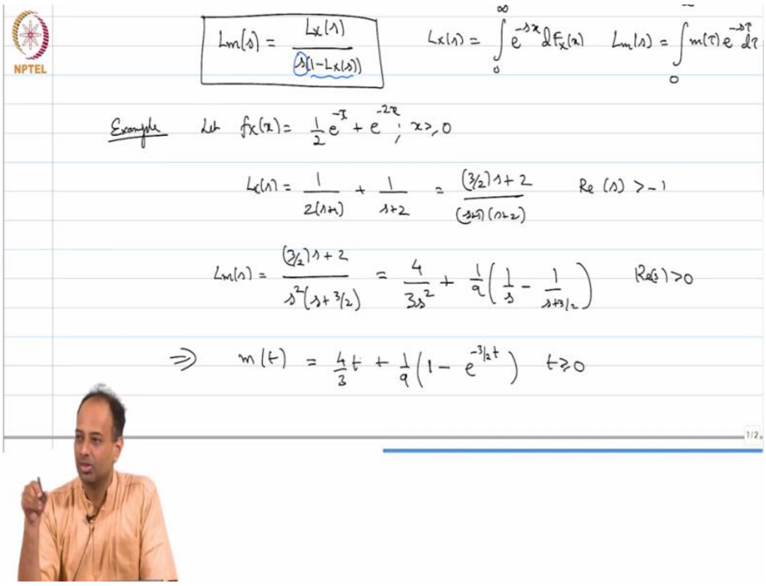

So, let us just do an example let f x of x = 1 / 2 e to the – x + e to the – 2 x for x greater than or equal to 0. So, this is my distribution. So, my renewal process has inter arrival distribution given by this PDF. So, you can verify that this is a valid PDF and all that. For this you can calculate L x of s. L x of s is simply what is the transform of so if you have 1 over 2 s + 1 is not it correct + what happens here 1 over s + 2.

So, this will just work out to 3 / 2 s + 2 over and this is for real path of s bigger than – 1. So, now you plug this into L m of s you get L m of s is equal to you use that equation. You get 3.2 s + 2 over s square times s + 3 over 2. So, this is just an exercise in putting this L x and manipulating. So, now we have to invert back. So, this is the Laplace Transform of m of t. How do you invert this?

You put into partial fractions so you have to write this as in this case you have to write it as a over s + 3 over 2 + b over s + c over s square find a, b, c. So, you have just two poles you have origin there is a double pole and – 3 over 2 there is a pole. So, you have to write this as some a over s + 3 over 2 + b over s square + c over s. So, you just work this out it comes to + 1 over 9 times 1 over s – 1 over s + 3 over 2.

This is for real part of s greater than 0. So you can write down what is m of t? See, if you have 1 over s that corresponds to a constant for positive t it is like 1 for positive t 1 over s square is t is not it is like a (()) (05:22) you know all this very well. So, this m of t will be 4 over 3 t + 1 over 9 1 – e to the - that correct. This is m of t for any t greater than 0 an explicit characterization of expectation of N of t for this renewal process.

Of course, even easier (()) (05:55) if you put F x of x is equal to lambda e power – lambda x which is Poisson's process. You should get back lambda t m of t should be lambda t you can check that it is an easier exercise than this even. So, it turns out that there is a term that goes linearly in t there is a constant term and then this is a decaying term throughout this looks like.

It of course it worked out very easily because this was a rational I mean this distribution had a rational transform. This is the easiest case I mean if L x of s is rational transform L m of s is also rational transform so you can do this trick to find m of t. In general this can be more complicated this is always correct, but it may not be practically so convenient to do it. And of course you can verify elementary renewal theorem very easily.

You can find the expectation of x of this distribution and there is that linear term this term will turn out to be 1 over x bar. You can just verify all that. Actually this is what you saw in this example is pretty standard for any rational transform you will get a 1 over s square pole under 1 over s pole plus other poles which are always on the left half plane which corresponds to decaying terms in t.

The poles on the left half plane you will have decaying poles. So, if you just look at this guy if you look at this term L m of s there is clearly a 1 over s term so there is definitely at least one pole at 0, but there is also another pole coming from 1 – L x of s because if you look at L x of 0 what is L x of 0? L x of 0 is just 1 L x of expectation of e power s x so L x of 0 is 1 so 1 – L x of 0 will be 0.

So, this term 1 – L x of s will contribute another pole at 0. There is already a pole at 0 coming from 1 over s and there will be another pole coming from 1 – L x of s always. So, there is your guarantee to have 1 over s square term that is all that I am saying which corresponds to this t sort of a behavior. This t term that came out so this will always come then there will also be 1 over s term whose residue you have to find out.

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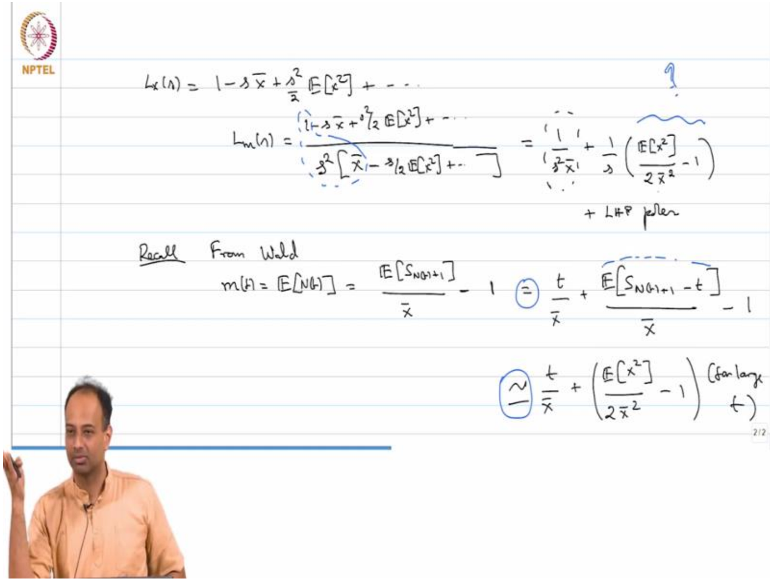

So, normally this you know so L x of s basically looks like 1 – s so 1 – s x bar + s square over 2 expectation of x square dot, dot, dot. So, you put this into L m of s will then be 1 – s x bar + s square upon 2 expectation of x square plus dot, dot, dot over you will get s times 1 – L x of s so 1 – L x of s will be there will be there will be an s square term coming out then you will have x bar – s over 2 expectation of x square plus dot, dot, dot.

So, if you just look at the leading term of this if you just look at this bit you will get so the 1 over s square term is clear it will be 1 over s square x bar in the transform domain which in the time domain will become t upon x bar which is very happy for us because we want m over t m t over t to go to 1 over x bar. So, you will always have this situation where m of t has a leading term which looks like t over x bar and that corresponds to 1 over s square term.

And then there will also be 1 over s term which is like the residue at 1 over s you have to calculate you can either do that so what is the 1 over s residue so 1 over s times what is the question? You will get a term that looks like this. So, you can either work this out from the series or you can look at it in another way which I will tell you in a minute.

You will get something like this plus left half plane poles let me just write it like that. The other terms will be decaying see all other poles of L m of s will be on the left half plane that is because you can in fact show the 1 – L x of s absolute value of L x of s is less than or equal to 1 you can prove that because it is a distribution.

So, this L m of s will not have any further poles on the right half plane or on the vertical axis if all the other poles will be on the left half plane which means that in time they will decay just like we had in the example. So, basically the terms that matter so you have a second order pole at 0 which will corresponds to a residue of 1 over s square a term corresponding to 1 over s square and the term corresponding to 1 over s plus the whole bunch of decaying terms.

This is a standard thing and if you are wondering where this come from? So, it is clear where this 1 over x bar comes from and if you are wondering where this comes from? You can algebraically see it by getting the working out basically this is the residue at this is the 1 over s residue we can calculate that either from the series or by other means, but the most straightforward way to see where this comes from is to recall the other equation which had from Wald you know this expectation of N t + 1.

I had an equation expectation of can you go look at that equal to expectation of S N t + 1 / x bar – I think I will have one more time – 1. This is from Wald so this will just work out to be t upon x bar + expectation of sorry S N t + 1 – t correct what have I done? I have just added and subtracted t upon x bar. See because t upon x bar term I am getting anyway over here. This is the t upon x bar term.

And I am trying to explain where this equation mark comes from this the other term 1 over s the term corresponding to 1 over s. So, that is the same as this. So, what is this term? This is just expectation of y t this is the expectation of the residual time. The (()) (13:37) average of the residual time, we have not yet calculated that, but we have calculated the time average and we have a strong sense that if they must be equal.

So, I am just giving you an intuitive reason. So, this guy must be t over x bar plus this guy must asymptotically look like what? Expectation of x square / 2 x bar square there is also 1 over x bar here – 1 except this relationship is only for large t it is not true for all t this is for large t. See the time average that we worked out is equal to this, but will be equal to (()) (14:40) average expectation of y of t for large t.

For t finite there will be other terms so this is not equal so I can say this is approximately equal. Why am I putting approximately equal? Because there will be other decaying terms.

See this equality is exact t over x bar plus expectation of all that over x bar – 1 is correct that is exact, but in writing this I have made some approximation. So, I am just trying to argue where this guy comes from this blue question mark comes from?

It is roughly coming from the expected residual time which has this term plus some transient terms which go down to 0 exponentially fast in the case of a rational transform. So, that is the intuitive justification so again this equality is exact this is not exact it is true only for large t because I have put asymptotic expected residual life to be the expectation of residual life for finite t which is not correct.

You will get some transient terms in general and this transient terms you have to work out by looking at the left half plane poles for rational transform. We could explicitly get those terms the transient terms in the example. So, this Wald regression is just to justify where this term comes from., So this question mark that I put where does it come from? That comes from the expected residual life which is roughly like this expected x square over 2 x bar, but there were also transient terms for finite t that is all that I am saying.

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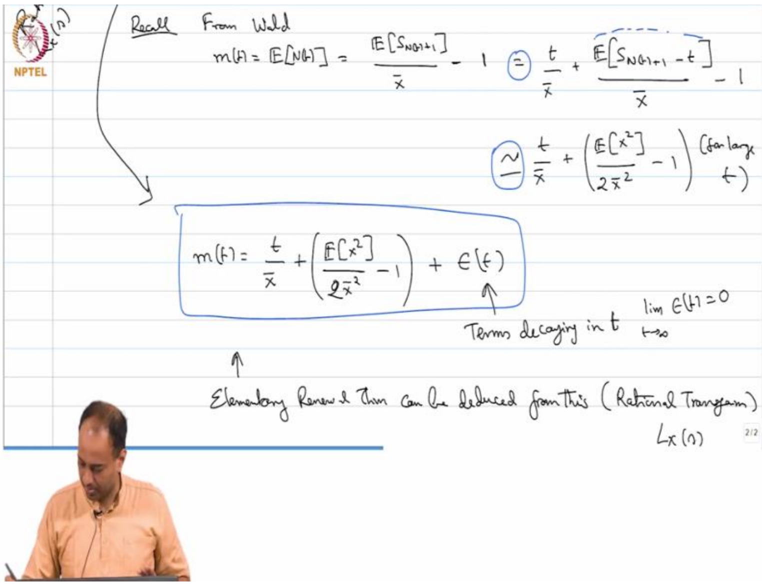

So, again get back to this L m of s you can transform back from here you can transform back and write m of t is equal to t upon x bar + expectation of x square over 2 x bar square – 1 plus sum epsilon t. These are terms decaying t. So, this is the inverse transform corresponding to left half plane poles. You see what I have done here 1 over s square x bar inverse transform is t upon x bar.

And 1 over s times this constant will just be the constant which is this guy plus they will be all this left half plane poles which in the example was like 1 over (()) (17:17) over 2 which turned out to be some decaying term. You will have a whole bunch of decaying terms depending on the particular example you are looking at I am just capturing it as epsilon t that is all.

These terms will go to 0. So, what I am saying is that limit t tending to infinity epsilon t will be 0. So, this is what m of t looks like, but of course this entire analysis is valid only for this rational transforms. Otherwise, if you have e power – s tau or something like that in the transform it is not rational then a simple structure like this would not emerge because those e power – s terms or whatever irrational terms you have in the transform will keep coming.

I mean the transform of L m of s is always valid it is just that it will not be so easy for you to invert back and get this form that is all. So, from here so this is only for I mean this whole analysis is only for rational transform everywhere this is this whole analysis is for that. So, here you already see that we have gotten here using the renewal equation and taking its transform and analyzing the transform.

So, if you take m t over t it is clear that m t over t is going to 1 over x bar because this epsilon t terms are dying anyway and you have a constant over t which will go to 0. So, elementary renewal theorem is clearly obvious from this can be deduced for again for rational transforms, but of course we proved elementary renewal theorem in full generality using a truncation argument.

I am just saying that if you had a rational transform you have a very explicit characterization of expectation of N t from which the elementary renewal theorem is obviously corollary, but this is not a very general proof so this is for rational transform L x of s. There is another very important result that actually comes from that you can actually deduce from this equation m of t.

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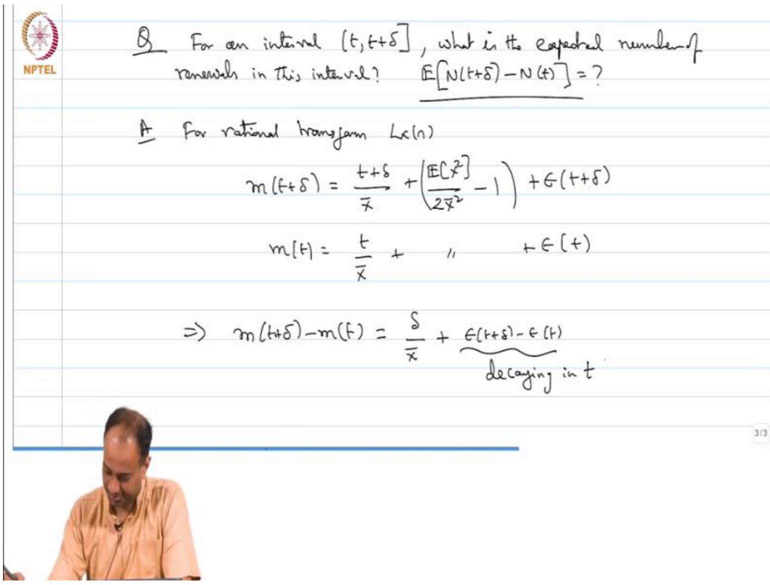

Question for an interval you fix this interval t, t + delta what is the expected number of renewals in this interval? So, I want to look at this. How many renewals happen in this interval t, t + delta? So this is answered by a theorem known as Blackwell theorem that is what we are heading to that theorem answers precisely this question. Again for a rational transform the answer I know.

It is for rational transform L x of s I have m of t + delta I know m of t + delta is simply t + delta over x bar you look at that term t + delta over x bar + expectation of x square over 2 x bar square – 1 + epsilon t + delta this is the decaying term and m of t is simply t over x bar plus the same term plus epsilon t. What have I written I have just copied what I know for the m of t?

M of t is just the expectation of N of t. Therefore, this guy m of t + delta – m of t which is what you want is equal to what? Delta over x bar + this constant term cancels plus this epsilon t + delta – epsilon t. Anyway this whole thing is decaying in t. So, for large t expectation of N t + delta – expectation of N t behaves like delta over x bar. So, this is Blackwell theorem. This is a very important theorem.

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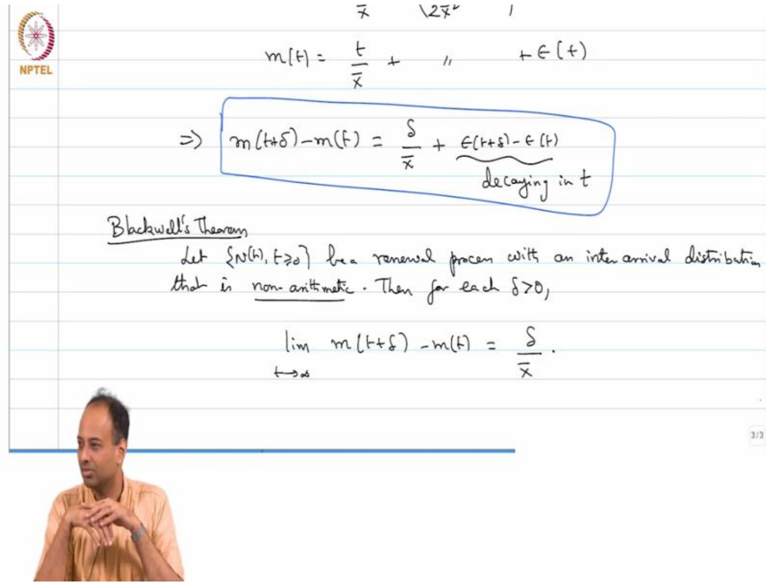

Let me put that down let me state the theorem first let N t be a renewal process with inter arrival distribution that is non-arithmetic. I will tell you what non arithmetic is then for each delta greater than 0 limit t tends to infinity is equal to delta over x bar. So, I am basically looking at m of t + delta – m of t which is the expected number of renewals between t, t and plus delta.

So, I derived all this if you look at this equation this guy I have send t to infinity I will get delta over x bar that is all I am saying, but I am saying this as a theorem for non arithmetic distributions. First of all I have not told you what arithmetic distribution is and the second thing is that does this derivation above the statement of the theorem actually prove the theorem.

Is it a proof of the theorem or no? It is not a proof of the theorem because it holds only for rational transforms and it turns out that rational transform once you have a rational transform the distribution will always be non arithmetic. So, the above derivation is sort of an appetizer I mean it sort of tells you why the theorem is true, but it is not a proof of the theorem and in fact Blackwell theorem proof is very non trivial.

We will not do the full proof it is not a good use last time actually even (()) (26:29) volume 2 for the proof. If I remember correctly the proof takes slightly circle this approach you have to start with it uses renewal equation and you start with F x the CDF which is first for proof it for simple random variables if you remember what simple variables are from your probability course then you generalize, but I will not do that.

But I will tell you what a non arithmetic distribution is. See I am looking at some t, t + delta I mean delta does not have to be small this is true for any delta greater than 0, but you are looking at some t and t + delta. Now, suppose I have a renewal process in which the inter arrival distributions let us say are always integers.

Renewal occurs at some integer let us say 1 or 2 or 3 or something like that. For example if your inter arrival distribution takes only integer values then all your subsequent renewals will take place at integer times. So, if your inter arrival t, t + delta does not include integer it does not include an integer point then for that renewal process which renews only at integer times then expectation of N t, t + delta – N of t will be 0.

So, clearly Blackwell theorem cannot hold in full generality. An example of an arithmetic distribution is what I just gave you only it only renews a multiple of integers. A non arithmetic distribution is basically the opposite of an arithmetic distribution. Now what is the arithmetic distribution? An arithmetic distribution it corresponds to a random variable which takes values with non negative probability which are just some multiples of some parameter kappa.

So, renewals will happen at kappa or integer multiples of kappa that kind of a situation Blackwell theorem cannot hold because what happens this t, t + delta does not have a multiple of kappa in it.

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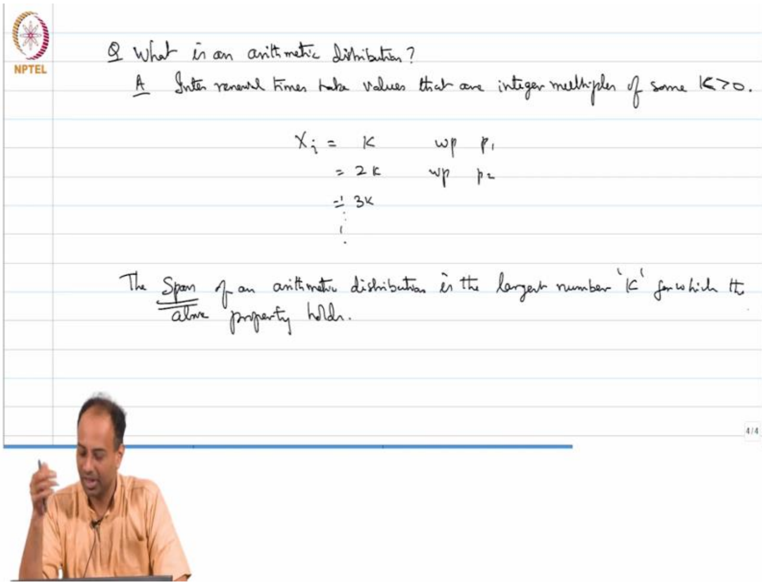

So, what is an arithmetic distribution? Answer inter renewal times take values that are integer multiples of some kappa greater than 0. So, it will be like x i will be kappa with parameter with probability p 1 = 2 kappa with probability p 2 dot, dot, dot. It take values only kappa to kappa 3 kappa etcetera for some kappa greater than 0 kappa can be any real number positive real number.

This sort of a distribution is known as an arithmetic distribution. So, please keep in mind that arithmetic distribution is not the same as a discrete distribution. Discrete distribution can take it not only that it takes only integer multiples of particular kappa. So, we will take x i = 1 with probability p and root 2 with probability 1 – p is that an arithmetic distribution? Not, at all. It is a discrete distribution, but not an arithmetic distribution is that clear.

So, Blackwell's theorem concerns itself with the case when the distribution is not arithmetic. With arithmetic it has no chance of holding because this t, t + delta may not include a kappa multiple in it and therefore it falls apart, but the distribution is not arithmetic so even if it is like 1 and root 2 or something like that then asymptotically the expected number of renewals in t, t + delta is delta / x bar that is what Blackwell's theorem says.

There is also Blackwell's theorem for arithmetic distributions. For arithmetic distributions what you want is that this delta should be some multiple of this kappa. So, let me talk about span. The span of an arithmetic distribution is the largest number kappa maybe I should call this I am reusing kappa, but (()) (32:39) suppose will be clear for which the above property holds.

See what do I mean by the span? See, the example I have given the x i take values which are multiples of kappa, but if it takes values which are multiples of kappa it also takes values which are multiples of kappa / 2. So, you basically look for the larger such kappa for which the inter arrival times are multiples of that number then you call that the span.

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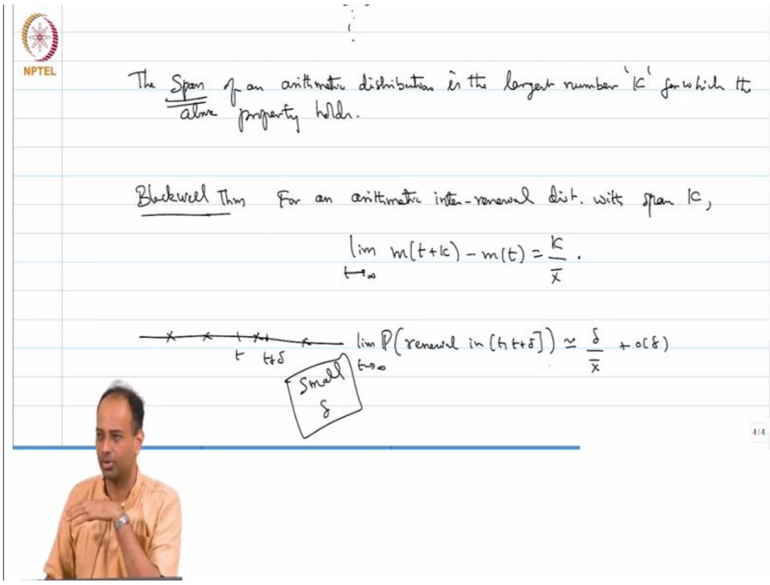

So, now I will state for the arithmetic version Blackwell's theorem for arithmetic inter renewal distribution with span kappa limit t tending to infinity. M of t + kappa – m of t = kappa over x bar. So, you are just requiring the delta be you are forcing the interval to include one of these kappa multiples now the Blackwell's theorem holds that is all that we are saying. So, this Blackwell's theorem has a nice interpretation.

So, if this delta were very small so you have this t and t + delta. We have this individual process running now what is the probability of actually having a renewal in this interval t, t + delta. If delta is very small assume non arithmetic for now okay. If delta is very small then the probability of having two arrivals in the interval is very small it will be little over delta we can ignore it.

So, this expected number of renewals as given by Blackwell's theorems is basically the probability that you have a renewal in t, t + delta. So, in t, t + delta when delta is really small you can have either zero renewals or one renewal. Zero renewals will not contribute anything to the expectation one renewal will contribute something and 2 renewals onwards has very low probability you can say it contributes little over delta.

So, what Blackwell is saying is that probability of so let us say small delta probability of having a renewal in t, t + delta is what is roughly delta over x bar. Yeah you are right it is lambda delta lambda is 1 over x bar lambda is the rate of the process (()) (36:08) little over delta terms will come, but this is for large t asymptotically.

So, this is what Blackwell sort of implies because if you have a small interval the expectation fully comes from that probability of that one renewal more than that two or more renewals has very little probability little over delta and zero renewals has substantial probability, but it will contribute zero to the expectation. So, this term is basically the probability of renewal of t, t + delta is essentially like expectation of N t + delta – N of t which is like lambda delta where lambda is 1 over x bar.

So, it is somewhat like a renewal we are saying this is somewhat like a Poisson process, but what is the key difference? The first key difference is that this is asymptotically true this is not true for every t. If you have large t the probability of having a renewal in a small interval delta is lambda delta plus little over delta. The second key difference this is very important is that you do not have IIP you do not have independent increment property.

So, the number of renewals in t, t + delta and an adjacent interval of another width delta they are not independent, they are dependent for renewal process in general. So, a renewal process has this lambda delta plus little over delta property asymptotically. So, it is all of the stationary increment process sort of a thing comes out asymptotically, but IIP does not hold for non arithmetic renewal process. I will stop here.