Stochastic Modeling and the Theory of Queues

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Lecture –42

Key Renewal Theorem & Ensemble Rewards

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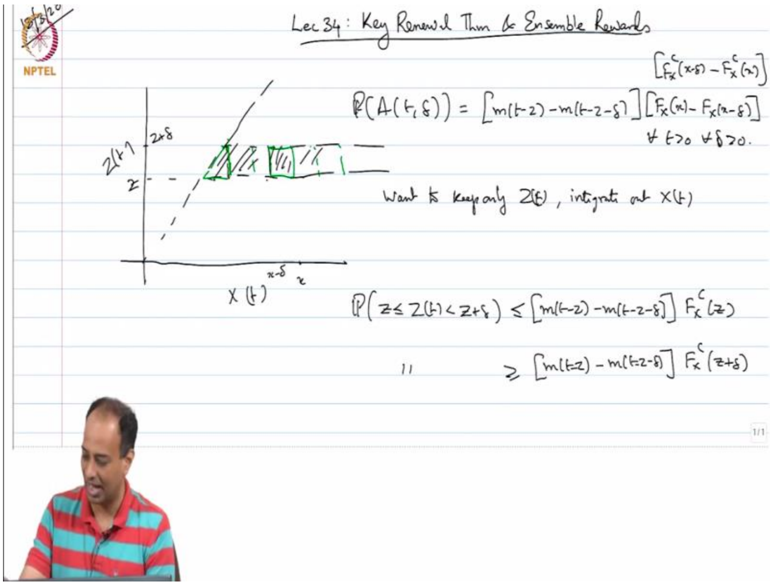

Recall from yesterday that we calculated so if this is renewal process let us say non-arithmetical renewal process and I take X t Z t of course if I am looking at the joint distribution of X t and Z t the joint distribution lies below this 45 degree line and yesterday we managed to calculate essentially what is the probability of if you put a square at x, x – delta and this is z, z + delta.

So, this I said this probability of A, t, delta the probability that your duration and age the joint will lie in this square recall from yesterday m of t – z I am just copying from yesterday – m of t – z – delta times F x of x – F x of x – delta this is what I got. So, this is roughly intuitively you can think of it as describing the joint distribution between Z t and X t for any finite t and any delta greater than 0.

So, this is true for all t greater than 0 and for all delta greater than 0. See at a high level if I am giving you a function R of t remember R of t is a renewal reward function R of t we took it as R of Z t – X t. Reward is a function of at time is a function of the age at time t and the duration at time t. So, if R is a function of Z t, X t and I know the joint distribution of Z t and X t.

I can calculate expectation of R t because I know the joint distribution (()) (02:42) known as joint distribution through this equation. Now except that there are some subtleties involved may be I will show it to you on what really happens. I will show it to you by calculating expected Z t or cdf of Z t for finite t and then taking t to infinity. So, if you look at suppose I want to take something like this.

I just want to keep only Z t basically integrate out X t. So, let us say I want to take that bit now. So, what will I do? I want to look at the probability that z is less than or equal to Z t less than z + delta. This is essentially give me the cdf of z for some finite t. So, this you can show is to be what will you do? You know the probability that your X t Z t lies in that square you can add many such squares over you can just add over all the axis.

So, what you can show is that this probability that Z t lies between little z and little z + delta can be upper bounded and lower bounded. So, an upper bound you can say that this is upper bounded by m of t – z – m of t – z – delta times F x compliment of z. This can also be written as F x compliment of x – delta – F x compliment of x is the same as that and the same quantity is lower bounded by m of t – z – m of t – z – delta F x compliment of z + delta.

See the upper bound and lower bound come by so for example the lower bound comes by summing all the squares like till here and then there will be little bit left here which I do not know what it is. So, I take a lower bound just sums still this point and the upper bound somehow upper bounds the area of this little triangle out here that is really what you do. This is a calculation which is just mechanical and you can get something like this.

This is not very difficult to believe at the end of the day. So now what do you have? Suppose, this delta is fairly small you have upper and lower bounds on the probability that Z t lies in z to z + delta. From this can I not get the cdf of Z of t for any finite t I can again sum this.

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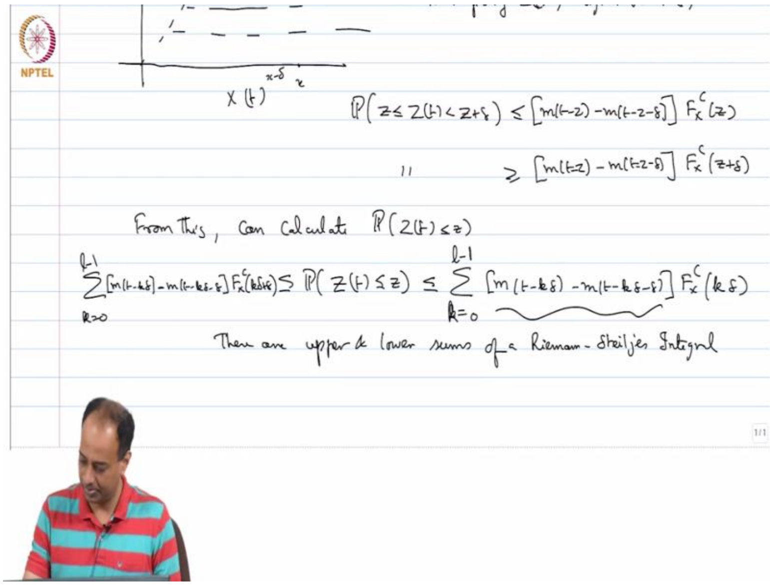

So, if you do that from this we can show the following. Can calculate probability that see the mechanics are little subtle, but you can calculate this and what it comes out is a sum like this I think you can if I write it down you will see that you can believe it you will get something like this probability that Z t less than or equal to little z is upper bounded and lower bounded like.

You will sum over all this z you will sum k = 0 to l – 1 m of t – k delta – m of t – k delta – delta F x compliment of k delta times F x compliment of k delta + delta. What I am essentially doing is I am summing over see I have a handle on the area I mean the probability mass under a strip like this. Now I am further dividing this into several other delta strips like that and I am summing all these areas.

And then I have lower bound and upper bound. Now, if I send delta to 0 what happens is the question I want to take probability of z less than equal to z and bound them between these two sums and I sent let us say delta to 0. Now what happens to these sums that is the question. The answer is that these sums that I put out it is a lower bounds and upper bounds to this cdf of z of t.

These upper bound and lower bound and S n delta to 0 they approach certain integrals. In fact these can be shown to be lower sum and upper sum corresponding to a special kind of an integral. It is not just a Riemann integral it looks like a little bit like a Riemann sum except that in a Riemann sum you will have something like so F x so F x compliment of k delta times delta itself.

Instead of delta you are having m of this is sum m t so this is like a difference of two functions. If these term were just delta in the upper sum and the lower sum you would just write it as a Riemann integral. This is not just a Riemann integral, this is known as a Riemann–Stieltjes integral, have you heard of a Stieltjes integral this is what it is you sent delta to 0.

You are basically integrating a function F x compliment in this case with respect to some function m tau or m t in this case m t – tau or whatever. So, instead of just delta you take m as the identity function for example you will have Riemann sum, but m is not identity m is sum function you will get the so called Riemann–Stieltjes integral. So, these are upper and lower sums of a Riemann–Stieltjes integral.

I have written some sort of thing like this before when I write integrate sum function g of x d f x that is also a Riemann–Stieltjes integral technically it is not just d x d f x I am writing. So, this is Stieltjes integral.

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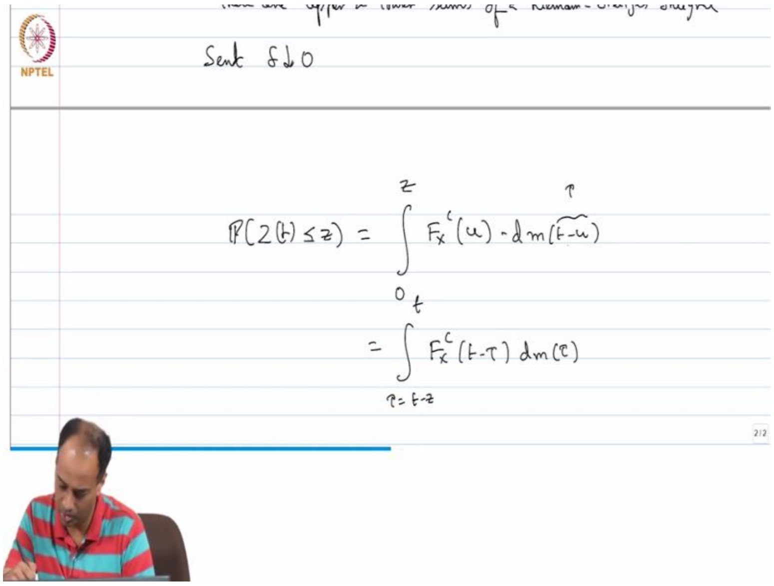

Like what happens is that you will get (()) (11:26) sent delta down to 0 you will get probability Z t less than or equal to z will be equal to integral F x compliment of u d m t – u where u goes from 0 to z or which you can again write by putting t – u as sum tau this is what you can read this off this is looking like F x compliment of some u k delta is the u variable so I am putting t – u.

It is not d u, but d mu m t – u this is the definition of this Riemann–Stieltjes integral if is the upper sum and lower sum converge to the same thing it is the integral. This is like a generalization of a Riemann integral that is all there is and if you just change this variable this becomes integral tau equals t – z to t F x compliment of t – tau d m tau. So, I will just put t – z = tau which means t – tau is u that is correct and u goes from u is 0 means there will be a negative sign which counteracts.

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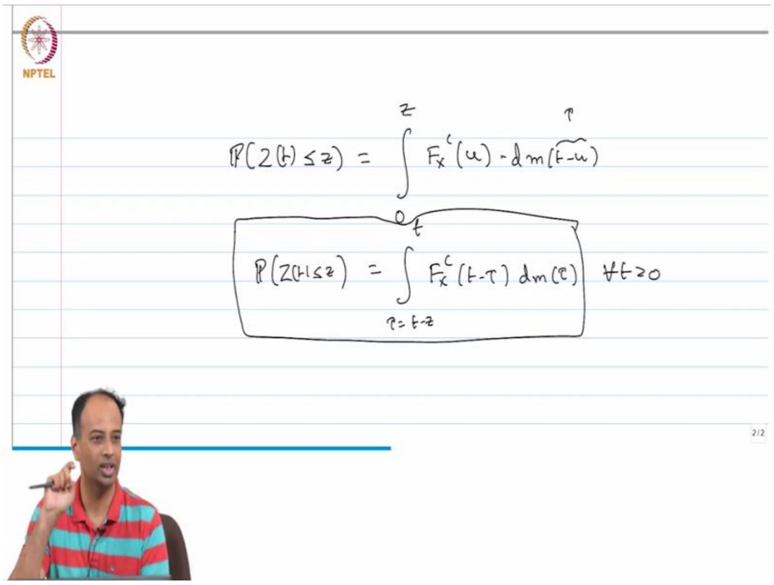

So this is probability that Z t greater than equal to z. So, let me just check F x compliment t – tau d m correct. So, this is what it is. So, this is the cdf of Z t for any finite t this is true for all t. I give you t is equal to 2.5 you tell me what the cdf of Z t is I can go calculate this Riemann–Stieltjes integral and give you the answer. Now, I need to integrate with respect to m tau. How do I find m tau renewal equation

Laplace transform whatever we have studied this. We can use Laplace transform and this solves the renewal integral equation somehow put m tau into this and solve. So, in principle get this.

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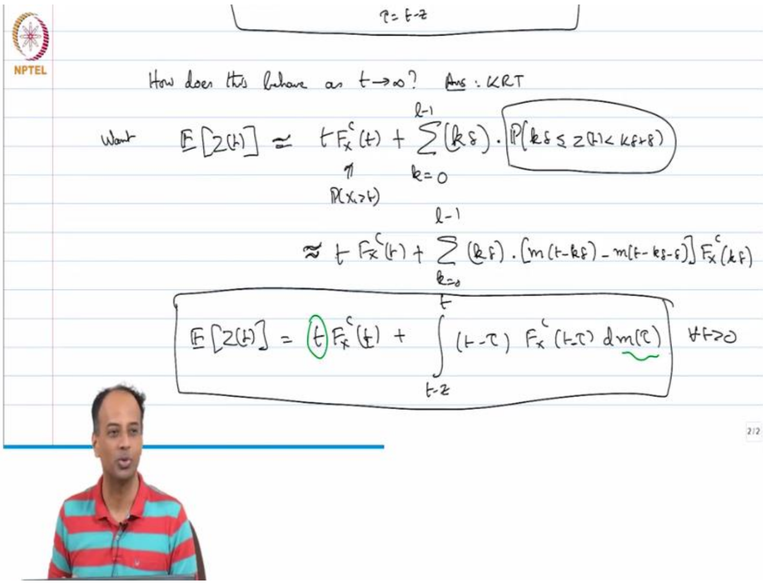

Now the question is how does this behave as t tends to infinity? So, these questions will be answered by the key renewal theorem. Similarly, if I want to calculate let us say this is just the cdf of Z of t. Suppose, I want to calculate expectation of Z of t. See the earlier cdf is expectation of indictor Z less than or equal to little z. Suppose, I want to calculate expectation of Z of t.

This is for finite t I am not taking t to infinity yet. So, the previous answer how does this behave as t tends to infinity will be given by key renewal theorem which will state in a bit. So, similarly expectation of Z of t you can write it as follows. Z of t is the age at time t. So, there are two possibilities now. The first possibility is that there has been no arrivals till time t no arrivals at all, no renewals at all which means the age will be equal to t.

So, with probability what is the probability there has been on arrivals till time t? Probability that X 1 greater than t which is F x compliment. So, with probability F x compliment of t Z of t will be t plus there is the other possibility that there have been renewals for this you have to write out an integral this again is done in your book. So, you have to take upper bounds and lower bounds so you will get this will roughly be like again some sort of a sum k = 0 to l – 1 you are dividing into delta, delta intervals.

It will be like k delta which is the value of the age times probability that X 1 greater than t probability that k delta less than or equal to Z t less than k delta + delta and this we already know what it is. So, this will be like t F x compliment of t + something that looks like k = 0 to l – 1 k delta times this we already know m of t – k delta – m of t – k delta – delta F x compliment of k delta.

So, I am just substituting this guy from here. I am just putting this here. When I write this approximately equal to I mean that I can upper bound, lower bound just like I did for F x compliment. Now you sent delta to 0. Now what will the sum become? What kind of integral will it become? It will become another Riemann–Stieltjes integral. So, bottom line is you can show that expected Z t the first term will be the same t F x compliment of t.

I think Gallagher Book is missing this t just please check it out I think there should be a t here plus integral t – tau so this will be a u F x compliment of u d m t – u. So, this will become similarly it will become t – z to t t – tau m of sorry F x compliment of t – tau d m tau similar sort of a calculations just the Riemann–Stieltjes. So this is another Riemann–Stieltjes integral you go ahead and find m tau put it in any finite t you can get those expectation of Z of t.

So, we can get cdf we can get expectation of Z of t for any finite t. Now again the question is how does this behave as t tends to infinity. See if t tends to infinity the first term will see the first term you can imagine will go to 0 because F x compliment of t is going to 0 and it is even has finite expectations so t times F x compliment of t will also go to 0. Now the first term will go to 0 and the second term what does it go to in the question.

Similarly, in this expression also we want to know as t tends to infinity what does this integral go to. Intuitively what does this m tau what does this behave like? See m t over t for large T behaves like 1 / X bar. So m t has t / X bar plus a bunch of other terms. So d m tau should behave like 1 / X bar d m tau should behave like d tau / X bar. So, that is what you would expect and that is what is true under some technical assumptions. So, that is what key renewal theorem says.

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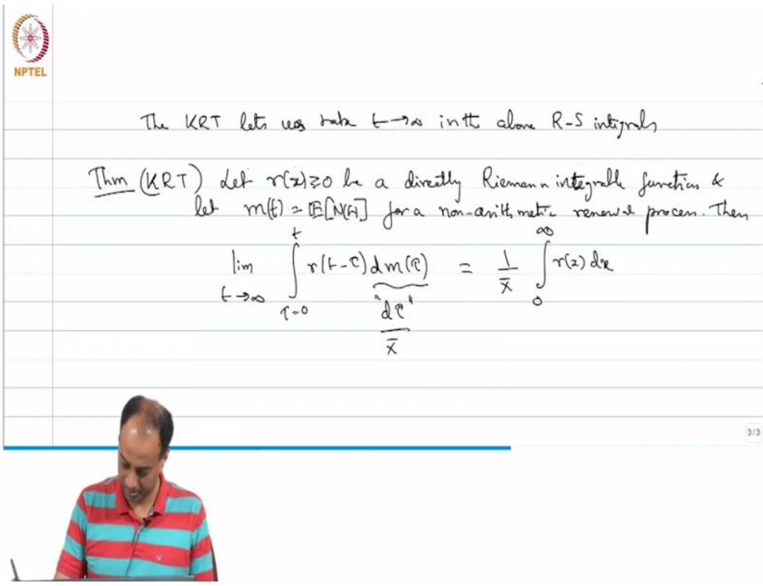

So, the key renewal theorem let just take t to infinity in the above Riemann–Stieltjes integral. It says key renewal theorem let r of x be a directly Riemann integrable function and let m t equal to expectation of N t for a non-arithmetic renewal process then limit t tends to infinity integral tau equal to 0 to t sorry I think I made a mistake so this should be 0 please correct this 0 to t.

I made a mistake here please correct that. So, tau = 0 to t r t – tau d m tau is equal to 1 over X bar integral 0 to infinity r of x d x. So, if you look at let us say this integral here thus looking like integral 0 to t sum function let us say this could be r of t – tau. You are looking at the integral 0 to t r of t – tau d m tau. In this kind of an integral I want to send t to infinity I want to know what happens.

Key renewal theorem tells you what happens. Key renewal theorem says if little r of x is any directly Riemann integrable function then this kind of an integral limit t tending to infinity r t – tau d m tau is simply 1 over X bar integral 0 to infinity r of x d x. Now I will tell you two things. One I have to tell you what directly Riemann integrable is, it is just a technical assumption.

And then I will just give you a very basic interpretation of this theorem. I am not going to proof this. This is a non-trivial result to proof. This actually comes from Blackwell. So, if you look at this what is directly Riemann integrable you know what Riemann integrable is. So, if I want to integrate a function little r of x from 0 to b I divide the x axis into small delta intervals take the upper Riemann sum, take the lower Riemann sum and send delta to 0.

I get integral 0 to be r of x d x. If I want integral 0 to infinity r of x d x I first calculate integral 0 to b using this division and then send b to infinity. So, you have Riemann integrable and then you send b to infinity, but if you were to partition the entire 0 to infinity using delta intervals and take upper and lower Riemann sums. If the lower Riemann sum for the entire 0 to infinity interval converges then such a function is said to be directly Riemann integral.

Meaning that you do not go from 0 to b and then calculate the integral and then send b to infinity you directly write a Riemann sum for the entire semi-infinite interval. What you can show is that directly Riemann integrable functions are always Riemann integrable, but not vice-versa. So, directly Riemann integrable is a little stronger than Riemann integrable, but nevertheless it is not too stronger restrictions.

So, if r of x is a nice enough function d m tau behaves like d tau / X bar that is all this is saying in the limit. So, this guy behaves like d tau / X bar that is what we are getting that is a one way X bar outside. So, the question is the other examples of a function which is Riemann integrable, but not directly Riemann integrable your book has an example exercise 5.28 apparently it gives an explicit example.

But these are really I mean these are just technicalities. All I want to know what happens on S N t to infinity that is my question. So, using KRT so if I want to take limit t tending to infinity in Z of t what will I get?

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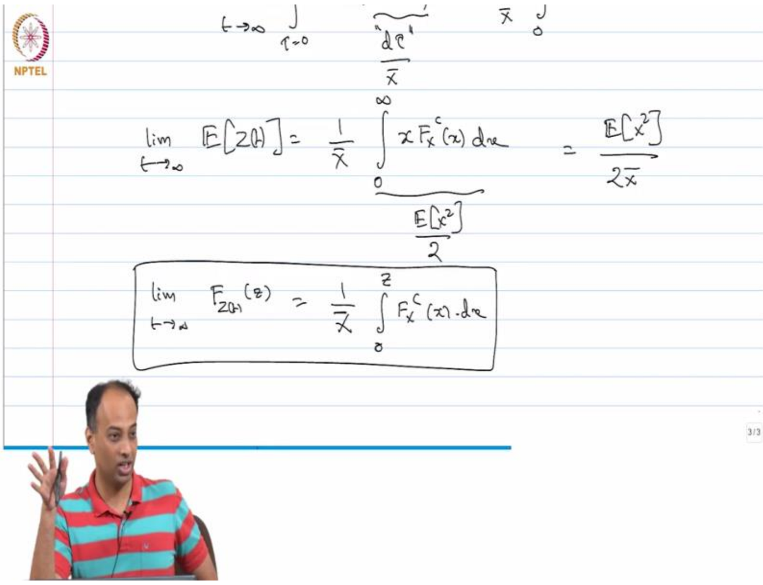

So, limit t tends to infinity expected Z of t equal to what. The first term goes to 0. Thus, for the second term I can use key renewal theorem. So, this is my r t – tau. So, I can simply write down the answer as 1 over X bar integral 0 to infinity r of x is x F x compliment simple x F x compliment of x d x. You have to show that x F x compliment is directly Riemann integrable. Now what is this integral?

Integral x F x compliment of x d x so you can show that this is equal to expectation of you can do integration by parts if you want. So, this will be the answer of course we always knew. This answer we knew should be the case why time average we know the answer. Just to get this answer involved calculating Stieltjes integral and then invoking this big hammer called key renewal theorem.

So, the bottom line and all this is that for non-arithmetic renewal processes limit t tending to infinity of expectation of r of t so this here r of t is just Z of t will be expectation of R n over X bar, but to get this you need to go through these Riemann–Stieltjes integral and then apply key renewal theorem and all that. So, likewise you can just similarly you can write t tending to infinity F z t of z is equal to you can write you can write it as 1 over X bar integral 0 to z F x compliment of x d x.

I am going back to this guy now. I am going back to this and I am now taking r of t – tau is F x compliment of t – tau and instead of going from 0 to t I am going from t – z to t and so if I apply key renewal theorem I will get an integral like this. This is integral 0 to z F x compliment of x, but this also you already knew so these are all answers that are not surprising.

But the method is through this Riemann–Stieltjes and then putting down this key renewal theorem and coming to this. So, now again so we can do this for any reward function. Reward function r of t is r of Z t, X t. I can write down see I know the joint distribution of X t and Z t for all practical purposes. So, I write down this Riemann–Stieltjes sums and delta to 0 I will get some Stieltjes integral and then invoke key renewal theorem.

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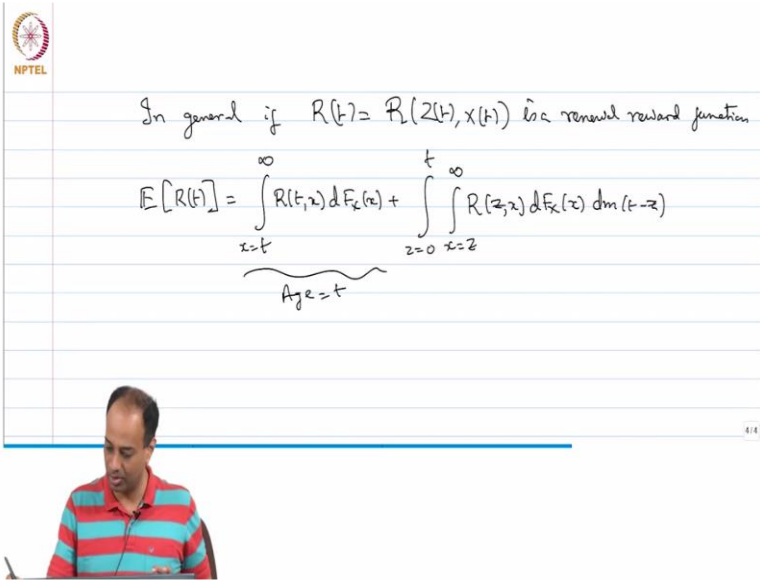

So, in general if R of t = R of Z t, X t is a renewal reward function. You can write out I will not again dwell on this too much, but you can write expectation of R of t = integral x = t to infinity R of t, x d F x of x + integral z = 0 to t x = z to infinity R of z, x d f x of x d m t – z. So, again this term corresponds to age is equal to t meaning there has been no renewals which is why I am putting z is equal to t and then I have x has to go from t to infinity R of t x d F x.

This is like we wrote down t times F x compliment for expected Z of t. So, first integral correspond to there being no renewal until time t. The second integral is a Riemann–Stieltjes integral you have to write R z x and then you have to sum out you had this probability of being in a small delta square. You will just have to write that out as a Riemann–Stieltjes integral.

The method is the same if you just look at what I did for expected Z t. So, I wrote down this big sum out here. So, the same sort of a thing you can do for I just did it for Z of t you can do it for any R of z of t to my X of t then you will get this kind of a result. So, this is for any t greater than 0. Now how do we send t to infinity using what theorem key renewal theorem.

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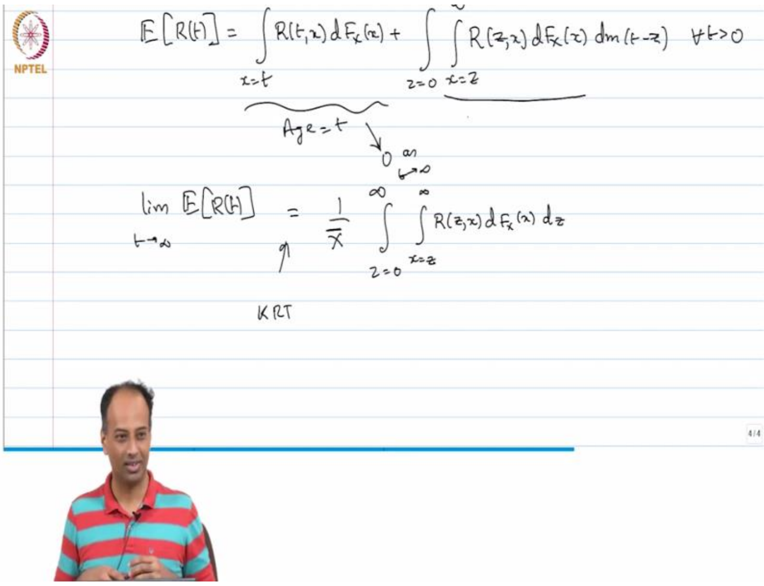

So, if you take limit t tending to infinity expected R of t what can be shown is that this guy will go to 0. If this if you look at this guy this is the function so if you look at this bit if this bit this is just a function of z. So, if this bit is directly Riemann integrable then I can use key renewal theorem and if that is the case this term will go to 0 first term will go to 0 and I will get what will I get I will get 1 over X bar integral z = 0 to infinity little r of x d x where my little r of x is z to infinity r of z d z.

So, it will be so R of z, x d F x of x d z. This is by key renewal theorem. I am going from here to here by key renewal theorem. By taking this inner integral as my little r function, but again if you go back to your time average reward processes this is exactly what we got. You remember this was a while ago, but if you remember we calculated limit t tending to infinity 1 over t R of integral 0 to t R of tau d tau this is exactly what we got.

So, bottom line is that for non-arithmetic processes the answers are you mean the time average reward ensemble sample reward are numerically the same. Time average rewards was very easy to calculate using strong law of large numbers. Ensemble average reward requires a lot more subtlety and lot more machinery. So, in particular you have to find out the joint probability distribution of F of Z t X t by looking at this little square.

Then for calculating any expected R of t which is R of Z t, X t you have to sum over all the probabilities of the little squares. These sums become Riemann–Stieltjes integral. So, that much is just calculus then for going from that Riemann–Stieltjes integral with respect to (()) (35:12) going to just a plane old Riemann integral requires this key renewal theorem, but if you do all this finally the answer you get is what you want which is reassuring.

So, that is really all you need to know I think for all practical engineering purposes in vast majority of cases of interest you will have directly Riemann integral little r and you will have ensemble average rewards is equal to time average reward, but just in case you are doubtful you know where to look. So, the purpose of this lecture is to not so much to tell you that the answers are the same.

Of course the answers are the same between ensemble reward and time average rewards. It is roughly to tell you how it goes. I have not proved anything rigorously. So that if in your research if you find some technical subtleties you can check for Riemann integrability and you know you go back to the Riemann–Stieltjes integral just in case you have this problem.