Stochastic Modeling and the Theory of Queues
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Lecture –54
Introduction to Countable-state DTMC

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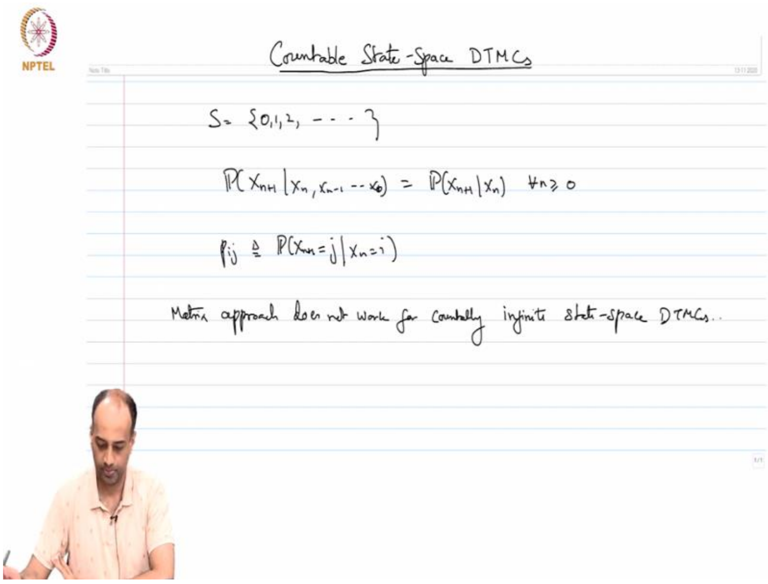

Welcome back. Today, we will start discussing the topic of DTMCs with countably infinite state space. So, we are going to discuss countably infinite state space Markov chains in discrete time. So, the Markov property is satisfied, but the state space S is something countably infinite. So, without loss of generality we can take it to be 0, 1, 2 dot, dot, dot and the Markov property is satisfied in the sense that P X n + 1 given X n, X n – 1 dot, dot, dot X 0 is still just equal to P X n + 1 given X n.

This is true for all values of n greater than or equal to 1 sorry n greater than or equal to 0 and for all values of X n in the state space. So, we will continue to denote P X n + 1 equals j given X n equals i. We will continue to denote it as P ij as before. Now, in this situation the transition probability matrix is not really a matrix it is an infinite by infinite dimensional object.

For this reason we cannot use these matrix analysis technique a linear algebra techniques or spectral techniques in a very straightforward manner as we did it in the finite state case. We cannot directly apply those kind of results to the countably infinite state space Markov chains. This is mainly because the spectral theory and the linear algebra of this infinite dimensional operators is far more involved.

So, the matrix approach does not work for countably infinite state space. So, if you want to study things like long term behaviour etcetera. So, we have to go for a different kind of an approach.

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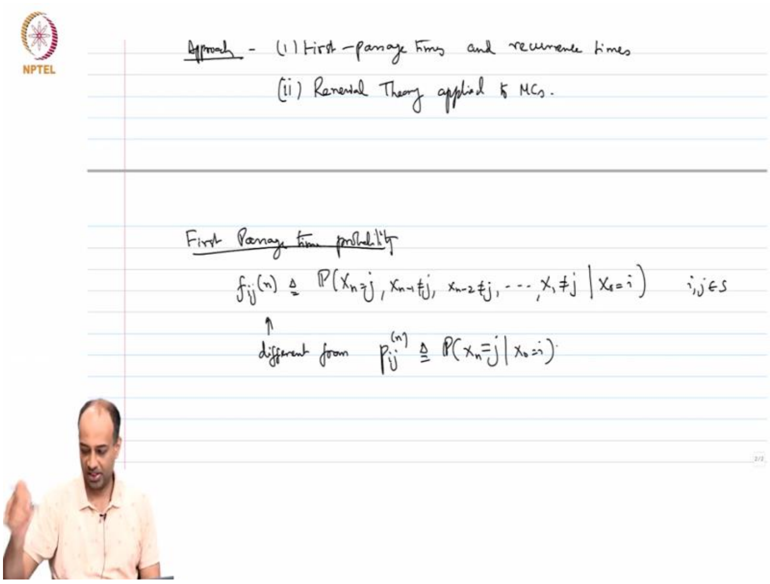

So, in fact the approach that work is through is by looking at first passage times and recurrence times. So, what you do is we already discussed first passage times in the context of in the previous lecture we discussed how to calculate expected first passage time and expected recurrence time. So, first passage time means you start at i and what is the first time at which you get to state j.

Recurrence time of a state j is you start at j and the next time you return to j that is the recurrence time. We learnt how to calculate the expected first passage time and expected recurrence time of a state in the previous module. In this particular approach we will study the random variable of the first passage time and the random variable of the recurrence time. As we shall see this random variable may not really be a random variable in the sense that there may be infinity also in that case they will be defective.

But these quantities the first passage time and recurrence times will help us study these properties of countably infinite state DTMCs. In fact, we will use first passage time and recurrence times in conjunction with results we know from renewal theory apply to Markov chains. These two tools together will help us study these countably infinite state Markov chains.

In fact, everything we are going to say for countably infinite state Markov chains also applies in a easy way to finite state Markov chains, but not everything we said for finite state Markov chains will apply for countably infinite state Markov chains. And we will see that the notion of recurrence transient etcetera are also defined very differently for a countably infinite state Markov chain.

So, let us define the first passage time probability. Let us say f ij of n is defined as the probability that X n equal to j X n – 1 not equal to j, X n – 2 not equal to j dot, dot, dot X 1 not equal to j given X 0 equals i. So, you are starting the Markov chain at time 0 in state i. First passage time probability means I should be in state j at time n for the first time. Please keep in mind that this is different from P ij n.

So, this is different P ij n which we defined as the probability that I am in state j at time n given that I started in state i at time 0. I am not requiring in this case a P ij n case I am not requiring that i be in state j at time n for the first time. I just want to be in state j for at time n I could have been in state j before also. This P ij n allows for it, but f ij n is looking at the probability that I am in state j at time n for the first n.

And in the previous time I was not in state j given that I started at state i. So, this f ij n is a first passage time probability. This will be useful for us to study. So, this f ij n also satisfies a equation very similar to the Chapman–Kolmogorov equation this is sort of an iterative sort of an equation.

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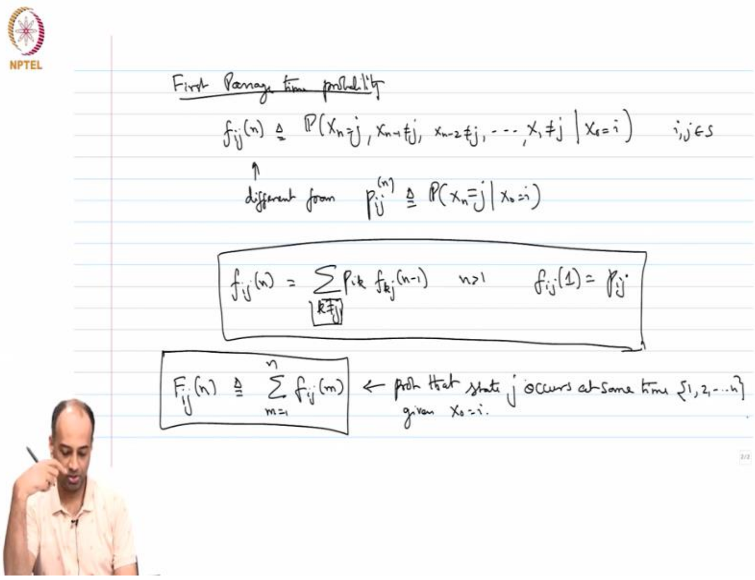

It is as follows. f ij n can be shown to satisfy sum over P ik, k not equal to j f kj of n – 1 and this is true for n greater than 1 and for f ij of 1 is simply P ij. So, f ij of 1 is nothing, but the probability that you will be in state j at time 1 given that you started in at time i at time 0 this is nothing, but P ij, but for n equals 2 onwards we have to exclude this k not equal to j is important. So, f ij of n is given by sum over k not equal to j P ik f kj n – 1. So, f jk n – 1 means you were in state k for the first time with at time n – 1.

The sum on the right should be interpreted as you started at time state i we went to some state k and from state k you went to state j at time n – 1 for the first time that is what this equation means. This is an iterative equation so you solve you said f ij = 1 f ij of 1 is equal to P ij then you iteratively solve for f ij of 2 and then you go ahead and solve for f ij of 3 and dot, dot, dot.

These are the iterative equations governing this first passage time probabilities this is not difficult to derive. Similarly, let us define big f ij of n as sum over m = 1 to n little f ij of m. So, F ij of n is the probability state j occurs at some time between 1, 2 dot, dot, dot n 1 through n given X 0 equals i. So, little f ij of m is the probability that I start at i and reach j for the first time at time m.

Now I am summing over all these times from m = 1 to n so this big F ij of n should be interpreted as the probability that state j occurs at some time between 1 and n inclusive 1, 2, 3 dot, dot, dot n given that I started at i X 0 = i so this is also a probability. So, remember the events that define f ij of n are disjoint over these n. So, I basically this probability interpretation for this big F ij comes from summing over disjoint events.

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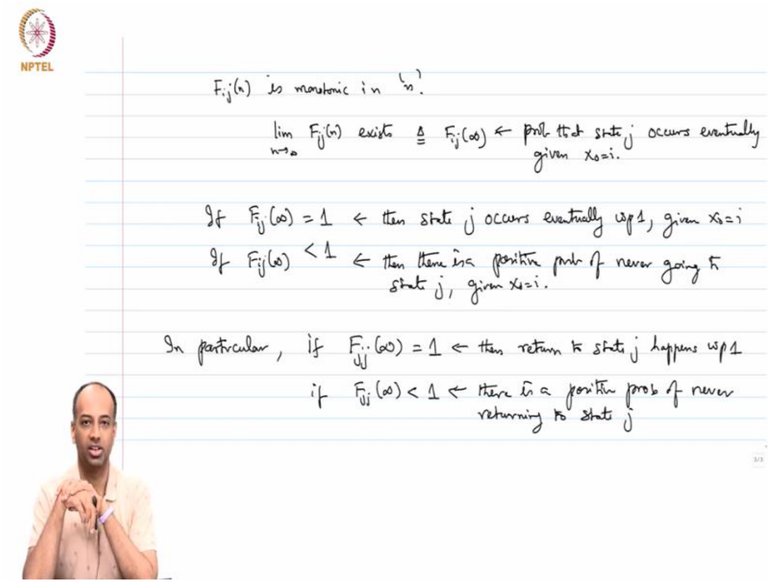

Now, if you look at this big F ij of n what can you say about it? F ij of n is a monotonic in n that is because you are adding these non-negative little f ij so f ij of n is monotonic in n and it is upper bounded by 1. So, why is it upper bounded by 1? Because if you start at i and reached j for the first time at time 3 you cannot also start at time i and reach j for the first time at time 5 so that is not possible.

So, what happens is that this F ij of n this quantity which is monotonic in n which is clear and is also upper bounded by 1 being a probability. So, what happens is that this limit n tending to infinity F ij of n exist. Let us call this limit as F ij infinity that is just notation for limit n tending to infinity F ij of n which exist because big F ij of n is monotonic and bounded above so the limit has to exist.

So, F ij of infinity should be interpreted as the probability that state j occurs eventually. Given X 0 equals i. F ij n is the probability that state j is reached between 1 through n starting at i. Now if you take limit n tending to infinity this F ij of infinity should be interpreted as the probability that state j is eventually occurs given X 0 equals i and in particular so if you can look at this way so F ij of infinity equals 1.

Then state j occurs eventually with probability 1 given X 0 equals i. If this is less than 1 then there is a positive probability of never returning to state j or never going to state j given X 0 equals i So, this F ij of infinity so this limit exists and F ij of infinity could be some number it is certainly a number which is utmost 1 it could be 1 or it could be less than 1. If it is 1 it means that the probability of eventually seeing state j.

Given that I started at state i is equal to 1 which means almost surely I will see state j given that I started at i. If F ij of infinity is a number which is strictly less than 1 which means that the probability of eventually going to state j given that I started at i is strictly less than 1 which means there is a strictly positive probability that I will never see state j given that I started at i.

Now these quantities these F ij infinity is a very useful thing to look at. In particular if you look at this guy F jj infinity. What is F jj infinity? Given that I start at j what is the probability that eventually see j again. So, this is the probability that I come back to j starting at j. If this probability is equal to 1 then return to state j happens with probability 1 having started at state j.

And if F jj infinity is less than 1 in this case there is a positive probability of never returning to state j having started at state j. So 1 – F jj infinity will be the probability that you never return to state j having started at state j. Now, using this characterization of F jj infinity whether or not it is equal to 1 or less than 1 we can define the recurrence of a state. Remember that in the finite state case we defined recurrence very differently.

We said that a state is recurrent if for every state that you can get to from that particular state j there is also a way to come back that is how we defined for finite state Markov chains that definition is inadequate for the case of countably infinite state DTMCs. So, we need an altogether different definition of recurrence and transients and we will give that definition using this F i f jj characterization.

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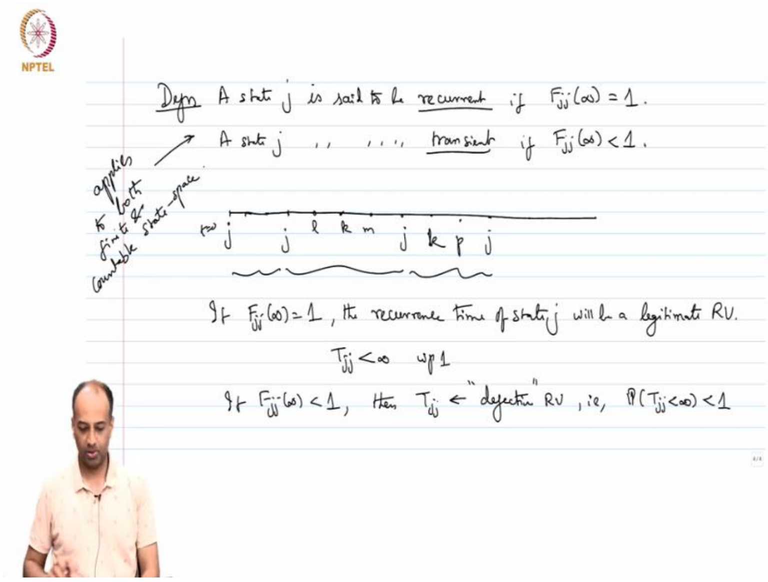

Definition the state j is said to be recurrent if F jj infinity is equal to 1 and a state j is said to be transient if it is recurrent that is F jj infinity is strictly less than 1. So, if this is your discrete time axis let us say you are in state j at this point so you may go through various states l, k, m and so on and return to j again. Again some other state k, p etcetera and then j to j again and so on.

So, you look at all these returns so you will start at time 0 you will start at j this is time t = 0 you start at j you go through bunch of states and then hit j again. So, if you look at if the situation is such that F jj infinity is equal to 1 then you are guaranteed with probability 1 you are guaranteed that you will come back to state j. If F jj infinity is less than 1 there is a positive probability that you will never come back to j.

If F jj infinity is equal to 1 which means that you are guaranteed to come back with probability 1 to state j having started at state j then the times between these times these case they can looked up as random variables because in finite time you are guaranteed to come back with probability 1. So, these time between consecutive occurrences of state j will be finite with probability 1.

If F jj infinity is equal to 1 the time interval between consecutive let me write it like this you know a term for this which is called the recurrence time the recurrence time of state j meaning that you start at j and then come back to j will be a legitimate random variable. It will be finite with probability 1 and I can denote this as T jj. So, this T jj will be less than infinity with probability 1 in this case.

If F jj infinity is strictly less than 1 then this T jj will be a defective random variable meaning that a probability that T jj is less than infinity will be less than 1. In other words, T jj can be infinity with positive probability and if you remember random variables are generally defined to be real valued, you are not allowed to have infinite value with any positive probability. Such a random variable is not really a random variable it is called a defective random variable.

So, this T jj in that case will be a defective random variable if you are not guaranteed return to j with probability 1. So, F jj infinity is equal to 1 with this consecutive this recurrence times between consecutive terms to j will be finite with probability 1 and you can define a random variable T jj and this will be finite with probability 1. So, this T jj is a very useful thing to look at.

Now we can show that this definition of recurrence is sufficient to accommodate both countably infinite state Markov chains as well as finite state Markov chains. This definition will fall back to your original is equivalent to your older definition for finite state S, but the older definition will not apply for the countably infinite state S this can be shown. So, maybe I should say applies to both finite and countable state space DTMCs. Okay.