Stochastic Modeling and the Theory of Queues
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Lecture –58
Stationary Distribution of a Countable State Space DTMC and Renewal Theory

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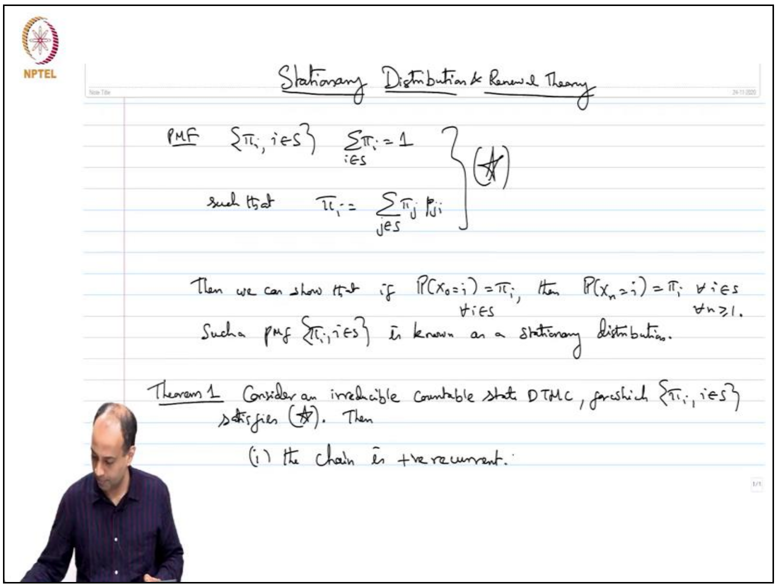

Welcome back. So, far we have look we have been looking at the connections between renewable theory and DTMC's. In particular we have said that consecutive returns to a particular state recurrent state j constitute renewal instances. Today we are going to use renewal theory renewal reward theory in particular to discover the stationary distribution of accountable state DTMC.

Recall that we say that let us say PMF pi is given to you. So, this sum over pi i equals 1 such that it satisfies pi i equals sum over pi j P ji. Suppose a PMF on the states satisfies pi i is equal to sum over j by j P j i then we can show that you can show that if the chain is started in the distribution pi then for all i belongs to S then probability that X n equals i will also be equal to pi i for all i belongs to S and for all n greater than or equal to 1 what does this mean?

We already saw this in the finite state context. So, essentially we are saying that pi is equal to pi P except that I am not writing it as pi is equal to pi P because the matrix P is not really a matrix now it is an infinite dimensional object but the fact remains the same that if you start off the Markov chain at time equal to 0 in state i with probability pi i then the probability of being in state i for any time n remains pi for every state i.

This is easy this is just by applying just applying total probability and Markov property. Now the thing is we are going to prove that this stationary distribution if is such a such a pi PMF pi is known as a stationary distribution. This we already seen of the DTMC such a PMF such a distribution is known as the stationary distribution of the DTMC. Now we are going to relate this pi if at all there exists such a pi we are going to relate it to recurrence times mean recurrence times of state i using renewal theory renewal regard theory.

So, this is the key result is the following theorem. So, theorem one consider an irreducible countable state DTMC for which pi i satisfies the star. Then let me write the following then number one the chain is positive recurrent.

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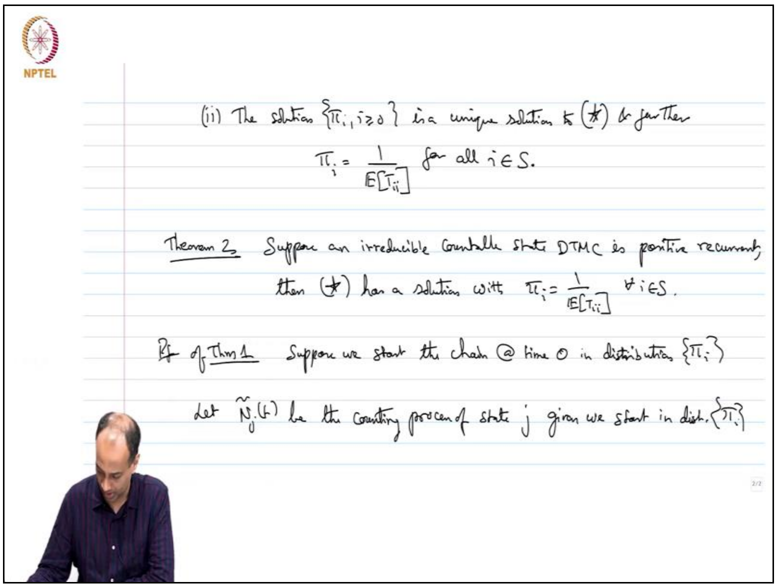

The solution pi is a unique solution to star and further pi i is equal to 1 over expected T ii for all i in S. So, this theorem is a very important theorem suppose. So, it says that suppose you manage to find a solution pi suppose if somehow by guess work or asking somebody or whatever means you manage to find a solution a distribution pi such that some over pi is are equal to 1 and some over j pi j P ji is equal to pi i.

So, these are the balance equations this is what we call the global balance equations. So, if you manage to find a solution to the global balance equation by whatever means then we can come to some important conclusions the conclusions are that the solution you manage to find is unique and the chain is positive recurrent. And furthermore the pi is you found out are the reciprocals of the mean recurrence time of each of the states.

So, there are three conclusions in this theorem. So, you somehow managed to find a pi. So this theorem says that a this chain is positive recurrent P. The solution you manage to find somehow is the only solution there cannot be any other solutions to pi is equal to pi P and this pi i has the interpretation as the reciprocal of the mean recurrence time of that particular state. So, this is a very important theorem.

Let me also state theorem two which is like the converse of the theorem one. So, these two theorems are central and then we will see how these theorems are proved using renewal reward theory. Suppose an irreducible countable state DTMC is positive recurrent then star has a solution with pi i is equal to 1 over expected T ii for all i in S . So, this is like the converse theorem two is like the converse theorem one says that if you somehow manage to find a solution to this pi is equal to pi P to these balance equations.

Then you get positive recurrence for free and the solution is unique and pi i will be equal to 1 over T ii bar when expected T ii for each state. Now we are saying the opposite. So, now I do not know pi but i know that the chain is positive recurrent. So, what I know is that. So, it is an irreducible DTMC it is positive recurrent. So, all states are positive recurrent. So, one over expected T ii a bar is something non-negative.

Now it is actually strictly positive if you think about it because expected T ii is finite because it is positive recurrent. So, one over expected ti is some strictly positive number we are saying that one over expected T ii is the solution to pi is equal to pi P that is what theorem two is saying. So, these two theorems can be viewed as I mean theorem two is a converse to theorem.

So positive recurrence implies there exists the this one over T ii bar the solution to pi is equal to pi P and theorem 1 says if pi is equal to pi P has a solution pi then its unique positive recurrence follows and this pi has the interpretation of 1 over expected T ii. So, bottom line is that the stationary distribution pi i for any positive record Markov chain is simply 1 over expected mean recurrence time.

We can prove all this using renewal reward proof of theorem one suppose we start the chain at time zero in distribution pipe let N j tilde t be the counting process of state j given we start in distribution pi N j tilde t simply a counting process.

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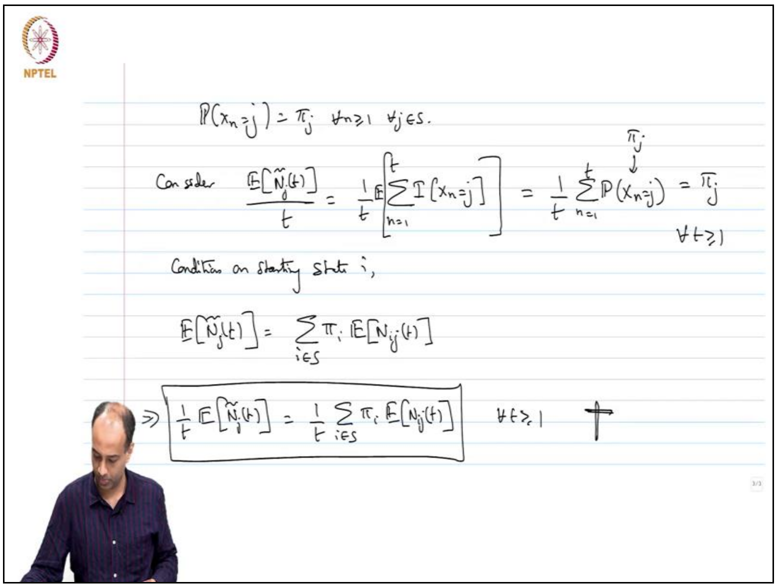

So, what am I doing I am starting in distribution pi at time equal to zero what does that mean. So, at time equal to 0 the probability that I am in state i is just pi i and then i let the chain run and I am going to count the number of times state j occurs until time t and I am going to call that N j tilde t. Now please note that probability that X n is equal to j is equal to pi j for all n greater than or equal to 1 and for all j and S.

So, if you look at this guy now consider expected N j tilde t over t that is just one over t sum over n is equal to 1 to t probability that X n equals j. So, you can look at this as you know maybe I should write one over one more step maybe I should write it as expectation of the indicator what is the what is N j tilde t? N j tilde t is simply the indicator that X n is equal to j summed over n is equal to 1 to t.

Now I can take the expectation inside because inside the summation because this is a finite sum. So, and if I take the expectation inside the summation I have the expectation of an indicator and the expectation of indicator is simply probability. So, this becomes 1 over t sum over n is equal to 1 to t probability that X n equals j but of course probability that X n equals j is simply pi j. So, this is pi j correct.

So this is true for all t correct. Now you can condition on starting state i and use total expectation to write expected n j tilde t is simply equal to sum over pi i sum over i pi I expected N ij of t. So, why is this? So, N j tilde t is the number of occurrences of state j until time t but you could have started in any of the states i and the probability of stating starting in state i is by assumption pi i am starting in the stationary distribution.

So, this is my total probability that the total expectations the law of total expectations. So, this implies what. So, this implies an important equation. So, one over t this implies expected N j tilde t is equal to 1 over t sum over i belongs to S pi i expected N ij of t for all t not equal to 1. So, this is an important equation let me call this dagger.

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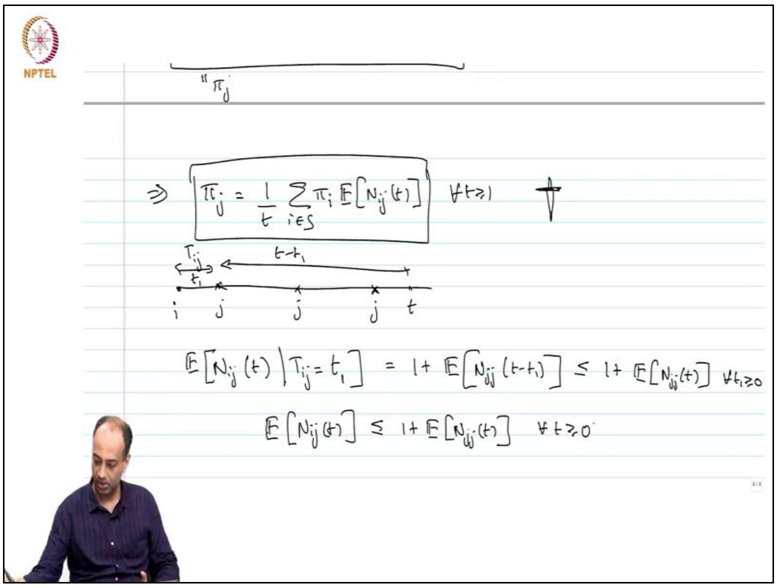

Now I have to bound. So, but I know that. So, the limit t tending to infinity of the left hand side is equal to pi j. So, that also I know. So, but let me let me do the following ah. So, this ij this i know to be equal to pi j but I will use that later maybe I should just write it down. So, this implies pi j is equal to one over t sum over i belongs to S expect pi i expected N ij of t actually let me call this is true for all t greater than or equal to 1.

So, this let me call this dagger this is an important equation now looking at expected N ij of t. So, if you are starting at i and then you are looking at the subsequent occurrences of j that is your N ij process. Now if you are looking at some time t you can write expected. So, let me call this time as T ij. So, it is the first time that the first passage time from i to j and then i have i am looking at a delay renewal process basically of returns to j.

Now this expected N ij of t can be written as. So, let me let me condition on expected n a j of t given T ij equal to some some t 1. So, I am taking this time to be equal to some t 1. This will be equal to 1 plus expected. So, you will have 1 occurrence of j and after which you are looking at n j j of t minus t 1. So, conditioning on T ij is equal to t 1 and then so, n ij of t will simply be 1 + N jj of t – t 1 correct.

So, this is t - t 1 that is t - t 1 and this is less than or equal to 1 plus expectation of N j of t. So, this is true for all t 1 and in the hand side. So, this is two for all t 1 and in the hand side there is no t 1. So, this seems to be true for all t 1. So, I can write expected i can just remove the conditioning and write expected N ij of t is less than or equal to 1 + expected N jj of t for all and this is true even if this t 1 had to be infinite this is still true you can easily verify that this equation will still trivially hold.

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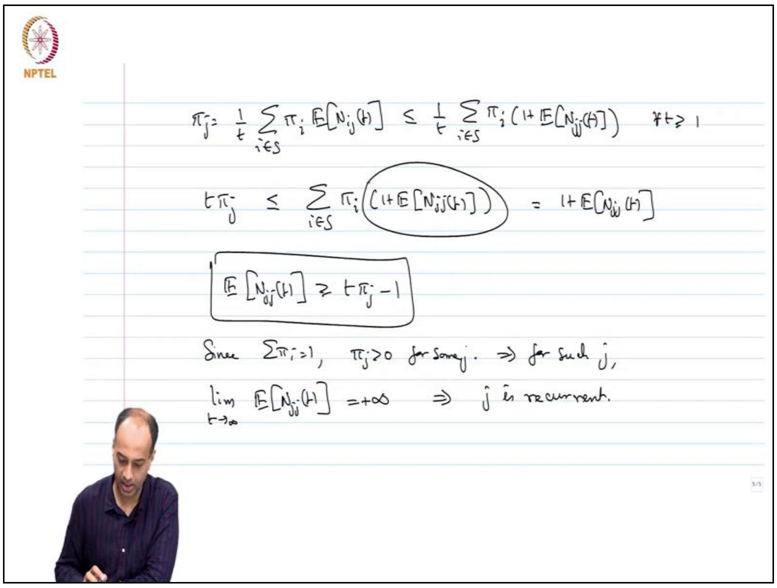

So, we have this equation which is nice. So, we can put this back into our dagger. So, then what happens. So then I have sub back in this dagger and then what happens I will get pi j equal to 1 over t sum over i belongs to S pi i expected N ij of t which is less than or equal to 1 over t sum over i belongs to S pi j 1 plus expected n j j of t this is true for all t and now what does this mean I have t pi j is less than or equal to sum over i belongs to S pi j 1 plus expectation of yeah that is expectation of N jj of t.

So, this is pi i sorry. So, this is the mistake i made i think sorry I made this mistake my apologies. So, I made this mistake. So, here this also should be pi i. So, now if you look at this guy this term is has no i in it. So, this is just sum over pi i which is one because I know that pi is a distribution. So, this becomes equal to 1 plus expectation of N jj of t which means that i can write expectation of N jj of t is greater than or equal to t pi j – 1.

This is true for all states j. Now since sum over pi i is equal to 1 pi j is strictly positive for some j otherwise they cannot sum to 1. So this implies for such j expected N jj t limit t tending to infinity is greater than or equal to t pi j - 1 and pi j is strictly positive. So, this becomes infinity plus infinity and this implies from our earlier theorem if this limit of N jj t is going up to infinity then this process is a legitimate renewal process the place that state j is is recurrent correct and if you go back to the above equation you get um pi j.

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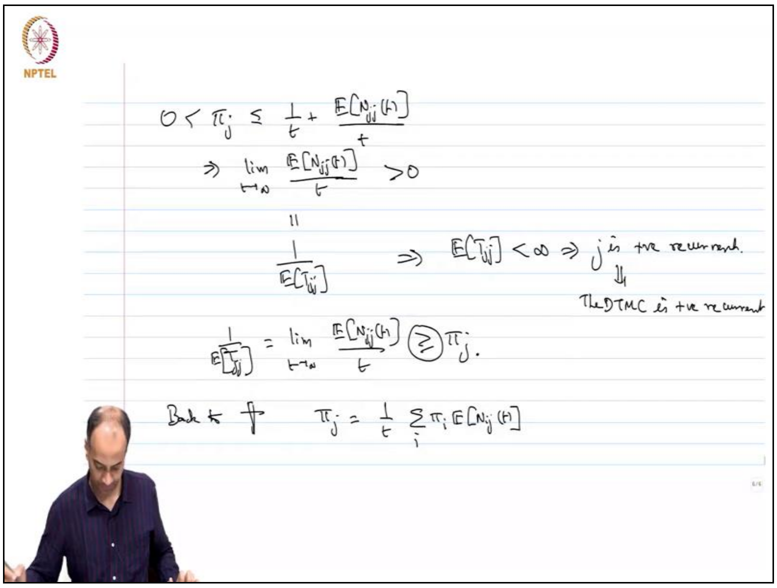

So, now you have you proven that j is recurrent you have to prove that j is positive or current now if you go back to this dagger you have pi j is less than or equal to 1 over t plus expected N jj of t this is just an upper bound on the dagger equation this is this i know to be true and this guy is strictly positive. So, this implies limit t tending to infinity expected N jj of t over t is strictly positive.

Now N jj t we already proved that N jj t is a renewal process that is because state j is recurrent we already proved the state j is recurrent out here. So, N jj t is a renewal process. Now we are saying that limit t tending to infinity expected N jj t by t is strictly positive but by elementary renewal theorem this is equal to the reciprocal of T jj. So, this implies that expected T jj is finite implies j is positive recurrent. So, not only is j recurrent j is also positive recurrent.

So, in fact we can prove that 1 over T jj expected 1 over T jj which is just limit t tending to infinity expected N jj of t over t which is by elementary inverse theorem is greater than or equal to pi j which i know is strictly positive in fact this greater than or equal to we will show holds with an equality that will be our next result in proving this theorem um. So, just to recap, so, I have proved that j is i follow the sequence of step first prove that j is recurrent by showing that expected N jj t goes off to infinity as t goes to infinity.

Then I prove that limit t tending to infinity expected N jj t by t is positively positive which means j is positive recurrent. Now the entire chain must be positive recurrent because you have one irreducible class. So, you know that all the states in the same class must be of the same type. So, if j is positive or current then the entire class must be positive recurrent. So, the DTMC itself is positive or correct.

So, from this we can prove that the DTMC is positive recurrent. Since the DTMC is reducible. So, we got that the DTMC is positive current and we got this lower bound on expect one over expected T jj is greater than or equal to pi j now we will show that there is also an upper bound. So, that pi j will be equal to 1 over expected T jj. Now you go back to dagger which says that pi j is equal to sum over.

So, 1 over t sum over i pi i sorry I keep making this mistake pi i expected N ij of t have written it correctly correct.

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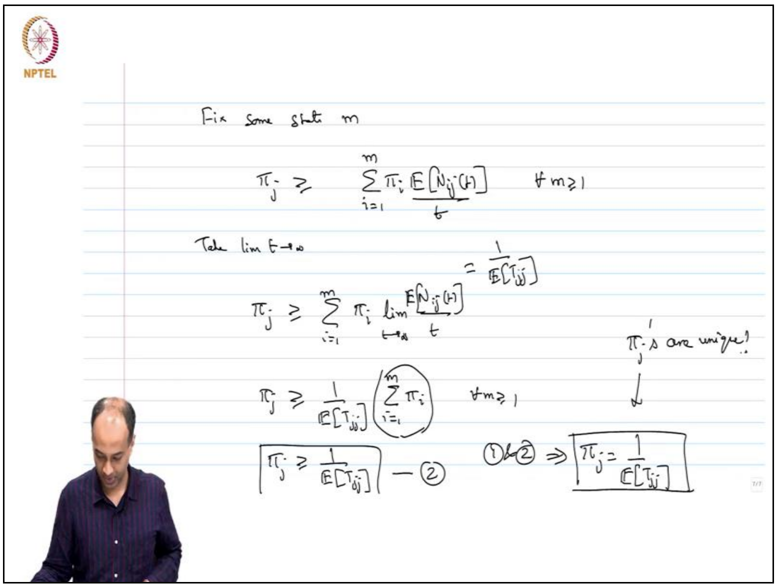

Now I am going to just go from fix some state m and then i am going to go the summation I am going to go to only till m. So, I have pi j is greater than or equal to 1 over t sum over i equal to 1 to m I am just going to go to m pi i expected N jj of t this is true for all m greater than or equal to 1. So, I am just fixing some. So, I want to take some limit inside. So, I am fixing some finite sum. The reason I am doing this is, so, that I can take.

So, take limit and turning to infinity. So, I can simply just take t inside. So, that is perfectly legitimate. Now take limit t tending to infinity then I will get pi j is greater than or equal to sum over i equals 1 to m. See I am taking the limit inside this finite sum which is why I took only finite m . So, pi j limit t tending to infinity N ij of t over t expectation expected N ij of t over t. So, now we know that the state j is positive recurrent.

So, this guy this limit by elementary renewal theorem for delayed renewal processes this will be equal to 1 over expected T jj. So, this is equal to. So, therefore pi j is greater than or equal to 1 over expected T jj times sum over i equals one to m pi i maybe I should say again i made the mistake. So, I have to write i here I keep making this mistake sorry sum over i pi this is true for all m greater than or equal to 1.

Since this is true for all greater than or equal to 1 m greater than or equal to 1 and so, I can take m as large as I want. So, and then this guy will get as close to 1 as I want. So, since this is true for all m greater than or equal to 1 i have that pi j is greater than or equal to 1 over expected T jj because if pi j were less than 1 over expected T jj I should be able to take little m large enough. So, that the scenic with this inequality above is not true.

So, I got this bound. So, I have this guy let me call this let me call this 2 and I have already upper bound is not it where do I have that this i have 1. So, I have 1 over expected T jj is greater than or equal to pi j and here I have 1 over x square T jj less than or equal to pi j. So, 1 and 2 imply that pi j is equal to 1 over expected T jj and this is true for all states j there is nothing special about this j that i fixed.

So, what have I proved? If I proved that if there is a PMF pi if there is a distribution pi i which satisfies the balance equations pi is equal to pi P and some over pi is equal to 1 then that pi i must be equal to 1 over expected T jj therefore pi j's are unique. So, whenever there is a solution pi j that pi j must be equal to 1 over expected T jj and the strain is positive recurrent. So, if you go back to theorem one I have proved theorem one.

So, I have proved this I have proved uniqueness and I have proved positive recurrence. So, I am done proving theorem 1.