Stochastic Modeling and the Theory of Queues
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Module - 1
Lecture - 6
Poisson Process - Introduction

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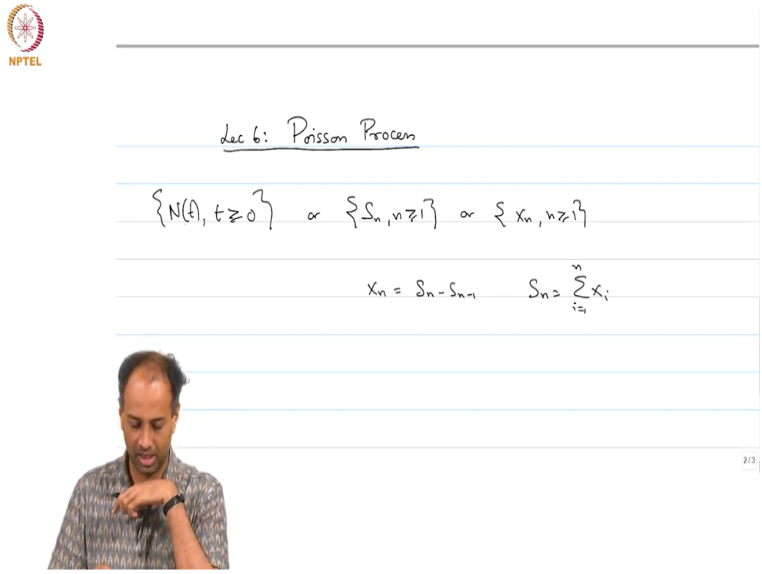

Lecture 6. So, in this, we will define what is a Poisson Process? So, recall that you can specify a counting process, either by specifying $\{N(t),\ t\ \geq 0\}$ or the sequence $\{S_n\}$ or the sequence $\{X_n\}$. We know how to go back and forth between the arrival epochs S_n and the inter-arrival times X_n. That is a simple relationship; in particular; that we know. And likewise, S_n can be obtained from the X_is using that. And also, there is some statistical equivalence that we just proved between S_n and the $N(t)$s. So, if you just go back to the proposition that we proved in the previous module;

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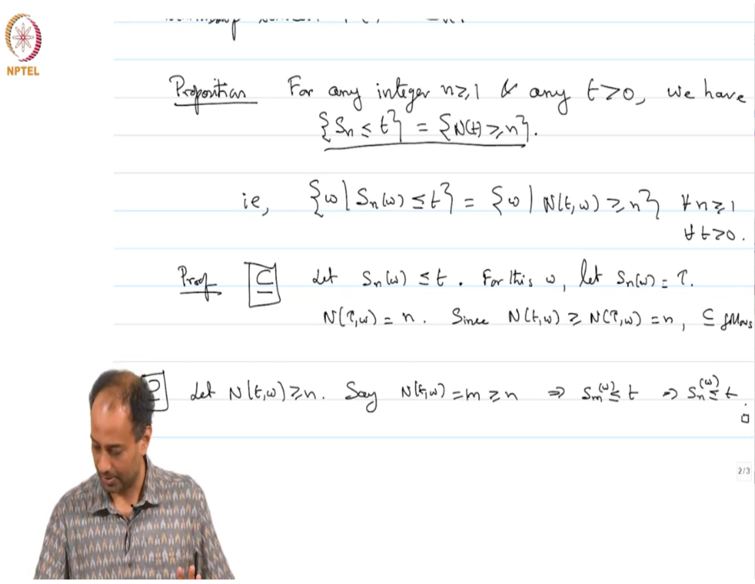

So, this event is equal to this event. So, in some sense, you can, in principle get the CDF of S_n. If you know the CDF of S_n or joint CDF of S_n, you can get the joint CDF of $N(t)$ and vice versa. So, this relationship in principle gives you a statistical equivalence between $N(t)$ and S_n. So, in principle, a statistical description of any one of these, you should be able to get the statistical description of the other two.

So, if I give you a X_i's, you should be able to go to S_i's, from which you can go to $N(t)$ and so on, or the other way around. So, what is a convenient way to define, if you want to study a particular counting process? What is the convenient sequence to study? Is it $N(t)$ or is it S_n or X_n's? Well, it depends on the situation usually. And in the case of a Poisson process, which is perhaps the most elementary counting process, you can approach it when there are; I know *at least* 3 equivalent definitions.

We will do 3 equivalent definitions in class. The one that has to do with these inter-arrival times X_n's is perhaps the easiest to understand. So, we will start with that, but they are also equivalent definitions in terms of $N(t)$'s or S_i's.

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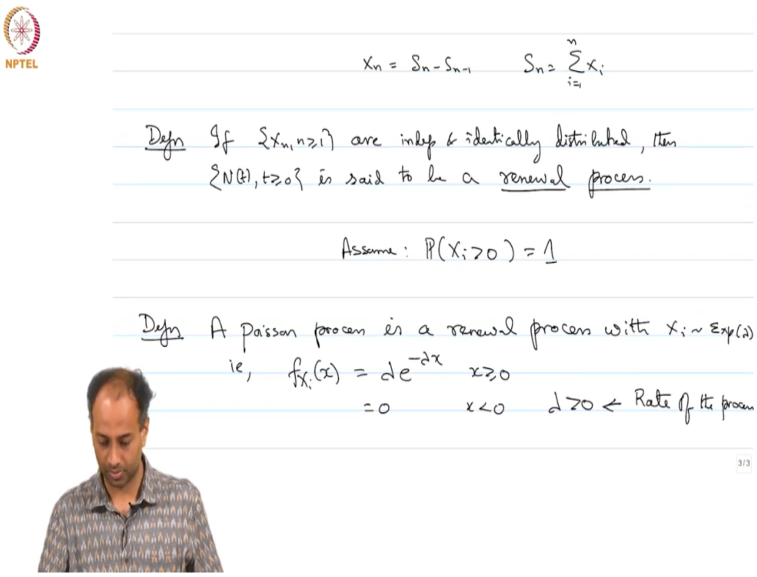

If X_i's or $\{X_n, n \geq 1\}$(inter-arrival times) are independent and identically distributed, then $N(t)$ is said to be a renewal process. So, these X_n's, remember, is the amount of time that elapses between the $(n - 1)^{th}$ arrival epoch and the n^{th} arrival epoch. It is the inter-arrival time. If these inter-arrival times; of course, these inter-arrival times are assumed to be; they are, of course, non-negative random variables.

If these X_i's are IID, independent and identically distributed, then we say that the counting process is a renewal process. It is a word. I am introducing a new terminology. So, if X_i's are IID, it is a renewal process. By the way, I should make a remark that throughout, we assume that two arrivals occurring at the same time, has 0 probability. So, in all this, we will assume that $X_n > 0$ with probability 1.

So, the probability that inter-arrival time is actually equal to 0, has 0 probability. So, with probability 1, you will have distinctly separated arrivals; you do not have 2 arrivals at the same point. This is something that we will assume throughout, because it is a little easier to deal with. It is technically easier, number 1; and number 2, in practice also, the probability of like 2 radioactive emissions happening exactly at the same time is practically 0.

In continuous time, you do not have arrivals at the same time. So, I think it makes sense to assume this. If this is not the case, if you have multiple arrivals at the same time, what you do is, you put a different distribution on a number of arrivals at that term, at a particular point.

That is, you can model it differently using a different random variable which; if a file comes to you with a certain number of packets in it, you can just model the number of files in the packet with a different random variable.

You can just say file arrival happens at this time, and a bunch of packets come in that file; something like that. So, long story cut short. X_n has; there is no atom at the origin. There is no probability mass at the origin. So, everywhere we assume that $P(X_i > 0) = 1$. This we will assume, for all i. So, that is what a renewal process is. And what is a Poisson process?

A Poisson process is a renewal process with X_i's exponentially distributed with some parameter λ. So, in other words; the density of X_i is $\lambda e^{-\lambda x}$ for $x \geq 0$and 0 for x < 0. This λ is a positive parameter called the rate of the process. So, what is a renewal process? A renewal process is a counting process in which the inter-arrival times are independent and identically distributed.

What is a Poisson process? A Poisson process is a very special renewal process. So, the inter-arrival times are not only IID, they are IID exponentially distributed, with some parameter λ. And this λ is called the rate of the Poisson process. So, that is what it is. So, for a Poisson process, it is a counting process defined on the positive time axis where the arrivals occur IID exponentially apart.

Now, what is so great about this exponential distribution? We are saying the renewal process means IID inter-arrival times. I am giving it a specific name when inter-arrival times are exponentially distributed, I am calling it by a very specific name, Poisson process, and I have a chapter dedicated to it. So, what is so special about exponential inter-arrival times? And what is so special about the exponential distribution? It satisfies a very important property called memorylessness. Exponential distribution is memoryless.

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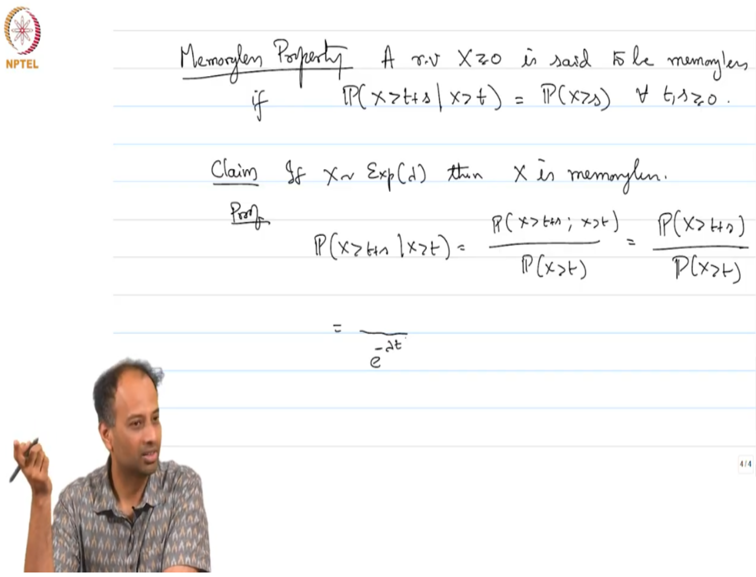

Let me just define this memoryless property. So, forget all these processes for now. Just for the purposes of the definition, non-negative random variable X is said to be memoryless if the $P(X > t + s \,|\, X > t) = P(X > s);\ \forall\, t, s \geq 0$.

So, you are looking at some random variable X. In our case, this X is an inter-arrival time. You are waiting for some arrival to come. So, the probability that you waited for t; you waited till t and there has been no arrival, meaning that the inter-arrival time has been bigger than t, given that you have waited longer than t, what is the probability that you will wait for a total time of $t + s$?

Which means, what is the probability given that you waited for time t, you will have to wait for another time s. It is the same as the probability that you waited for time s from the beginning, and this should be true for every t and every s. So, in some sense, this random variable X forgets how long it has been. That is why it is called memoryless. So, the fact that you waited a very long time or very short time, does not make it any more likely or less likely for an arrival to come soon.

So, if X could model, let us say, the lifetime of a light bulb, given that it has lasted, I do not know, 1 year, the probability it will last another month is the same as the probability that a new bulb lasts 1 month. That is what we are saying intuitively. So, this is the memoryless property. So, we can easily argue that. So, claim: You can easily argue this, if X is exponentially distributed with parameter λ.

So, which, this means X is distributed according to the exponential distribution λ. I hope you have seen the exponential distribution. Which means that X just has that sort of a distribution. If X is exponentially distributed, then X is memoryless. So, you can easily prove this. You take the LHS, which is the probability that X bigger than $t + s$, given X bigger than t; this probability of A given B.

This is probability of A intersection B over probability of B by definition of conditional probability. Now, this is the probability of $\{X > t + s\}$and $\{X > t\}$. But event that $\{X > t + s\}$ is contained in the event $\{X > t\}$. Because $\{X > t + s\}$ it is certainly bigger than t. So, this just becomes; it is like probability of A intersection B when A is contained in B. When A is contained in B, A intersection B is A, the smaller set.

So, this should just be $P(X > t + s)$ over $P(X > t)$. And you have assumed that $X$is exponentially; so far I have not assumed anything, I have just manipulated the conditional probability expression. Now, I assume X is exponential with parameter λ. Then, the denominator becomes $e^{-\lambda t}$, because the CDF will be $1 - e^{-\lambda t} = P(X \leq t)$.

So, $P(X > t) = e^{-\lambda t}$. You go back from, you know, the density. You can always calculate CDF and the complementary CDF. I hope you can do this much. The numerator will of course be $e^{-\lambda(t+s)}$ which is equal to $e^{-\lambda s}$. This cancels. This is nothing but the probability that; so, this is true for all s and t. So, I have memoryless property.

So, if X is an exponential random variable, then it is memoryless. So, if a light bulb whose lifetime is exponentially distributed, then, if I tell you that this light bulb is really old, does not mean anything. It is as good as the bulb is new. Of course, light bulbs in real life are not like that. But in this Poisson process.

So, you have these inter-arrival times which are exponentially distributed. Let us say, there is a bus process; you are waiting at a bus stop and a bus process that is going by you is a Poisson process, then the fact that you waited for a very long time does not make it any more likely for the bus to come soon. And of course, in the real world, the buses do not behave this way.

Also, what can be shown is that, among the continuous distributions, this exponential random variable is the only memoryless random variable. If X is memoryless, then it must be exponential. What we have shown is that, if X is exponential, then it is memoryless. But the converse is also true. If X is a memoryless continuous random variable, it must be exponential. So, you can add that as a remark.

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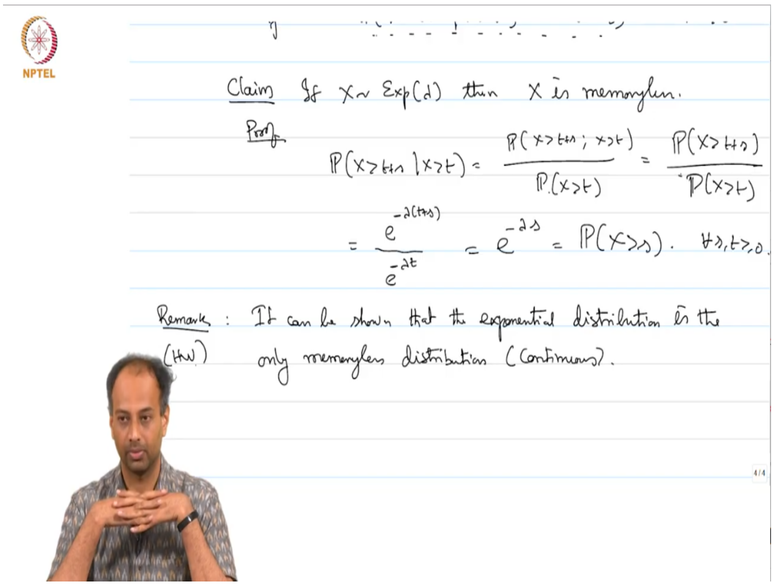

Remark: It can be shown that the exponential distribution is the only memoryless distribution, in the continuous world. If you are looking at the discrete world, discrete integer value random variables, you can have a definition of memorylessness, which is very similar to this, except the t and s will be replaced by integers m and n. In that case, for discrete random variables, you can prove that a geometric random variable is memoryless and it is the only in that world.

So, continuous world: exponential is memoryless; discrete world: geometric is memoryless. The way you prove that; since I made this remark, you can prove this. You basically go like this. So, this step to here; to here, remains the same. So, essentially, you have some sort of a functional equation. You have $\frac{P(X > t+s)}{P(X>t)} = P(X > s)$.

So, you turn that around. So, this is the complementary CDF. Complementary CDF at $t + s$ is equal to complementary CDF at t times complementary CDF at s. Now, you go ahead, take logarithm; and solve this functional equation. You can prove that this; you will get an exponential distribution. It is a little bit harder than; the forward proof is very easy, I did it in 2 steps, 3 steps. The reverse proof also you can try.

Maybe this, I will leave as homework. It is a little bit more involved, but I think you can do it. Essentially, what I am saying is that, you prove the converse of this claim that we have proved. You start with this equation that relates CCDF, the complementary CDF at $t + s$ is equal to complementary CDF at t times complementary CDF at s. You take logarithm. Then the logarithm of the CCDF will be linear, and you can get an exponential.

Try doing this yourself. So, that is why this Poisson process is so special. So, if you are looking at a Poisson process, the fact that you have not seen an arrival in a very long time does not make it any more likely or less likely to see an arrival soon. I said that buses do not behave this way, generally. What does behave very close to a Poisson process is in fact radioactive emissions.

If you are looking at a; if you take a radioactive sample and count with a counter, how many emissions do I see, these, the clicks of the counter will constitute, very closely constitute a Poisson process. The inter-arrival time, the time arrivals between any two clicks will be exponentially distributed and independent. This is very good; radioactive decay closely follows a Poisson process. The reason this is the case, we will discuss later.

So, in the bus stop example case, because it does not behave this way. Ideally, if you waited for a long time, you would like the bus to come soon. So, suppose the bus arrivals, let us say they are uniformly distributed between 5 minutes and 10 minutes, the inter-arrival times. So, the time arrival between 2 buses could be anywhere between 5 and 10 minutes, let us say for the sake of argument.

So, if I waited for, let us say, 8 minutes, I know for sure that there will be a bus in the next 2 minutes. Just to give you an example. So, it makes it more likely, in this case. But of course, the other extreme is also possible that you may have a small probability of the bus breaking down, which means the bus may not come for 2 hours. So, if you have waited for a very long time, the chances are that the bus is broken down and you will wait even longer.

So, that is also possible. This Poisson process is in between. It does not make it more likely or less likely to see an arrival soon. And what is more? This is the only continuous time counting process which has this property. That is why it is so important. And it is a good model, as I said, for radioactive decay. And it is a very good model for the arrival of phone calls into telephone exchanges.

Lot of people are connected to its exchange but only a very few people are actually making calls at any given time. So, if you look at the arrival of calls to a telephone exchange over some fixed period of time, it is reasonably well approximated by a Poisson process. So, it has many of these applications, and analytically, it has some very beautiful properties as we will see. I will stop here for today. Thank you.