Stochastic Modeling and the Theory of Queues
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Lecture –60
Stationary Distribution and the Steady State Behaviour of a Countable-state DTMC - P1

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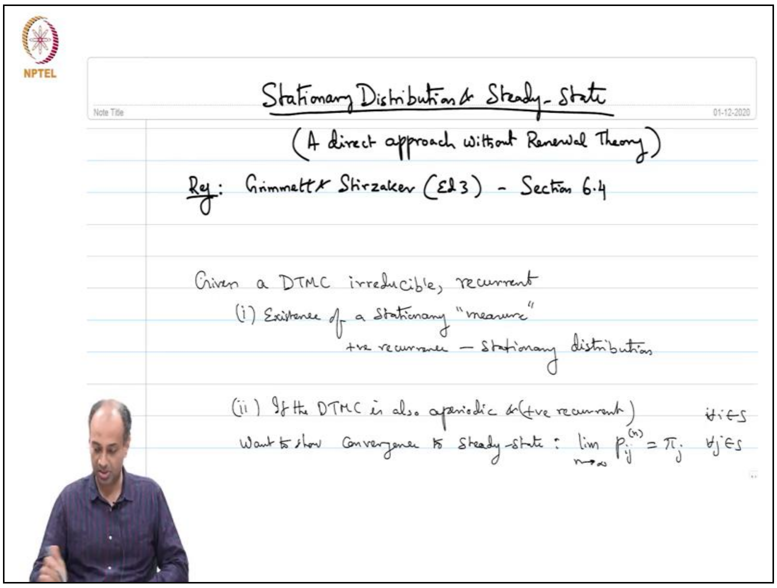

Welcome back till the last module we discussed stationary distribution and steady state behaviour of countable state Markov chains using renewal theory. So, we used renewal reward theory to interpret the steady state probability with pi i as 1 over T ii which is the expected recurrence time. We also use Blackwell's theorem to prove that P ij n converges to pi j if you have an irreducible a periodic positive recurrent Markov chain. Now all this is perfectly valid what we have done.

So, far is perfectly valid however it makes heavy use of renewal theory. Now one might argue that is renewal theory really required if it is valid to use it we have studied renewal theory but you may argue that hey so, this I do not know any renewal theory suppose can you help me learn Markov chains. It turns out that you can derive some of these results some of the key results about stationary distribution and steady state probability and all that without using renewable theory at all.

We can just use a direct elementary approach and derive some of these fundamental results about stationary distribution and steady state behaviour of Markov chains. So, that is something I want to do today not in great detail. But I will just point out how the arguments go this part is from the book by Gramat and Sterzeker's book on probability and random processes third edition section 6.4 which is this is a lovely book.

And this section is a good exposition on the stationary distribution and renewal without renewal theory all. So, my agenda for today is to roughly tell you how the stationary distribution can be constructed explicitly without using any renewal theory and how the convergence to steady state can be proved. So, I am given let us say given DTMC which is irreducible and recurrent.

I want to prove that I want to prove the existence of a stationary distribution I want to prove the existence of a stationary measure. And in the case of positive recurrence the stationary measure will be a stationary distribution I will tell you what I mean by a stationary measure it is a counting measure that we will introduce. And also for an aperiodic if the DTMC is also aperiodic. So what I mean is that it is irreducible recurrent and aperiodic positive recurrent any periodic.

Then I want to show that there is a convergence to steady state that is what i want to do is I want to show that limit n tending to infinity P ij n is equal to pi j all this is for all j in s for all in s and for all j in s. So, this is convergence to steady state. So, remember the for the first for the existence and the uniqueness of stationary measure and all that we have used for the stationary distribution we used renewable reward and for the second we use Blackwell's theorem. I want to do it without using any of these renewable theory arguments.

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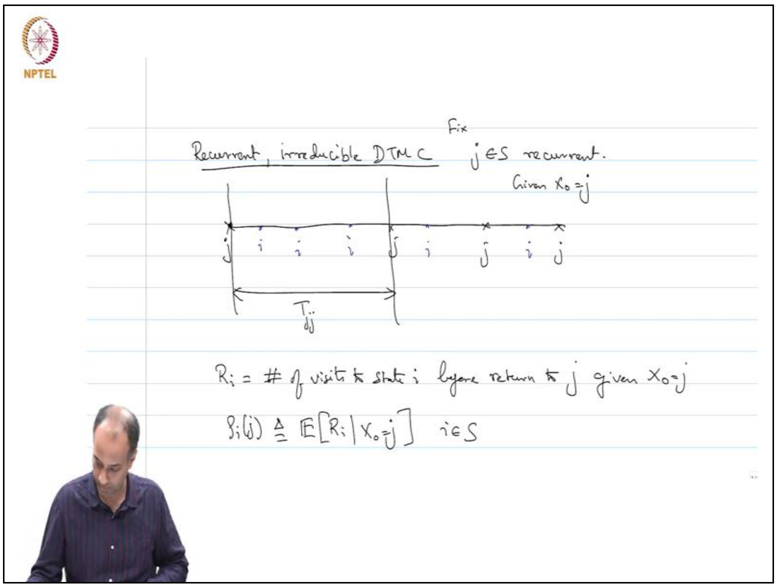

Suppose so, you are given a DTMC are given a recurrent irreducible DTMC. So, I am going to say let us say I fix j recurrent. So, I am just going to fix a recurrent state j. So that is my Markov chain running in time let us say I started j let us say given x naught equals j let us say I have a j which is a recurrent state and i start the Markov chain at state j then this is a recurrent state. So, I have to return to state j at some point and I will in fact with probability 1 I will keep returning to state j infinitely many times.

Earlier said that we said that these subsequent consecutive return times constitute a renewal process but I do not want to. I am going to pretend that I know nothing about renewal theory at all in this lecture. So, what I am going to do is you take any other state i let me mark it differently. So, let me say that i am hitting state i in these places this at these times I am going to say Ri.

So, I am going to just consider this interval between two consecutive visits to j. So, i started j and I look at the first return to j and i used to call this T jj. So, I will keep the notation but I will not use any renewal terminology I am going to define R i as the number of visits to state i. So, state i is in the same class, state is in the same class as j. Number of visits to state i before return to j given x naught equals j.

So, this is some this is some number. So, earlier we used to say R i is some kind of a random variable we said this is some expected reward inside a renewal interval and all that but forget all that it is just a random variable at this point and I am going to say I am going to call rho i of j defined as expected R i which is the expected number of visits to i given that i started at j. So, it is the expected number of visits to i before I come to j given that i started with j.

So, this I will define for I am fixing a state j and this is defined for the other states S including state j. Now I am going to use this rho i of j as some kind of a it is some kind of a counting measure. So, I am going to argue that there is a stationary measure can be con stationary measure and eventually a stationary distribution can be constructed using these rho ij's.

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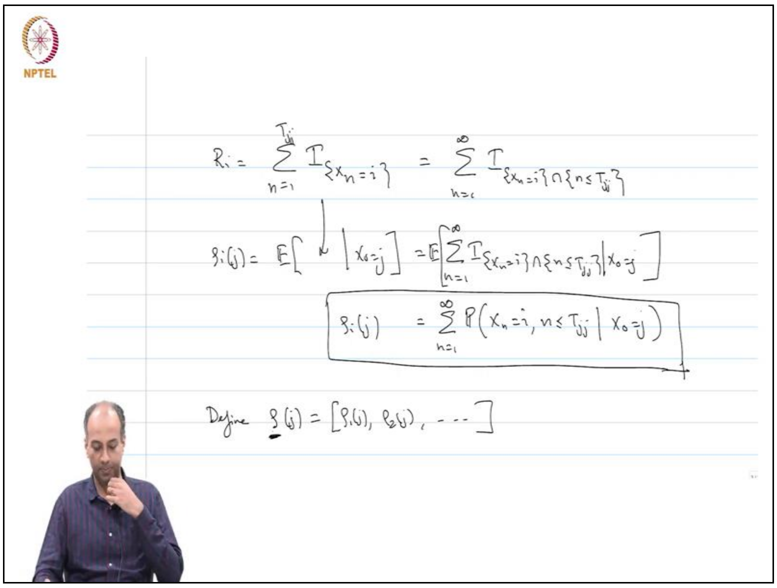

So, to be little more precise maybe I should define R i as sum over i is equal to 1 to T jj indicator x i maybe I should say n is equal to 1 to T jj indicator x n is equal to i. So, it is the number of times I am in state i before this before this stopping time T jj or I can also write this as sum over n equals 1 to infinity indicator x n equals i intersection n less than or equal to T jj this is my expected this is R i.

I can write then rho ij as the expectation of all this expected that given x naught equals j which is just sum over n equals 1 to infinity expectation of sum over 1 equals indicator x n equals i indicator sorry intersection n less than or equal to T jj. This I can take the expectation inside because all the terms are non-negative. So, this just becomes n equals 1 to infinity probability that x n sorry all this given x naught equals to bigger button x naught equals t x n is equal to i and n less than or equal to T jj given x naught equals j.

So, this is my rho ij let me go i of j. So, now my claim is that this rho i of j which is the expected number of visits to state i between two successive returns to j is some kind of a stationary measure in the sense that it satisfies the balance equations see this rho ij is will turn out to be a number I am going to argue that this is a stationary measure. So let me also define the vector rho vector rho of j as rho 1 of j rho 2 of j and all that over all the states.

So, this is a vector which contains all the rho i of j as you run over i. So, I am going to make a few claims.

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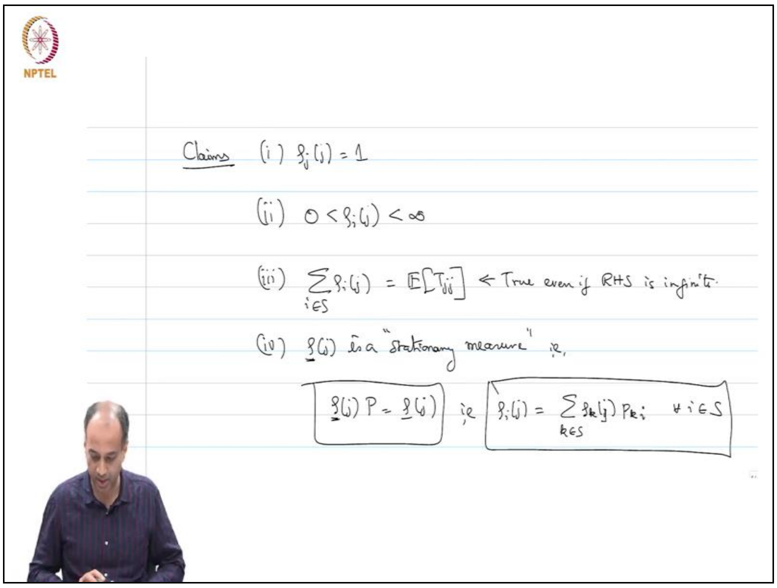

Now claims first claim is that row j of j equals 1. So, this is the expected number of visits to state j between the two visits to j and one of the final return to j you count the last return. So, in fact you he you return to j when you turn to j. So, that is the only time you count this particular reward. So, if you put I am saying reward but it is not I mean I am not really in the I mean in the renewable world we used to think of it as the reward.

You used to you basically collect one unit of reward when you return to j but it is here we are just counting and that is. So, we get one expected unit of reward. So, if you look at in fact r j will be equal to 1 in this the number of times you visit j between successive returns to j. So, that will be equal to 1 this is claim 1. Claim 2 is that these rho ij's rho i of js are strictly between zero and infinity.

Claim 3 is that if you sum over all the states is rho i of j you get expected T jj. So, the expected time between successive returns to j remember that j is recurrent. So, you do have T jj is a random variable what we are saying is that the expected T jj the expectation of this random variable is equal to sum over all the states i of rho ij and this is true even if RHS is infinite of course if T jj is a random variable expected T jj could be finite or infinite expected T jj is finite then we know that the state the chain is positive recurrent expected data is infinite the chain is null recurrent.

But fact is that if you sum over the rho ij's you get expected T jj this is true for both positive recurrent and different chains. And most important of all this vector rho, rho of j is a stationary measure ie what we can show is that rho j P equals rho j whereas I am writing this pi P is equal to pi kind of a thing except this pi I am writing instead of pi I am writing row j and this row j is not a probability distribution.

It is a stationary measure in the sense it is stationary you put rho j into p you get another you get another rho j but it is not a distribution because it does not have to sum to one it does not sum to one in general it does not come to one at all actually. Now so, this is a shorthand for saying that maybe I should write this see this P need not be a matrix P is infinite dimensional object here. So, I should really write rho i of j is equal to sum over k rho k of j P kj.

Sorry P Pk I am sorry Pk i rho k of j Pk i this is true for any this is true for this stationary measure. Let us say this rho j vector this is true for all i in S. So, what I am claiming is that this rho ij which i defined as the expected number of visits to state i between two successive which is to j. This rho ij satisfies rho jj equal to 1 which is reasonably clear because you have you basically return to j and that is the only time you count that.

And the second claim is saying that this rho ij is strictly positive and strictly finite and third is saying that the sum over all i rho ij is equal to expected time of return between j to j expected T jj and finally it is the most important it is saying that this rho j vector satisfies rho j of P rho jp is equal to rho j. We have to prove this I have indicate you how this is proved.

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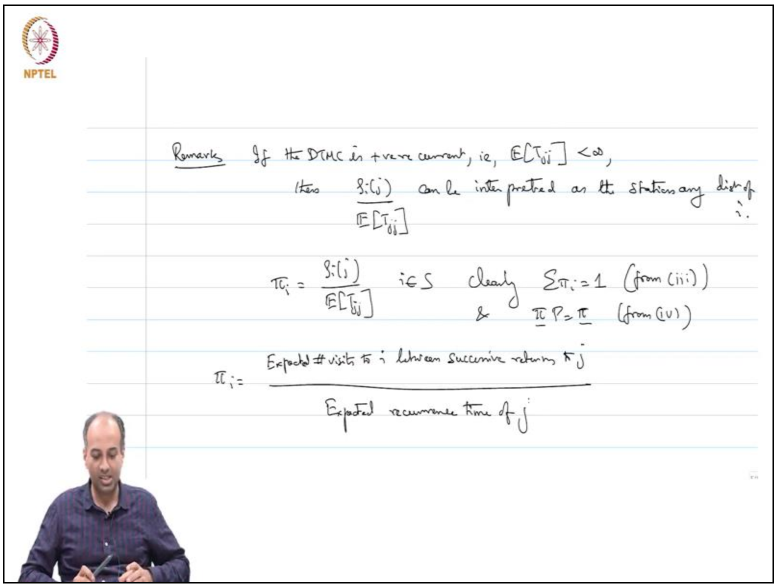

So, now let us look at what the consequences if the DTMC is positive recurrent ie if expected T jj is finite then if you take rho i of j over expected T jj can be interpreted as the stationary distribution of i. So, that what i am saying is that if you call pi i as rho i of j over expected T jj. So, I am saying. So, this is finite the denominator is finite. So, this pi i will turn out to be something strictly positive because rho j is strictly non-zero and expected T jj because i have positive recurrence is finite.

So, rho ij over expected T jj will be some positive number now this if you take this pi i and you calculated for all i and S then clearly sum over pi i is equal to one that is because I am normalizing by expected T jj and that is because sum over rho ij over x sum over rho ij is equal to expected T jj and I am normalizing by expected T jj. So, sum over equal to sum over pi equal to one this is by due to 2 sorry due to 3 which we have not proven and phi p is equal to pi from 4, property 4.

So, further if the DTMC is positive recurrent I can take this rho ij and divide it by expected T jj and interpret this as pi i call this pi i and this pi will be a distribution over all the states in the sense that it sums to 1 and pi p will be equal to pi it will be a stationary distribution of course if the state if the Markov chain is null recurrent you have the stationary measure rho ij if it still satisfies rho jp is equal to rho j.

But you know if you divide it by expected T jj you get all zeros. So, it does not sum to one. So, this pi i will all be 0 and it is not really a distribution but so, even for a null recurrent chain everything I have said before about rho ij still holds except that you do not get a probability distribution over the states in a null recurrent case. In a positive recurrent case you do get a distribution over the states.

So, what we are saying is that this is the steady state. So, pi i as the expected number of visits. So the steady state probability of steady state distribution of state i has the interpretation as expected number of visits to i between successive returns to j divided by expected say expected recurrence time of j which we are taking to be finite. So this we already knew from Renewal Reward theory.

In fact we considered a reward process which gives a reward of 1 when you visit I but the renewal process we considered was between with us to j and we used renewal reward theory to come up with this. So, this is not surprising to us at all because we have seen this using Renewal theory. But all I am saying is that you can explicitly construct this pi like. So, and from elementary principles not using any annual theory or anything we can use we can prove these claims one two three and four which I have not done which I will indicate how to do.

So, what I am saying is that you define the rho ij and if you are expected T jj is finite you divide rho ij by expected T jj. And you call it that turns out to be pi i which is your stationary distribution which we already knew from because we know renewable theory but we are constructing from first principles now.