Stochastic Modeling and the Theory of Queues
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Lecture-65
The Reversibility Markov Chains Contd…

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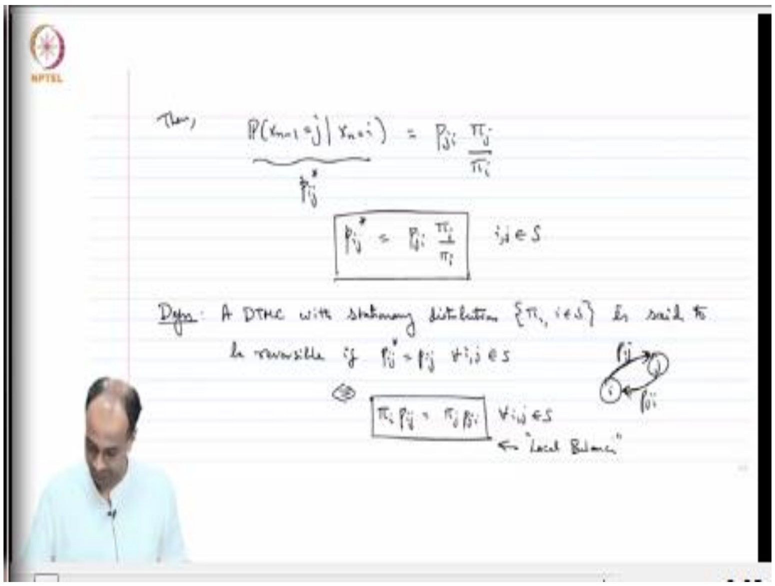

Definition, a DTMC, this definition holds for finite as well as countable states maybe I should say DTMC with stationary distribution; pi i is said to be reversible, if P ij star derived above = P ij for all ij belongs to s, i.e is the same as saying pi i, P ij = pi j P ji for all ij belongs to s. So, if pi i P ij = Pi j P ji for all ij belongs to S then we will have P ij star = P ij, which means that that transition probability from i to j in the forward chain is the same as the transition probability from i to j in the reverse chain for every ij.

Such a Markov chain is said to be reversible. So, what this is saying is? If you look at a picture this is i, this is j so you have some this is the detail balance, this is P ij, P ji. So, for any 2 pairs of states the rate of transition from i to j is the same as the rate of transition from j to i for any 2 i either it could be i to k, j to k, k to l any 2 pairs of states in the Markov chain, this should hold. This is again the local balance or sometimes known as detailed balance, local balance equations or detailed balance equations.

So, reversibility is basically local balance holding for any 2 pairs of states, it is defined in terms of local balance, excellent. So, that is what reversibility is. So, if pi j P ji = pi i P ij, then we know that the forward running Markov chain and the reverse running Markov chain are statistically completely indistinguishable. So, if somebody shows you a videotape of transitions going forward let us say i go from state i, j, k in the forward chain. And I run it in reverse the statistical properties will be identical.

So, you will not be able to tell whether the tape is running forward or tape is running backward for the reversible Markov chain, in general you can. So, in general if you take some Markov chain the reverse transition probabilities will be like P ij star which are derived here. It so happens that for reversible Markov chains by definition P ij star = P ij and you will not be able to tell by looking at the statistics of transitions, you will not be able to tell whether the time is running in this direction or whether it is running in this direction.

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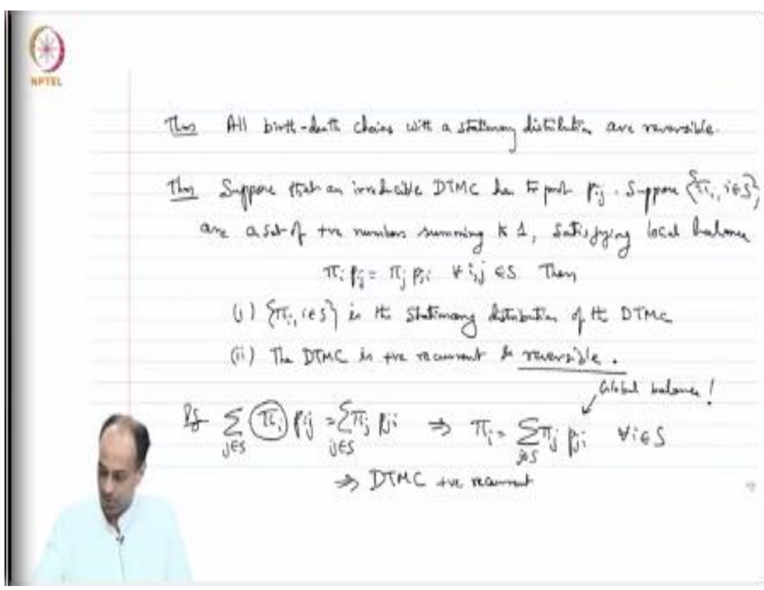

Now clearly this local balance equation is satisfied by Birth-Death chains. So, we can straightaway write down all Birth-Death chains with a stationary distribution or reversible. So, if just from our previous discussion we know that local balance is satisfied between any 2 pairs of states in Birth-Death chains. So, as long as the Birth-Death chain has a stationary distribution, has a pi, which means that the Birth-Death chain is positive recurrent.

We are just saying that all positive recurrent Birth-Death chains are reversible. So, if you look at an evolution in the forward time of a Birth-Death chain or the reverse time you will not be able to tell the difference statistically. Now it looks like to establish, so how do you establish that a Markov chain is reversible or otherwise? You need to verify whether pi i P ij = pi j P ji for every 2 parts of states i and j.

So, it looks like we are in an unhappy situation where you need to first figure out the stationary distribution pi i before you can even tell whether the Markov chain is reversible or not. It turns out that this is not necessary. There is a theorem that says that if you manage to guess some probability is pi i which satisfies local balance then these pi i's are in fact your stationary probabilities, stationary distributions and that the underlying Markov chain is positive recurrent and reversible.

Suppose that an irreducible DTMC has transition probabilities P ij and suppose pi i are a set of positive numbers summing to 1, satisfying local balance i, pi i P ij = pi j P ji for all i, j in S. Then 3 things come for free, pi i are the, so pi i may be pi is the stationary distribution of the DTMC, the DTMC is positive recurrent and reversible. So, this theorem is saying that see pi i you somehow find some pi i's which sum to 1, satisfying local balance.

So, you are given P ij the transition probabilities and you somehow managed to guess these pi i's which are a probability distribution, such that local balance is satisfied between any 2 pairs of states. So, you do not know this is what these pi i's are, you just manage to guess some numbers which sum to 1 and satisfying local balance. Then what happens is? This pi i are in fact the stationary distribution of the Markov chain, this pi i's are unique and this is the stationary distribution.

And then the Markov chain is positive recurrent and reversible, so you get all of this for free. So, if you somehow manage to guess these pi i's by looking at the structure of your chain or looking at your P ij's then you immediately get reversibility and you have automatically found out the stationary distribution. So, proving this is actually easy, so since you have pi i P ij = pi j P ji. So, you know that this sum to 1, so you just sum this over j, this is true for all ij.

So, I am just summing both sides over j, so if we look at the left hand side sum over, so pi i this is independent, this has no, nothing to do with the ji at all. So, I can pull that out of summation and then I have some P ij which is simply 1, sum over j P ij is 1, so I get pi i = sum over pi j P ji, now what is that? This is global balance. So, what have we shown? Local balance automatically implies global balance, because sum over pi i = 1, it is already given they sum to 1.

And we are looking at sum over j of the local balance equations I have pi i = sum over j pi j P ji. So, I am able to recover global balance from local balance, which means that local balance equations imply global balance. Converse is not true; if global balance is satisfied it is absolutely not the case that local balance should be satisfy, not at all the case. In that case if you satisfy global balance it does not imply your local balance, if it did then all Markov chains would be reversible, that is absolutely not true. But local balance always implies global balance.

And since we have these pi i's which satisfy local balance and sum over pi i = 1, local balance implied global balance, so this pi i satisfies sum over pi = 1 and global balance. And we already know that if pi i satisfy global balance, then these pi i's are unique and the underlying Markov chain is positive recurrent. And these pi i's are the unique stationary distribution of the Markov chain. So, this implies DTMC's positive recurrent and it also implies that pi i is the stationary distribution.

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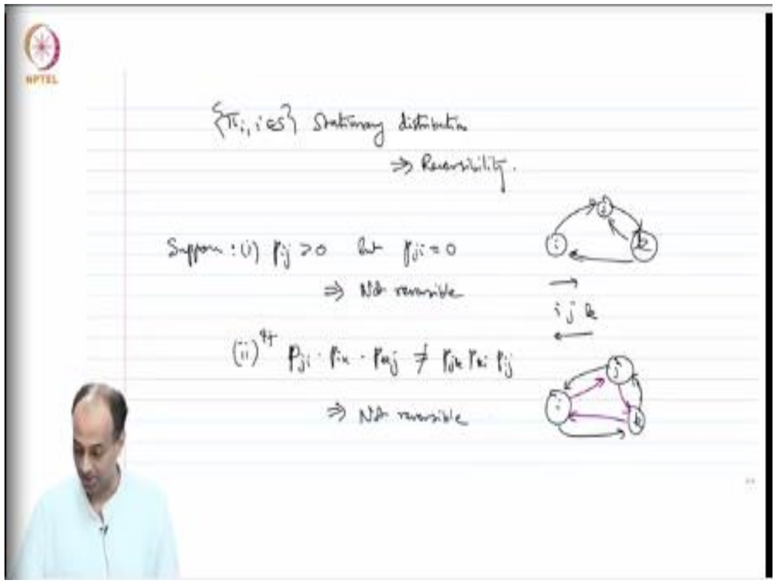

So, the pi that you manage to guess which satisfies local balance or normalization automatically becomes the stationary distribution of the Markov chain. Now that pi i have the interpretation of stationary distribution, you go back to local balance that implies reversibility. Now the local balance, so this implies reversibility also. Because now pi i's are the interpretation of this stationary distribution and then this pi i P ij = pi j P ji is satisfied for any 2 pairs of states and therefore the Markov chain is reversible.

So, this is a very nice elegant theorem which says that if you somehow manage to find these pi i's which are normalized and satisfy local balance. Then these pi i's are the steady state probabilities and that the underlying Markov chain is positive recurrent and reversible, this is a very useful theorem. There is one more related theorem which there is a theorem maybe I should not spend too much time on this.

So, if you find this theorem 6.5.3 in Gallagher which says that if you fine is to find any pi i's and some other P ji star with satisfy P ji star = pi i P ij over pi j. Then the Markov chain is such that it is positive recurrent pi i's are the steady state probabilities and P ji star that you found are the transition probabilities of the reverse Markov chain, that also can be proven. But I think this theorem is much more useful although the next theorem I stated is more general, I think this is probably more useful.

So, this pi i P ij = pi j P ji is quite useful. Now if you have a reverse, so there are some quick tests for reversibility that you can make. So, suppose P ij greater than 0 but P ji = 0, so you can go from i to j, so there is a k and all that. So, you can go from i to j but you cannot go from j to i, let us say there is a Markov chain like that. This Markov chain can never be reversible, why? Because pi j P ji will never be equal to pi i P jj, intuitively the forward chain you will say see transitions from i to j. But in the reverse chain, you will never find transitions from i to j, you will only see transitions from j to i in the reverse chain.

Because in the forward chain you will have transitions like i, j, k and all that, but in the reverse chain you will never be able to find transitions from i to j, you will always find transform j to i. Because reverse chain transition from i to j would mean a forward transition from j to i and that is not possible because P ji is 0. So, if you have a situation where P ij is greater than 0 but P ji is 0 then you straightaway know that the Markov chain is not reversible.

And similarly for example if it so happens that P let us say for this for 3 states i, j, k, if it so happens that P ji P ik P kj is not equal to the opposite P jk P ki P ij. Suppose this is the case, so what am I saying? So, this is i, j, k, so P j, so probability of going from j to i and then i to k and k to j. If you are multiplying; all these probabilities if this product is not the same as P jk P ki P ij. If this product on the left is not equal to product on the right, then this chain can never be reversible.

Because the probability of going from i to j to k is different from the probability of going, so this probability in going the cycle in one direction is different from going in the other direction. So, the forward statistics or reverse statistic will not be the same in which case you can tell the difference, so the Markov chain cannot be reversible.

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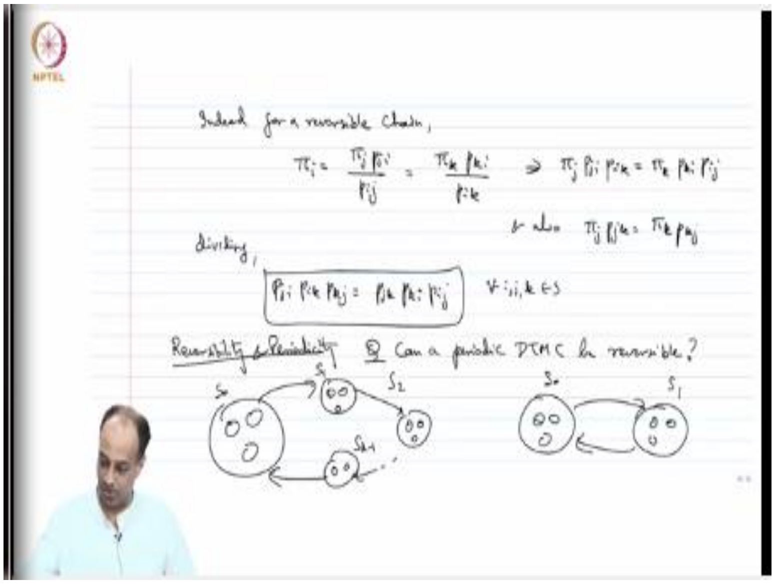

So, indeed if it were reversible you have that pi i should be equal to pi j P ji over P ij, that is equal to if it were reversible it should satisfy pi k P ki over P ik. This should be true if the chain were reversible this should be true which implies pi j P ji P ik = pi k P ki P ij, so this should be true. And also we should have pi j P jk = pi k P kj. So, if you divide these 2 equations, so these 2 equations have to hold if the chain were reversible. And if you divide this equation you will get P ji P ik P kj = P jk P ki P ij, so what this is saying is that?

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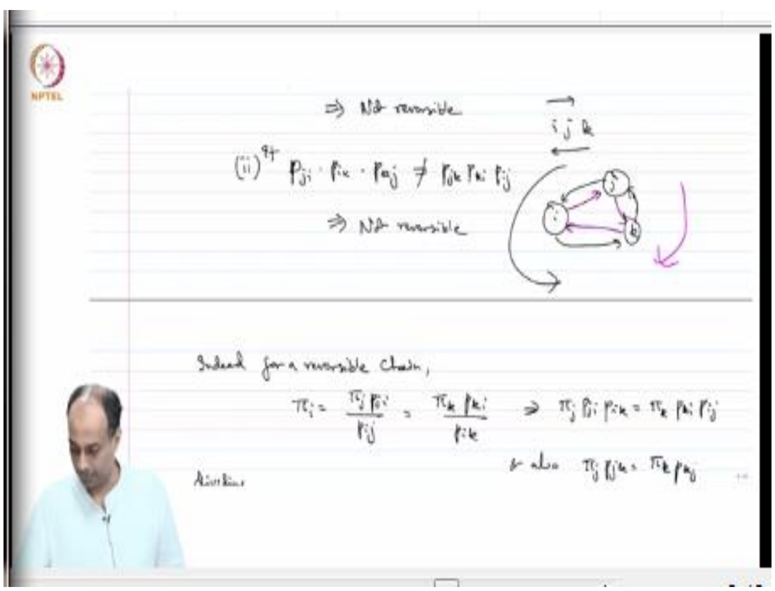

If you product the probability is going that way and the probability is going that way, they should be equal, this should be true for all i, j, k. Even if for 1 triple i, j, k if this is not true then the chain will not be reversible. Similarly you can show this for cycles of length 4 cycles of length 5, for any cycle of length 4 you should have the probability of going in this way is equal to probability of going the other way.

Even if there is some cycle where the probability of going in this way is not equal going in the opposite way then the chain cannot be reversible. So, it is necessary what is surprising? So, for every cycle a reversibility would; so it is necessary that the product of probabilities this way should be equal to the product of probabilities in the opposite direction. But it turns out that it is also sufficient, if for every cycle if the product of probability is going one way is equal to the product of going the other way for every cycle.

Then you can show that the chain is in fact reversible that is a more non-trivial result. But it turns out that it is both necessary and sufficient, that is anyway just an aside. So, before I conclude this discussion, I want to talk about reversibility and periodicity. The question is, can a periodic DTMC be reversible? So far we discussed periodicity long ago, which means that returns to the same state happen only in multiples of it is certain d.

If that is the case then we say that the state is periodic with period d, where d is greater than 1. Now can a periodic DTMC be reversible? Remember, that for a periodic DTMC let us say of period d, I can subdivide the states into these classes, where all the transitions happen from S 0 to S 1, S 1 to S 2 dot, dot, dot and then S d - 1 to S 0. You can always subdivide the states into these subclasses, where all the transitions go from S 0 to S 1, S 1 to S 2, S 2 to S 3, S d - 1 to S 0, all transitions go this way. Now imagine what reversibility means? You cannot have a, see then it is not possible to go from S 0 to S d - 1, S d - 1 to S d - 2 and so on.

Because all transition only go this way, if I run the time in reverse all the transitions will go in the reverse direction and therefore the statistics will not be the same. So, in this picture it appears that a periodic Markov chain cannot be reversible except if d = 2, if d is = 2 what happens? You have only a partition of 2 that is S 0, that is S 1. So, all transitions go from S 0 to S 1 and S 1 to S 0. So, you will get S 0, S 1, S 0, S 1, S 0, S 1 reverse change will be S 1, S 0, S 1, S 0 which is exactly the same sequence. So, the answer is a periodic DTMC can be reversible only if the period is 2.

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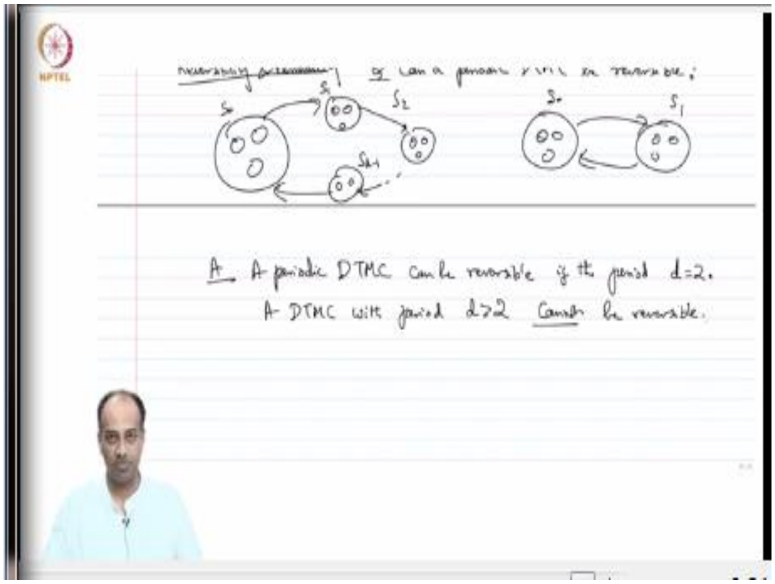

This argument is enough to show, but answer a periodic DTMC, a DTMC with period d greater than 2 cannot be reversible due to the argument that we just made. Because the sequence S 0, S 1, S 2 will occur in the reverse direction and you can tell the difference. So, with that I will stop the discussion on reversibility.