Stochastic Modeling and the Theory of Queues
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Lecture-66
Time Sampled M/M/1 Queue and the Burke's Theorem

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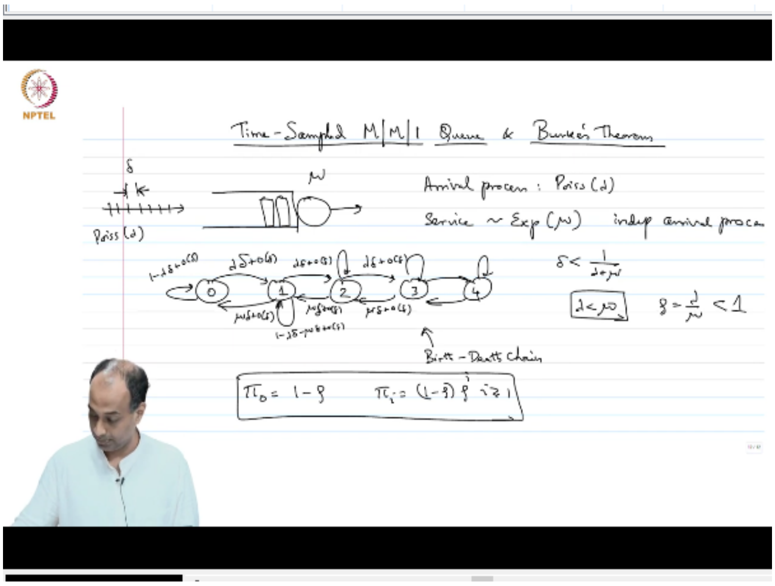

Welcome back. In this module we will discuss a time-sampled M/M/1 queue and we will discuss a very important theorem about the M/M/1 queue known as Burke's theorem. So, you already know what an M/M/1 queue is so you have a arrival process which is poisson of rate lambda, service is exponentially distributed, IID exponentially distributed across customers with some parameter mu.

And this service process is independent of arrival process. So, this is what an M/M/1 queue is; there is one server. Now if you look at some small delta time interval delta you can look at the system evolving in the small time steps delta and that will lead us to a countable state markov chain as we will see. So, in each of these little intervals delta the probability of getting an arrival is lambda delta to o delta.

That is because of the poisson process. So, in each of these small micro intervals you have probability lambda delta plus o delta of having an arrival and probability 1 - lambda delta o plus of having no arrival. Likewise the service process is also exponential it is a memory of this process. So, if there is a customer in service the probability that particular customer complete service in a particular interval lambda delta is mu delta or mu delta will be the o delta.

So, this can be used to derive a markov chain like so, so I am going to denote the state of the queue by the number of customers in the system. So, there is no limit to how many customers there are in M/M/1 queue. When you are in state zero the queue is empty; you can have an arrival in this tiny little delta interval; you can have an arrival with probability lambda delta plus o delta and no arrival the probability 1 - lambda delta plus o delta.

If you are in state 1 which means there is 1 customer 3 things can happen, you can have an arrival with probability lambda delta plus o delta, the customer in service can finish his or her service and leave with probability mu delta plus o delta or you can just stay put you can have neither which is with probability 1 - lambda delta - mu delta plus o delta. So, this is also mu delta plus o delta and so on.

So, all the forward transition probabilities are lambda delta plus o delta the reverse transition probabilities are mu delta plus o delta. So, I am just assuming that delta is very small, so that mu delta is still a probability. So, this picture is valid as long as delta is less than 1 over lambda + mu. Otherwise this will not be probabilities; this will not make any sense. So, if I choose delta small enough this is a valid picture.

So, this is basically a Birth-Death chain, if you are willing to neglect these o probabilities of having 2 arrivals. For example you can jump from 1 to 3, in a small interval probability o delta. The probability is not 0 but I am not drawing these transitions here because I am just neglecting them. So, I am neglecting o delta terms you can theoretically have 2 arrivals, but I am neglecting it.

With that understanding this becomes a fairly good approximation of an M/M/1 queue and it is a very good approximation if delta is very small. So, with the approximation that delta is very small you get a Birth-Death chain. This is a Birth-Death chain. So, already know quite a bit about Birth-Death chains, we know that the markov chain is reversible. We also know what its steady state probability is stationary distribution is.

So, we are going to assume that lambda is less than mu, under this regime when lambda is less than mu you will get positive recurrence. So, rho = lambda over mu. You know the forward transition probability divided by the reverse transition probability with this ratio I am taking to be rho which is less than 1. In this case from earlier calculations I know that pi 0 is simply 1 - rho and pi is simply 1 - rho times dou to the i; for i greater than or equal to 1. So, in this markov chain the stationary distribution corresponding to the i customers being present is simply 1 - rho times rho to i which is like a geometric distribution except it also takes the value 0.

It is like a shifted geometric distribution and with this distribution you can calculate the expected number of customers and all that. And once you know the expected number of customers you can calculate expected waiting time using Little's law all of that we have studied before. So, this formula is useful. My main topic of discussion here is reversibility, this is a Birth-Death chain with this pi I know that Birth-Death chains are reversible.

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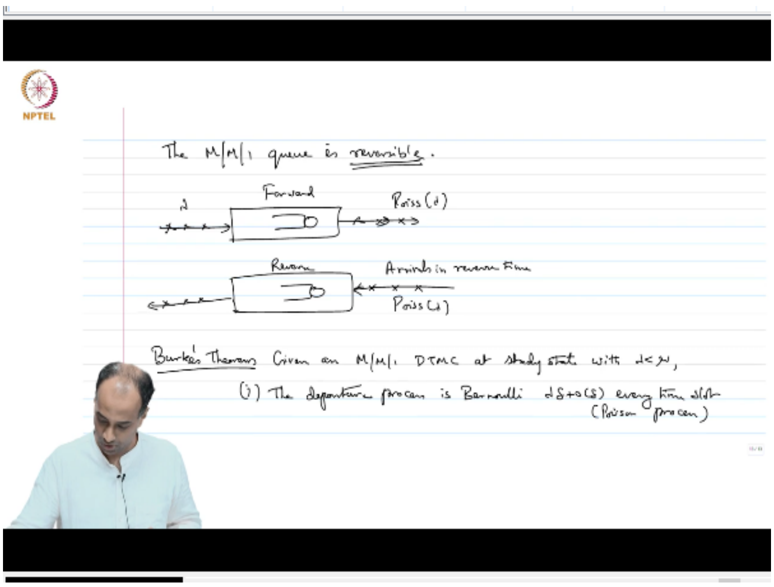

So, the M/M/1 queue is reversible, I should really be saying the time sample markov chain of the M/M/1 queue is reversible, I am just asserting that the M/M/1 queue is reversible, in fact M/M/1 queue corresponds to a continuous time markov process which we will study later but the statement is correct as a continuous time process in fact the M/M/1 queue is reversible also.

What really I am saying in this context is that the time sample M/M/1**,** markov chain is reversible. So, what does that mean? That means that if an M/M/1 queue is running in forward time and I tape the queue running and then I play the tape in reverse, I should not be able to distinguish the forward tape and the reverse tape. That is what reversibility means.

So, suppose I have a box, there is some M/M/1 queue inside, I do not know that there is an M/M/1 queue inside, I do not know what is happening inside and all I do not know the only thing I see is that there is a arrival process coming at me at rate lambda. In forward time and then customers are leaving. So, when I play the tape in forward direction so there is a box, customers are coming and then customers are leaving. That is all that I will see.

In reverse time what happens if I just play the time in reverse I do not know what is inside, all the departures from the queue will seem like they are arriving into the queue and all the arrivals in forward time will seem like departures out of the reverse queue, what we are saying is that if you do not look inside the box this process looks I mean if you just play the process of arrivals and departures you will not be able to tell whether the chain is running this way or time is running this way.

So, it is the chain is same M/M/1 queue inside, this is just I am just reversing the tape, this is forward, this is reverse. So, these look like arrivals, reverse time. Now what am I saying the reverse process is indistinguishable from the forward process which means the arrivals here have to be poisson of rate lambda, because the reverse chain is also a M/M/1 queue. So, this looks like the arrival to the reverse process should also be a poisson process because the reverse process is an M/M/1 queue, the statistically indistinguishable.

So, the arrival process to the reverse queue should be indistinguishable statistically from the arrival process to the forward queue. But the arrival process to the reverse queue is the same as the departure process from the forward queue, because you are seeing this departure process in forward time or in reverse time. So, the departure process from the forward queue looks like the arrival process to the reverse queue.

The arrival process to the reverse queue has to be a poisson process because of reversibility. So, the departure process from the forward queue has to be a position process of rate lambda or it has to be Bernoulli lambda delta plus o delta. So, this is a very interesting finding I said this in a somewhat hand waving way but this can be formalized the departure process from a M/M/1 queue is a poisson process of rate lambda.

This may be surprising to you, I mean the departure process has to be of rate lambda because all the customers that come have to leave but what is surprising is the departure process is not only a process of rate lambda it is a poisson process of rate lambda. This may be little confusing because you may think that the departures are happening at exponential mu, but departures happen at exponential mu when only when the customers are present.

When the customers are not present there is no departures, but what we are saying is that the unconditional departure process is poison lambda from an M/M/1 queue. So, this is formalized in a theorem called Burke's theorem which I will just state given an M/M/1 queue time sampled DTMC at steady state with lambda less than mu Burke's theorem says 2 things the departure process is Bernoulli with probability of departure lambda delta to o delta every IID across time slot so this is in fact a poisson process.

So, the reason I did not say is the poisson process directly is because I am looking at the time sampled chain. So, if I look at this time sample little delta intervals the departure process looks like a Bernoulli IID process with probability of departure equal to lambda delta with o delta which implies that the departure process in continuous time will be a poisson process.

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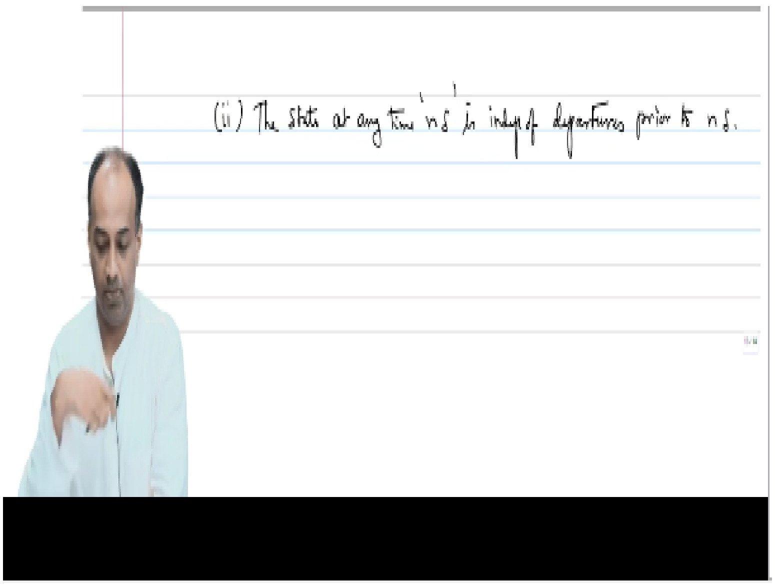

And second which is also quite interesting; the state at any time n delta is independent of departures prior to handle that. So, let us say you are looking at some time n delta. What we are saying is that the number of customers the state of the system at time n delta is independent of past departures in the M/M/1 queue. This may seem a little surprising because you may think that you know if I just told you that in the last let us say a few slots I had a lot of departures.

Let us say in the last few slots a lot of departures you may think that well the queue must be relatively empty. That is not true in an M/M/1 queue, past departures are independent of the current state in an M/M/1 queue and this is a consequence of our reversible of the M/M/1 queue chain. In fact if you look at the forward chain; in the forward chain we want to prove that the state at current time is independent of past departures.

But if you reverse the chain if you run it in reverse time so we want to show that the occupancy of the queue in the forward chain is independent of the number of departures in the past and the time of their departures. But if you look at the reverse chain past departures in the forward chain are nothing but future arrivals in the reverse chain, future arrivals for the reverse chain.

So, the reverse chain is an M/M/1 queue. Future arrivals are independent of past arrivals and past services in an M/M/1 queue. So, the numbers of people in the queue in the reverse chain is only a function of past arrivals and past services. So, for the reverse queue future arrivals are independent of the current state, but in the reverse queue future arrivals are nothing but the past departures in the forward queue.

So, at any time n delta the state of the forward queue is independent of the departures in the past. So, this seems like a very reasonable straightforward argument if you use reversibility and use the fact that the reverse process is also a legitimate M/M/1 queue. Proving these things directly is fairly hard, even proving that the departure process from an M/M/1 queue is a poisson process directly is not easy, it is quite hard.

But using reversibility it becomes very natural; likewise proving that past departures and current state of the queue are independent in an M/M/1 queue is very direct using reversibility arguments. These 2 statements are given by this very important theorem called Burke's theorem, which is a consequence of reversibility, I stop here.