Stochastic Modeling and the Theory of Queues
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Lecture-68
Introduction to CTMC (Contd.)

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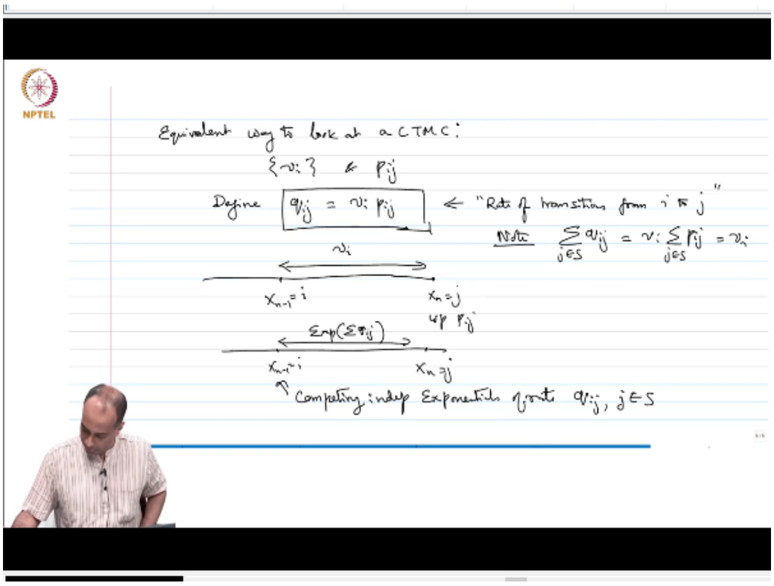

There is another equivalent way of looking at this the CTMC's that I want to talk about. So, we have talked about the embedded DTMC with transition probabilities P ij and the exponential holding times of rate nu i if you are in state i. Equivalent way to look at a CTMC. Let us say so you have this nu i's for each state which are exponential and P ij which is given to you.

Now you define q ij = nu i P ij, I am just defining something, q ij = nu i times P ij. Now what does this q ij represent? So, you can view this markov chain CTMC as being spending an exponential amount of time nu i in each state and then going to some state j with probability P ij. So, you can try look at this nu i times P ij which is now our q ij as the rate at which transitions happen from i to j. So, q ij can be interpreted as the rate of transitions from i to j.

So, an equivalent way to look at this is that let us say this is my CTMC time axis. Let us say that X n - 1 = i. Now so one way to look at it is that there is a exponential of rate nu i which is about to fire and after the end of the exponential I will go with probability P ij to state j and so you can say that X n = j with probability P ij. This is one perspective. Another perspective is to say that I am in state i X n - 1 I got to state i.

Now there are a whole bunch of competing exponentials with rates q ij. So, whenever you are in state i think of q ij where j runs over all the other states as being independent competing exponentials. Now so this q ij’s you fix an i this q ij’s where j runs over all the other states are and you think of these exponentials of rate q ij as competing now whichever exponential wins you go to that state.

Now what is the probability that of all these q ij exponentials which are competing they are independent exponentials, what is the probability that a particular exponential into state k wins? It will be q ik over sum over all the q ij's. Now what is sum over all the q ij's, sum over all the q ij’s is just nu i. So, note sum over q ij of all the other states is just ah nu i sum over P ij sum over all the states which is just nu i.

So, you have all these q ij competing exponentials. So, let me say I am in state i at X n - 1. So, I can have competing exponentials, independent exponentials of rate q ij, j belongs to the states. Now what is the probability that I actually transition into a particular state let us say k. The probability that the kth exponential wins is nothing but q ik over sum over all the q ij's. Some over all the q ij is nu i as we just saw. So, it is the probability that the q ik exponential wins will be q ik over nu i which will be P ik which is the probability of transition into state k.

So, instead of having an embedded markov chain with transition probabilities P ik and this holding time exponential which is nu i you can equivalently think of you can forget about the P ij’s, you can just say I have independent competing exponentials of rates q ij and this q ij sum to nu i. So, this is basically like splitting and merging of independent poisson processes if you will.

So, this perspective is also equally valid, you can either have exponential holding times and then P ij transitions or you can just have q ij exponentials independent exponentials where j ranges over all the other states competing. Whenever you are in state i you have independent competing exponentials of rate q ij.

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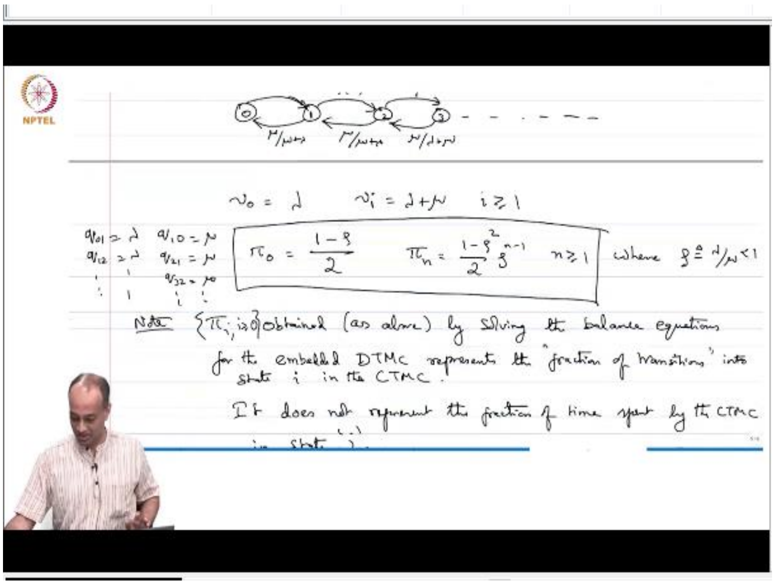

So, if you will go back to the M/M/1 example these are the nu i’s. What are the q's, q ij, for this example q 01 will be lambda, q 12 will also be lambda and so on and q 10, q 21 all that will be mu, q 32 all that will be mu. So, q ij is just nu i times P ij you can look at it that way or you can look at it as whenever I am in state 0 I have only one exponential a lambda exponential. In all other states I have a competition between the arrival exponential and the service exponential which are of rates lambda and mu respectively.

So, in any CTMC this perspective is valid, you can just say that whenever I am in state i I have competing exponentials q ij for every state j there is an exponential q ij independent for across the states and whichever exponential wins I will transition to that state and we know that the probability of transitioning to that state will be according to competing poisson processes, it is basically q ij over nu i.

And conditioned on transitioning to one of these q ij exponential winning the holding time is still exponential with parameter sum over q ij; that also we know from poisson processes. So, it is back to our original perspective, it is not as though if the jth exponential wins versus if the kth exponential wins the conditional holding times are no different; this is a fundamental property of the poisson process if you remember, you can go back and check if you want. So, if the jth exponential wins the conditional holding time is still exponential of rate nu i, it is not exponential of rate q ij.

Because all it happens to win among a bunch of exponentials, this is a fundamental property that we have studied in poisson processes. So, these are 2 equivalent perspectives and depending on whichever exponential wins you go to that state. If jth exponential wins you get X n = j and this holding time will be you can show that it will be exponential with parameter sum over q ij which is simply nu i. Good, this is just a equivalent way of looking at CTMC's.

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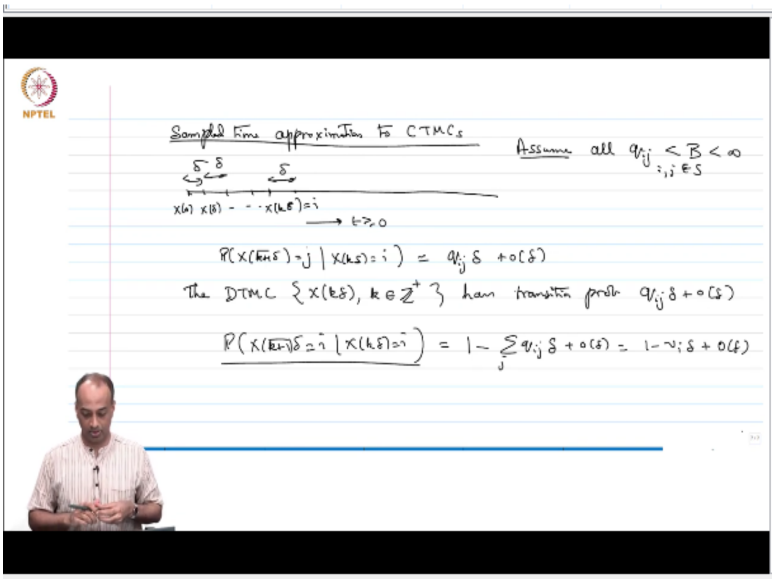

There is one more concept I want to introduce which is time sampled markov chain. See this is the perspective here is that see I have a CTMC. Now I am going to divide time into some very, very small intervals of time delta and look at what really happens in these small time intervals. So, I am going to divide time like this little, little deltas. Time is continuous the CTMC after all, but I am going to divide my time into these little delta intervals.

Now I am going to look at how do these transitions happen in these little delta intervals. Suppose that some particular so maybe I should call this see I do not want to call it again X n because X ns are the embedded markov chain. Now this is a time sample approximation. This is also will turn out to be a DTMC but it is a different DTMC from the embedded DTMC. This is the sample time approximation to the CTMC.

So, maybe I should call this X of 0, X of delta, X of k delta and so on. So, let us say that I am in some state i, X of k delta = i, I mean some state i. Now what is the probability? That let us say in another little delta interval what is the probability that I go to some state j. If I look at this probability that X of k + 1 delta = j given X of k delta = i. Of course I can also condition on the previous guys.

I can also talk about what happened in k - 1 delta and all that. But because of the markov property all that is irrelevant; we already know that I can just take out all the previous condition x, only X of k delta = i that is the only thing that matters. So, you can clearly I mean you can easily argue that this process satisfies a markov property in this index k. Whenever you are looking at this time intervals delta there is a markov property and what is this equal to?

So, this is the probability, so remember now we can look at this competing exponential perspective. So, you are at X of k delta = i, it does not matter how long you been in the state i because the exponential is memory less. So, now you have a bunch of competing exponentials. All these q ij exponentials. You will have probability of X k + delta k + 1 delta = j if the q ij exponential happens to fire in this little delta interval and of course the probability of 2 exponentials firing in this little delta interval is very small it is o delta.

So, this can be written as q ij delta + o delta. So, q ij times delta I mean if you ignore this o delta term q ij times delta gives you some kind of a transition probability from state i to state j in this tiny little delta intervals. So, you can view the markov chain, so you can view the DTMC which is the process X of k delta. See remember this X of t is the CTMC. I am looking at the CTMC of these k delta intervals.

X of k delta, where k is greater than or equal to k is takes non negative integer values. Let me write it as z +, it is of course a DTMC has transition probabilities q ij delta + o delta. We are going to assume all q ijs is less than some b which is finite, for all ij. So, remember this state space can be infinite so this q ij’s could potentially become very large could become unbounded there is an infinitely many of these q ij’s.

So, we are not going to consider those kind of processes. We are going to stick to q ij’s being upper bounded by some number b, it could be a 100 or a 1000 I do not care but it is something finite. So, I can take delta small enough, you take delta small enough such that q ij times delta is a valid probability, obviously q ij times delta is bigger than 1 this is not the delta is too big.

You take a small delta and you can fix a small delta such that q ij times delta is a valid probability. And then q ij times delta becomes the transition probability matrix of this time sample DTMC. Now what is the probability and similarly you can say what is the probability that X k + 1 delta = i given X k delta = i. That is the probability that you are in the same state. So, you are going to some other state with probability q ij times delta + o delta and the probability of not going to any of the other states will simply be the probability that none of these competing exponentials actually successfully fired in that little delta interval.

That will be you can say it will you can prove this it will be sum over q ij times delta + o delta which is nothing but what is sum over q ij is just nu i. So, 1 - nu i delta + o delta. So, if you look at this time sample markov chain you have transitions from state i to state j with probability q ij delta in this little delta intervals and no transition to some other state is basically you remain in the same state with probability 1 - nu i delta + o delta.

So, remember that **so** there is a see in the time sampled approximation I want to make this very clear in the sample time approximation ah to the CTMC which is this DTMC will have self transitions. We said that the embedded DTMC does not have self transitions. So, this term is in fact the self transition term between k delta and k + 1 delta and it will have a substantial probability of 1 - nu i delta, because there is a substantial probability of none of these exponentials actually firing in this little time interval.

So, basically there are 2 DTMC's, one is the embedded DTMC which looks at only the transitions at state changes and there is a sample time approximation which looks at the progression of the process are these little tiny delta intervals and this sample time approximation which is a DTMC for some small delta will have cell transitions and its transition probabilities are given by q ij delta for other states j and 1 - nu i delta for self transitions.

The embedded markov chain does not have self transitions. I hope this is clear. So, we know that for the embedded markov chain you can find some pi i's. Now for the time sampled approximation which is a different DTMC with transition probabilities q ij delta you can also find its steady state probabilities. The question is are they the same? You have the embedded markov chains transition probabilities from which you can find some pi i's which we already did.

We said the pi i's represent the average fraction of transition into a particular state. Now we have a different DTMC which is the time sampled approximation, you have all these little time little deltas and look at the transitions. You can go ahead and calculate the steady state probabilities of this DTMC also, are they the same as the steady state probabilities of the embedded markov chain? The answer is no.

In fact we will see that if you write out the transition probabilities of this time sample markov chain and work out its steady state probabilities you will in fact get the average fraction of time spent in state i, if you solve for the steady state probabilities of this particular transition probability matrix in front of you in the screen you will in fact get the fraction of time spent in state i.

Whereas if you solve for the steady state of the embedded markov chain you will get the fraction of transitions into state i. So, there are 2 different things. So, we will see this in more detail; at high level I just want you to understand that there are 2 different DTMC's corresponding to a CTMC. One is the embedded markov chain; other is the time sample markov chain. And their steady state probabilities are different, they are not the same and they may mean very different things. So, today we will stop here.