Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology-Madras

Lecture-69
The Steady State Behaviour of CTMC-Part 1

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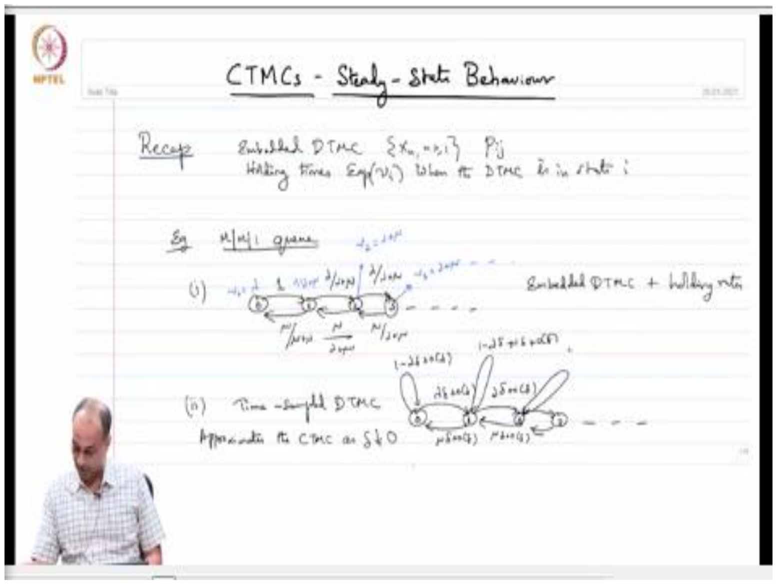

Welcome back, we were discussing continuous time Markov chains. Today we will continue to discuss CTMCs and proceed to discuss their steady state behaviour. Before that I would like to begin with a recap of what we have studied so far. CTMCs are characterized by an embedded Markov chain, embedded DTMC let us say denoted by X n which we assume to be reducible and without having any self transitions.

And holding times exponentially distributed with parameters of nu i when the embedded DTMC is in state i, so this is what a CTMC is. So, the embedded DTMC has some transition probabilities P ij and it is some generic possibly countably infinite state Markov chain. Now there are many ways of notating CTMC. One is to draw out, so let me just take maybe I should take an example of the M/M/1 queue which is a continuous time Markov process, you have seen this before.

Let me just use this example to show you how to variously notate graphically a CTMC. The one is to draw out the embedded DTMC which we already did. So, this is the embedded DTMC always go to from 0 to 1 and you know these are mu over nu + lambda and lambda over and so on. And so this is just the underlying the embedded Markov chain embedded DTMC.

Now for the CTMC you can put on top of each of these states the holding times or the holding rates the nu i's. So, which maybe I will use in a different colour, so on stage 0 how long do you stay in state 0? You are basically expecting an arrival, so the rate at which this the nu 0, the rate at which you get out of a state 0 is lambda.

And for the all other states the holding times are, so here nu 2 would be lambda + mu, the nu 1 would be lambda + mu, mu is over 0 is lambda, mu 3 will be equal to lambda + mu and so on. So, this is one way of notating the CTMC corresponding to the M/M/1 queue, this is one kind of condition. Another kind of notation is to use the, so the notation 1 you are using, so you are drawing out the embedded DTMC + holding rates, holding times or holding rates.

So, you are explicitly writing that out. In the second case you can use a time sampled DTMC you basically divide time into delta intervals. And you look at the transition probability from time k delta to time k + 1 delta. And what you will see is that for example in state 0 you have a forward going probability of lambda delta + o delta. And the self transition probability of 1 - lambda delta + o delta.

And in state 1, again say all the forward going probabilities are lambda delta + o delta and all the reverse probabilities are mu delta + o delta and so on, dot, dot, dot. And of course in all these states you will have a self transition of probability 1 - lambda delta - mu delta + o delta, all these you will have ditto. So, this is just time sampled DTMC, so and this approximates the CTMC as this delta becomes really small.

As delta goes to 0, this o delta term will be very small and you can just look at the forward transition probability as just lambda, delta reverse transition probability as mu delta. Of course please note that they will also be these sort of, these transitions are possible but they all have probability o delta.

And as delta goes to 0 these 2 jumps and 3 jumps are ignored, which is why I have not drawn it in the picture. They are there but for delta small I am going to just ignore it, these double jumps and triple jumps. So, this is the second notation, here you are basically drawing out looking at o delta intervals and looking at the way the Markov chain jumps in these delta intervals. This is a time sample DTMC approximation to the CTMC and the approximation is increasingly good as delta goes to 0.

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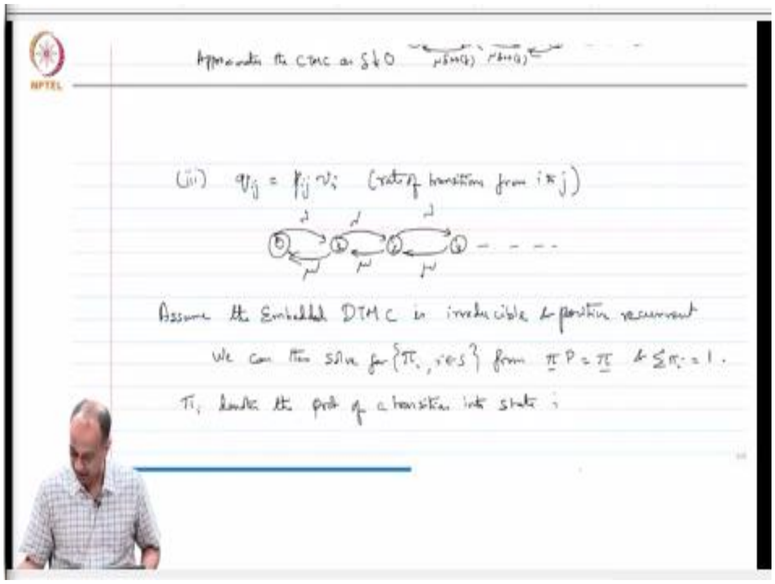

There is another 3rd way of looking at it which is in terms of q ij which we defined yesterday, q ij as we defined yesterday is P ij times nu i**.** So, q ij can be interpreted as the rate of transitions from i to j. So, you can view whenever you are in state i you can view all these q ij exponentials as competing with each other. One of these q ij, they are independent exponential, so it is like raising Poisson processes, raising exponentials.

And the probability that a particular exponential say the q ik exponential will win with probability q ik over sum over q ij which is q ik over nu i which is P ik. So, that probability of going from i to k will be P ik as you would want. And the holding times will be exponential with parameter nu i, so this you can show. So, you can actually draw out between states i and j instead of drawing probabilities you can actually draw the rates.

So, in previous 2 notations what we drew out was a DTMC, in the first case we drew out the DTMC which is embedded and then in blue we wrote out the holding rates. In the second case we drew the DTMC which is the time sample DTMC, the third case I am saying that we can just draw out not a DTMC but a CTMC with all the transition rates. Now what I am going to draw is not discrete times, it is the continuous time process.

But nevertheless you can look at the state transition; you can just do this lambda mu, that is it. So, here you are just notating the q ij's. So, this is the third way of drawing a CTMC. And these 3 you can go from any one to the other easily, this is just an example you can do it for any CTMC. So, M/M/1 is probably the most widely encountered CTMC, so I am just drawing that out.

Now remember that, so I am going to get back to the question of the steady state behaviour, which is very important aspect of CTMCs. Please remember that for the embedded Markov chain we already derived the pi i's. So, assume that the embedded Markov chain, embedded DTMC is irreducible and positive recurrent, then you can derive pi i's. You can then solve for pi i's, how will you do it? We can then solve for pi i's from pi P = pi and sum over pi i = 1.

So, we did that already for the embedded DTMC of the M/M/1 queue we already solved that in the last lecture I indicated how the pi i's look. So, this pi i denotes the probability of transition into state i. And it is not the fraction of time spent in state i by the CTMC, pi i is the steady state probabilities of the embedded DTMC. So, in the DTMC case the fraction or the probability of transitions going into a state is the same as the fraction of time spent in that state.

That is because for a DTMC each states you spend only one unit of time. But for a CTMC that is not true, you spend different amounts of times in different states depending on the holding times and holding rates nu i. So, while pi i denotes the fraction of transitions going into a state i for the CTMC, it does not denote the fraction of time spent in state i. So, our consideration now is when you are talking about the steady state probabilities how do you relate these pi is which is the fraction of transitions going to state i to the fraction of time spent in state i.

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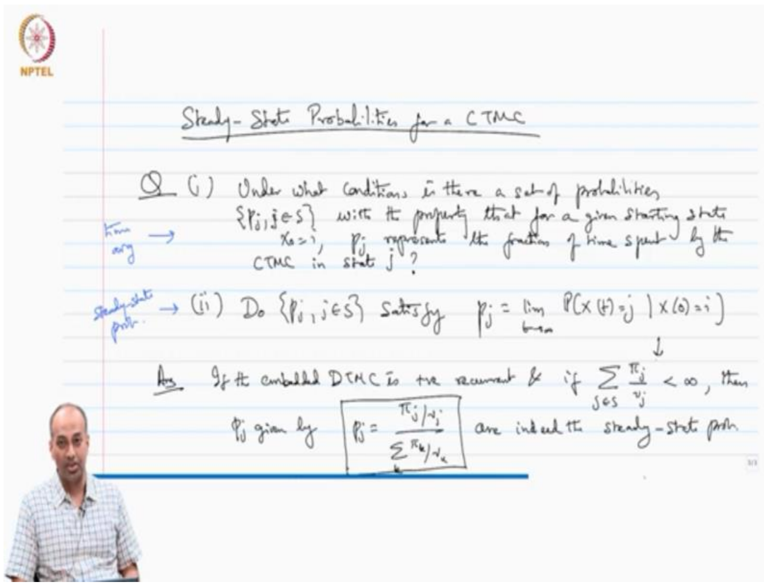

So, we are now going to discuss steady state probabilities for a CTMC. So, throughout I am going to assume that the embedded DTMC is irreducible and positive recurrent. Now the question is, now we want to answer 2 questions. Under what conditions is there a set of probabilities? Let us now call them P j, j belongs to S with the property for a given starting state X 0 = i, P j represents the fraction of time spent by the CTMC in state j?

So, I am going to use, so I am looking for a set of probabilities P j, I am calling them P j to distinguish them from pi j. pi j is the steady state probabilities of the embedded Markov chain which represents the fraction of transitions going into state j, P j represents the fraction of time spent by the CTMC in state j. And I want to find does there exist such a P j? How is it related to pi j and all that?

And also the second question I want to answer is that if the double probability P j is do exist do P j satisfy P j = limit t tending to infinity probability of X t = j given X 0 = i. So, this is if you can view this as time average interpretation and we are saying that the same P j under some nice conditions which we have yet to discover that this is an ensemble average or a steady state probability interpretation.

The same P j’s ideally we would like to have a P j as that average fraction of time spent in stage j and also the long term probability. If I start the process in CTMC in whatever state I want i is there does this P j satisfy that as t tends to infinity the probability of being in state j at time t as t tends to infinity is in fact P j. We want to answer these questions. Again there are a few ways to do this, we can since we already know renewal processes; we can use the renewal process approach.

So, that is probably I mean most straightforward way to see why these P j’s exist in the first place and how they are related to the pi j’s? In fact what we will see maybe I should give you the answer. So, the answer is this, if the embedded DTMC is positive recurrent and irreducible, we are assuming that and if sum over pi j over nu j sum over j is finite, then P j given by P j = pi j over nu j over sum over k pi k over nu k are indeed the steady state probabilities of the CTMC.

So, I am just throwing the answer at you before we derive it, so we need 2 things. We of course need that the DTMC be positive recurrent otherwise you will not even find these pi j's. Suppose this pi j's are well defined and sum over pi j over nu j is finite. Then these P j’s exist P j’s defined like so, like here. Of course these are very clearly probabilities because they sum to 1, notice that the denominator is assumed to be finite, so P j’s are some probability distribution over the states.

In fact these P j’s we will show are the steady state probabilities and also the average fraction of time spent in state j, so this is what we want to show. Notice that this assumption is required, the sum over pi j over nu j is infinite then even if the underlying DTMC under embedded DTMC is positive recurrent. Then you will not find these P j's the probability distribution P j will not exist which corresponds to the fraction of time spent in state j or a steady state probability interpretation neither is possible.

So, in fact this sum over pi j over nu j if it is infinite it is a slightly pathological case. The intuitive way to think about it is that if pi j over nu j sum over j pi j over nu j is infinite then some of these nu j’s are so small that the rate of the transitions in these states is becomes so sluggish, that the continuous time Markov chain actually does not move forward much. It just kind of although the embedded Markov chain keeps running in discrete time, the nu j's are so small that the CTMC is so sluggish.

That it makes only finitely many transitions in infinitely many times at some intuitive level. So, you will not have the interpretation of steady state probabilities but that is anyway it is a somewhat pathological case. All I am saying is that the sum over pi j over nu j should be finite for this steady state probability interpretation to even work.