Stochastic Modeling and the Theory of Queues
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Module - 1
Lecture - 7
Poisson Process - Memorylessness

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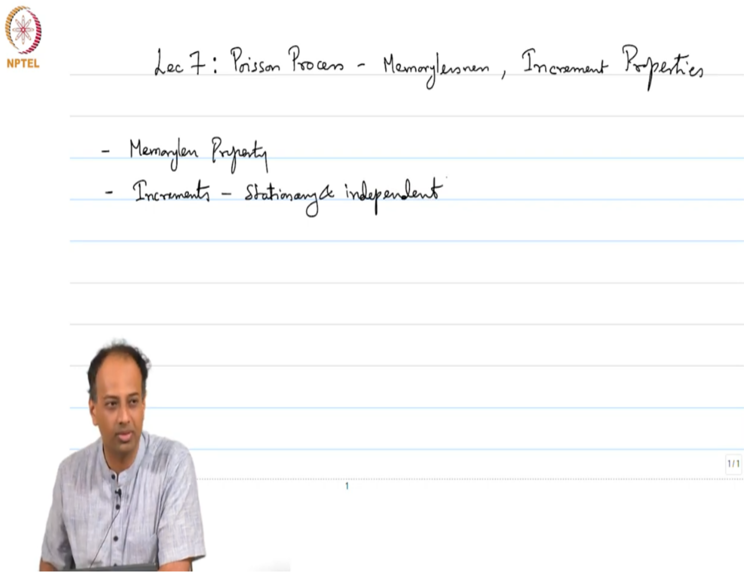

Welcome back. Good morning. Yesterday we introduced a Poisson process. We said that if you have a counting process, you can think about the counting process in terms of $N(t)$ which is the number of arrivals up to and including time t or you can look at it in terms of the arrival epochs S_i's or the inter-arrival times X_i's. If these inter-arrival times X_i are independent and identically distributed, you say that the process is a renewal process. The counting process is a renewal process.

Now, a Poisson process is a very special renewal process in which these X_i's are IID exponentially distributed with some parameter λ. And, we also introduced this memoryless property of the exponential random variable. So, in some sense, this exponential distribution forgets what happened in the past, which is why this Poisson process is very special.

We will formalise this in this lecture by proving, in what sense this memoryless property actually holds. So, in this lecture, we will prove this memoryless property in a very, in a more precise sense. We will also discuss, we will also basically prove that the increments in a

Poisson process are stationary and independent. At this point, these terms may not make sense to you, but we will define what these mean. So, this is where we are heading.

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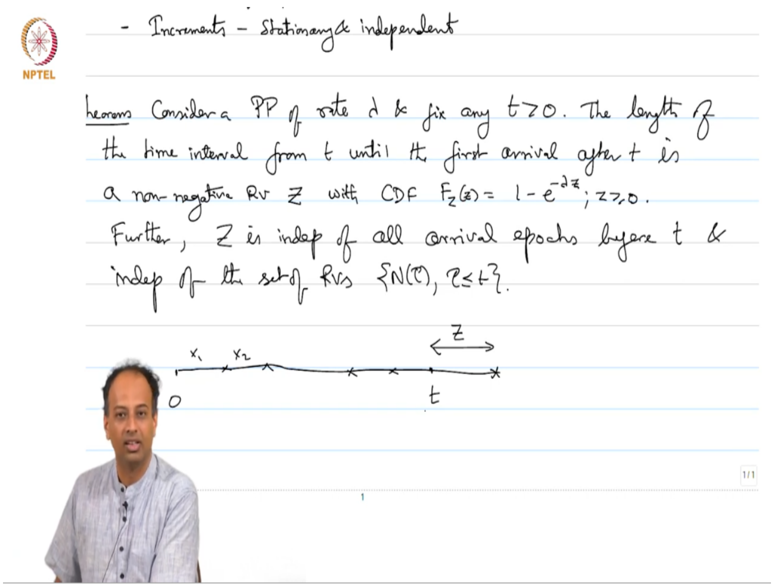

So, let me state a theorem. Consider a Poisson process of rate λ and fix any $t > 0$. The length of the time interval from t until the first arrival after t is a non-negative random variable Z with CDF $F_Z(z) = 1 - e^{-\lambda z}$ for $z \geq 0$. Further, Z is independent of all arrival epochs before t and independent of the set of random variables $\{N(\tau), \tau \leq t\}$.

This is a slightly long statement, but it is not, it is certainly not difficult to understand. So, what you are doing is, you consider a Poisson process of rate λ and you fix any t. So, you are looking at t. As far as you are concerned, your observation starts at t. So, a Poisson process has been running and you show up at some t, and you are looking at when does the next arrival after t come?

Now, this t is something fixed; it is not random. t is something fixed. And you are looking at the next arrival to come after t. And, from t until the next arrival, this interval is denoted by Z . It is some random variable; it is some non-negative random variable. What this theorem is saying is that this time to the subsequent arrival after t is exponentially distributed with parameter λ.

It is the same as an inter-arrival distribution, same as the original inter-arrival distribution. And furthermore, Z is independent of everything that happened in the past. It is independent of how many arrivals came so far, it is independent of when those arrivals came, until, at time t, some arrivals would have already come. And the time to the subsequent arrival after you show up at t, is independent of how many arrivals came and when they came.

That is what this theorem is saying. So, it is like, buses are going at a bus stop according to some Poisson process; you show up at some particular time. The time that you wait for the next bus will still be an exponential random variable, and it will be independent of how many buses came in the past and when they came. If the bus arrival process is a Poisson process, which it is usually not. Clear? Statement is clear?

So, just to show it in pictures. So, that is your time axis. So, this is 0. Some arrivals have shown up. So, this guy is X_1; this guy is X_2 and so on. These are inter-arrival times. You are picking some time t. The arrival after this, comes here. And the Z that is of interest is this guy. So, you are showing up at time t, and you are looking at the time to the next arrival.

What this theorem is saying is that Z is exponentially distributed with parameter λ. And Z is independent of the arrival epochs of all these previous guys, and also the number of those arrivals. So, it is independent of everything that happened in the past; that is what this theorem is saying.

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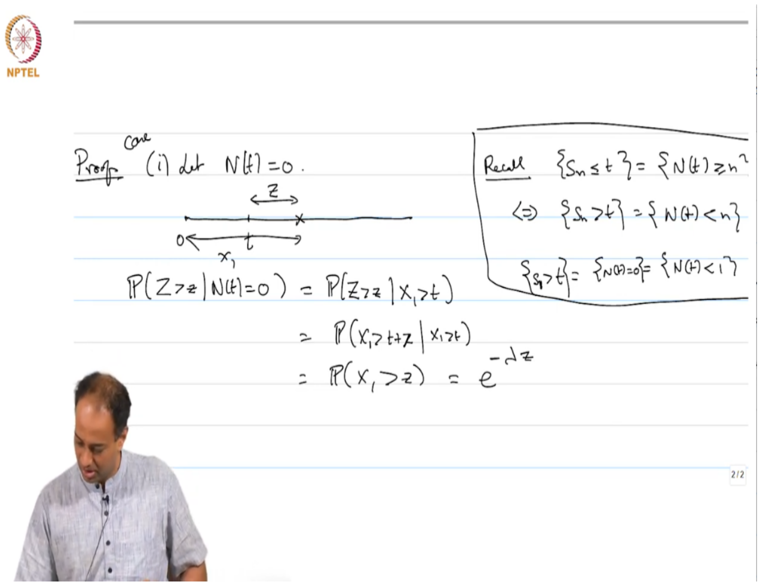

Now, let us prove this. So, we will first start with; so, let us say, let t be such that $N(t) = 0$. This is case 1. We will consider two different cases, $N(t) = 0$ and $N(t) > 0$. So, we will look at the following. So, you look at; let us draw; I will just draw a picture to make this clear. This is 0, time $t = 0$. Your time t is here. No arrivals have come so far, $N(t) = 0$. Your first arrival comes somewhere here.

That is the case we are considering. And that is your Z at this point. Now, we are looking at; let us consider the $P(Z > z \,|\, N(t) = 0)$. Now, you recall the following. We showed yesterday that the event $\{S_n \leq t\}$ is same as the event $\{N(t) \geq n\}$. This we showed yesterday. Did I write it down correctly? So, now, this is the same as saying that the event $\{S_n > t\}$; I am just taking complement; is the same as the event $\{N(t) < n\}$; just taking complements.

A = B; $A^c = B^c$. So, from here, you can see that $N(t)$; so, in particular, you can write that the event $\{N(t) < 1\}$, which is just equal to the event $\{N(t) = 0\}$is same as the event that $\{S_1 > t\}$. So, $\{N(t) < 1\}$is same as the event $\{N(t) = 0\}$, because $N(t)$ takes only non-negative integer values. And $\{N(t) < 1\}$is same as $\{S_1 > t\}$, from here. But $S_1$is same as X_1.

So, I can just write this guy as $P(Z > z \mid X_1 > t)$ because, the event $\{N(t) = 0\}$ is same as the event $\{X_1 > t\}$. So, it comes from that box, from the *recall* box. So, this is simply just equal to what? So, $Z = X_1 - t$. This is the same as $P(X_1 > t + z \mid X_1 > t)$. From the picture it is clear.

So, this guy is X_1. So, what is $P(X_1 > t + z \mid X_1 > t)$? This X_1 is exponentially distributed and we know that it follows the memoryless property. So, this is equal to $P(X_1 > z)$. This equality is due to memorylessness. And this is nothing but $e^{-\lambda z}$. So, for conditions on $\{N(t) = 0\}$, Z is exponentially distributed with parameter λ.

Now, we also have to look at $\{N(t) = n, n \geq 1\}$. In both these cases, if this holds, then the unconditional distribution of Z is exponential. That can be; that follows. So, let us consider case 2.

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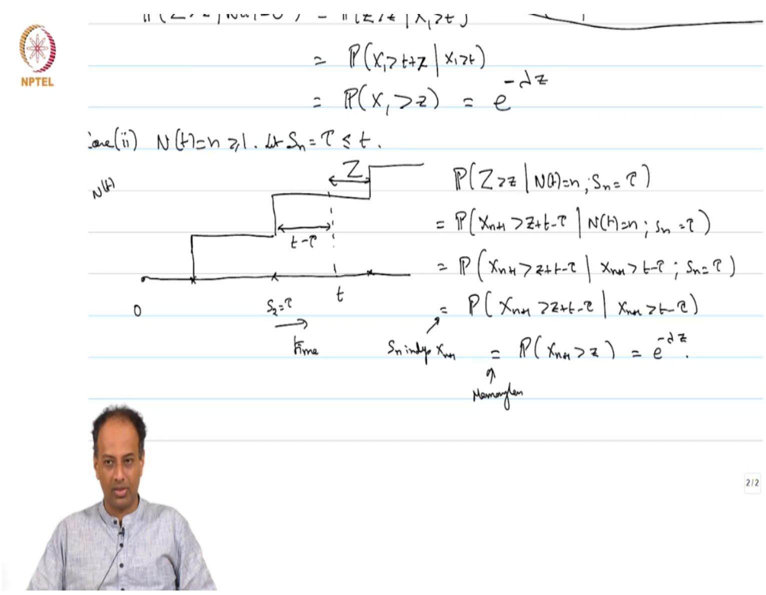

Let us say $N(t) = n$, some bigger number, $n \geq 1$. So, let us say; so, $N(t)$, as there have been some n arrivals. n arrivals have already occurred at time t. Now, let us say that $S_n = \tau$, $\tau \leq t$. So, I have said that there are n arrivals. The time at which the n^{th} arrival occured, I am calling it $S_n = \tau$. And this has to be before t, less than or equal to t. So, here the picture is as follows.

So, this is 0. Some arrivals occur here, some arrivals occur here, some arrivals occur here and so on. So, you are looking at; this is $N(t)$; this is t. This is the time axis. I am looking at a particular t; so, this is time; this is a particular t. In this case, with what I have drawn, $N(t) = 2$, but does not have to be 2; it can be anything that is not 0. And of course, your Z here is the time to the next arrival.

And this is S_2, which is; this I am calling τ. See, all random variables are big Z. So, I think my Z and z are looking very similar. Maybe, if you can write the cursive z, maybe it is better. See, always, random variables are big. So, what I want to put here is Z. So, this is Z. Now, you consider the following, $P(Z > z \mid N(t) = n, S_n = \tau)$.

Now, we can argue similarly, that this is equal to the $P(X_{n+1} > z + t - \tau \mid N(t) = n, S_n = \tau)$. That is because, this, what is this width? This is $t - \tau$. So, this random variable will be bigger than z. So; this is the n^{th} arrival; this is the $n + 1^{th}$arrival. This inter-arrival time is nothing but X_{n+1}. So, $X_{n+1} > z + t - \tau$ comes from the picture directly.

So, this guy, you wanted to be bigger than z, so, $X_{n+1} > z + t - \tau$. Obvious from the figure. It is useful to draw these figures, just so that you do not make any mistakes. Now, again you can argue this event. So, you are conditioning on $S_n = \tau$ and $N(t) = n$. So, the n^{th} arrival arrived at some τand the $N(t) = n$, which means that there have been no more arrivals.

You can again argue just like in the previous case, this just boils down to; This is the same; $t - \tau$, given $X_{n+1} > t - \tau$, because there has not been any arrival since, and $S_n = \tau$. Now, it is an important step. See, X_{n+1} by the definition of a Poisson process, is independent of S_n. Why is that? See, actually this is true for any renewal process. $S_n = X_1 + X_2 + ... + X_n$. But X_1 through X_n will be; they are all IID random variables.

So, X_1 through X_n will be independent of X_{n+1}. So, X_n is independent of S_n. So, this conditioning can be dropped. So, since, S_n is independent of X_{n+1}. I am going to write this as; $X_{n+1} > z + t - \tau$ given $X_{n+1} > t - \tau$. And this is equal to $P(X_{n+1} > z)$. This is because X_i's are exponentials; X_i's are memoryless. Actually, you see that the conditional distribution of Z given $S_n = \tau$ boils down to an unconditional distribution.

You can just drop this. So, you can see that Z is independent of S_n. Likewise, if you were to condition on $S_n = \tau, S_{n-1} = \tau', S_{n-2} = \tau''$ and all that; all those conditionings will drop off because X_{n+1} is after all independent of everything in the past by the definition of a Poisson process. So, this proof itself can be easily reworked to show that this Z is independent of S_n, all the previous arrival epochs.

Also $N(t)$ drops off; your condition $N(t) = n$ also drops off. So, nothing would change if you were to condition on $N(t) = n$ and previous $N(t')$, something before t to be n' and all that, it will still drop off. You can rework all this, to prove.

"Professor - student conversation starts"

This guy? This one? This guy? So, $N(t) = n$. So, we are looking at some time t. We are conditioning on $N(t) = n$, right?

And $S_n = \tau$; the previous arrival occurred at τ. So, the previous arrival occurred at τ, and $N(t) = n$, which means that there have been no further arrivals. So, this inter-arrival time cannot be smaller than $t - \tau$. So, what we are saying is that conditioning on $N(t) = n$ is same as conditioning on $X_{n+1} > t - \tau$. These two events can be shown to be the same. This would imply this and this would imply that.

Similar to, I mean, you can argue like this if you want, in the recall box, right? I did it for the case $n = 1$. We can argue similarly.

"Professor - student conversation ends"

So, hence, you are done proving that the time to the next arrival is exponentially distributed. And this exponential random variable is independent of the number of arrivals that have happened so far and the epochs of those arrivals. So, essentially, it is independent of everything in the past. So, this Poisson process has this; at every instant of time, it statistically regenerates. It is as though a new process is starting. Every instant t, it is like a new process starts. It forgets everything that happened in the past. So, there will be no way to statistically tell the difference between; so, I start a Poisson process now, and I keep running it. Somebody comes in at a particular time and starts watching the process. Statistically, there is no way to tell the difference. So, that finishes the memoryless property, in this module. Next module, we will discuss increments of a Poisson process.