Stochastic Modeling and the Theory of Queues
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Lecture-71
The Steady State Behaviour of CTMC-Part 3

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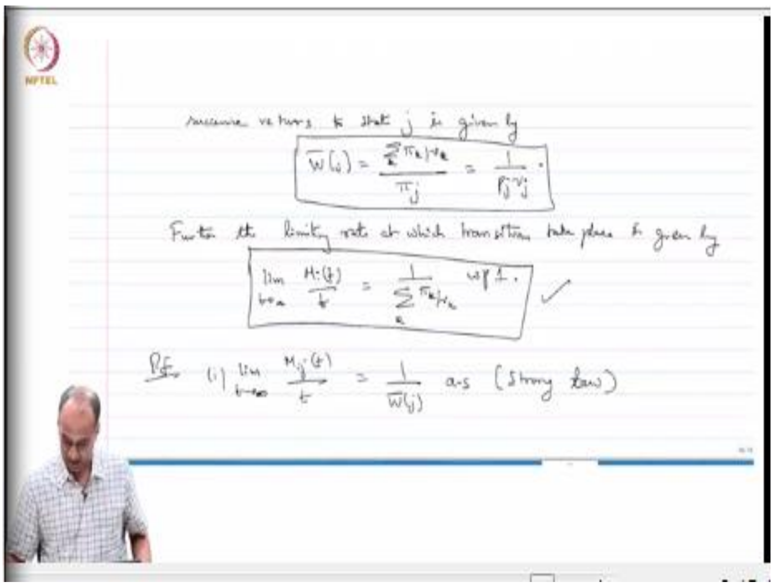

We have to prove this, so imagine this. So, rough proof sketch, so how does this go? So, I know this, what is limit t tending to infinity M ij of t over t, remember that M ij of t which it is a delayed renewal process counting the number of returns to j by time t this will be equal to 1 over w bar of j, almost surely. This is just strong law for delayed renewal processes, so this I know.

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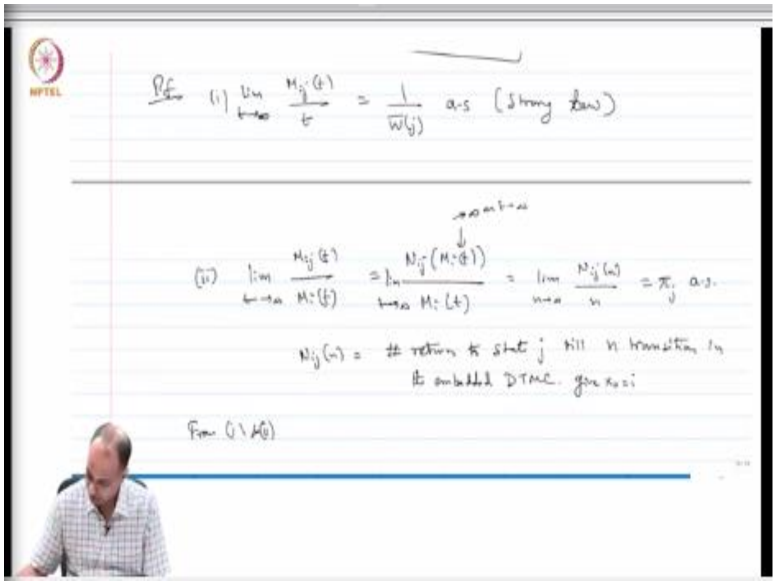

What else do I know? So, this is 1, if I look at limit t tending to infinity M ij of t over M i of t. So, M ij of t is the number of returns to j until time t, M i of t is the number of transition made by the process until time t. Now this also has a limit, let us see what it is? I am going to write this as N ij of M i of t over M i of t, this can be shown limit t tending to infinity. Now what is this N ij? N ij of n is equal to number of returns to state j till n transitions in the embedded DTMC.

So, the embedded DTMC keeps making transition and I am going to say that N ij of n is the also given X 0 = i, even that I start at i what is the number of times I return to j until there are n transitions in the embedded Markov chain? Now until time t there are M i of t transitions in the CTMC, which corresponds to N ij of M i of t transitions to state j in the DTMC, is that clear? So, if you look at this guy, so now remember that this argument this guy goes to infinity as t tends to infinity that we have already shown.

So, this is like limit n tending to infinity N ij of n over n, correct. Now what is limit n tending to infinity N ij of n over n? It is simply the fraction of transitions that go into state j in the embedded DTMC. And that we know is equal to pi j almost surely, correct. So, that is very nice then, so this is number 2. So, what does if you put these 2 together, what do we get? From, so 1 and 2 I can get something.

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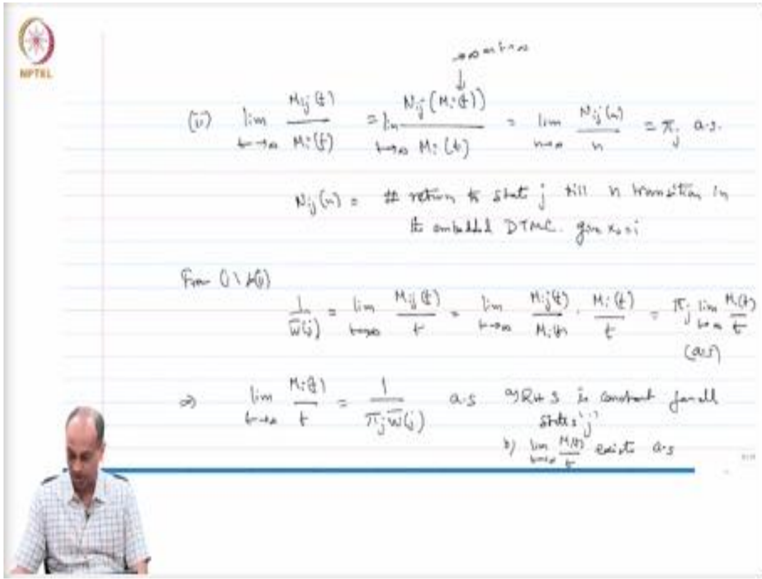

So, I can get 1 over w bar of j = limit t tends to infinity M ij of t over t which is simply limit t tends to infinity M ij of t over M i of t times M i of t over t. That is simply pi j times limit t tending to infinity M i of t over t, all of this holds almost surely. Therefore what does this say? It says that limit t tends to infinity M i of t over t is almost surely equal to 1 over pi j w bar of j. So, what have we shown? In this calculation we have shown that there is such thing as a average rate at which transitions happened in the CTMC.

So, there is an almost sure limit to M i t over t and that is equal to 1 over pi j w j bar. So, this also means that 1 over pi j w j bar is constant across all the states. So, this 1 over pi j w j bar has to be equal to 1 over pi k w k bar all of this will be equal to M i of t over t. So, the RHS is independent of is constant for all j, for all states j that is the first conclusion. The second conclusion is that limit t tends to infinity M i of t over t exist almost early, so these 2 conclusions follow. So, this is a nice conclusion.

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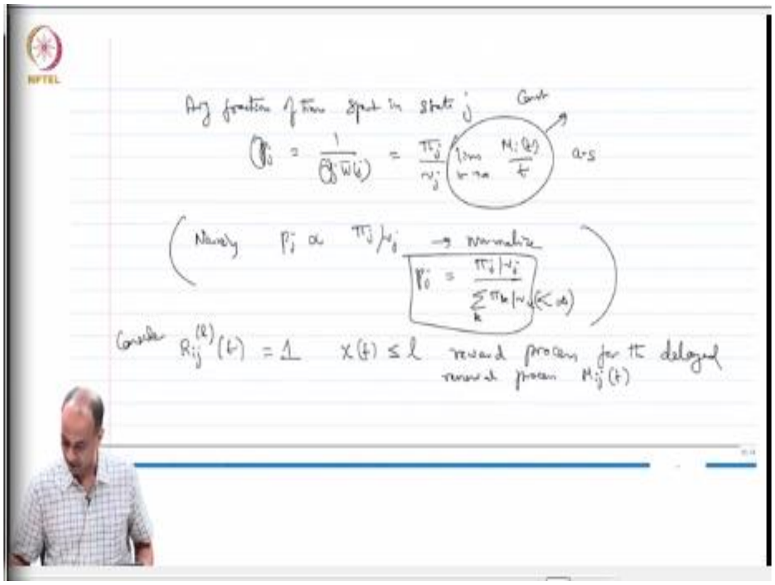

So, now if you go back to the average fraction of time, which is P j, we called it P j of i and then we got rid of the i dependence, P j of i which is we are just calling it P j = 1 over nu j w bar of j. So, which is equal to if you bring this up this will be equal to pi j over nu j limit t tends to infinity M i of t over t almost surely. So, this guy is just a constant which is 1 over pi j w j bar, we know that it is constant and is independent of any state j.

So, what we have shown is that P j is just proportional to pi j over nu j and there is a multiplying constant which is the limit of the rate at which the transitions happen. Now so if you were naively when you think about it you would say that, so basically naively. So, P j is proportional to pi j over nu j and there is a constant in front which is that limit. So, if you were to normalize, so P j is the fraction of time spent in some state j and the CTMC has to be in some state or the other.

So, you might argue if you were to be naive about it normalize and say that P j should be equal to pi j over nu j over sum over pi j over nu j sum over let us say pi k over nu k, which is sort of the answer we want. Well, this answer is correct but to show this properly requires some amount of work. Also you need that the denominator should be finite, this guy should be finite otherwise you will not get a probability distribution at all.

So, this is naively is you can put this in a bracket. To show this more properly you have to again use some renewal arguments. I am going to give a reward, so I am going to say I am going to look at a truncated reward R ij l of t = 1, whenever X of t is less than or equal to l. And this is a reward process, so consider this reward process for the delayed renewal process M ij of t. So, I am going to give a reward of 1, whenever my state is anything from 1 to l or 0 to l.

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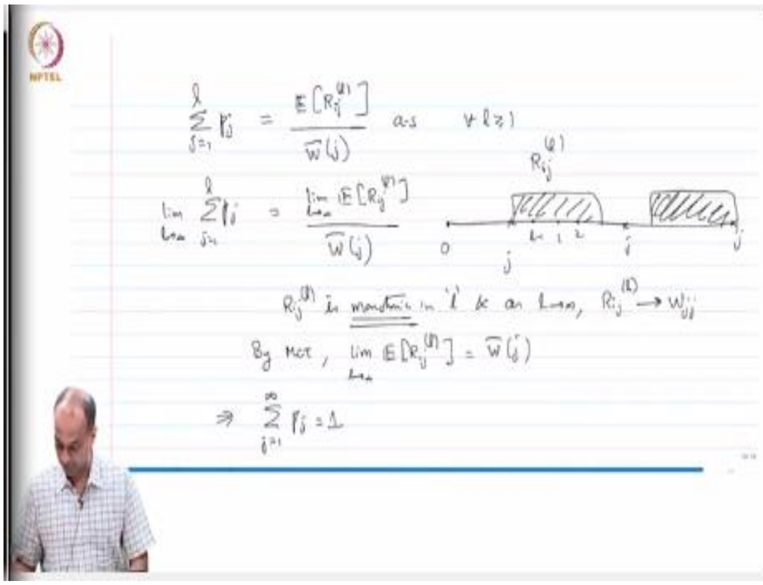

So, for this what happens? So, the fraction of time spent in anywhere from 0 to l which is simply P j j = 1 to I am applying renewal reward now j = 1 to l P j = expected R ij l which is the expected reward per annual interval divided by the expected width of the renewable interval, which is w bar j. Now I want to really show that this P j's in fact sum to 1, the fraction of time spent in state j P j when I sum over all states I want to show that that is equal to 1.

It is sort of intuitively reasonable but I am trying to show this rigorously, that is what I am trying to prove. Now I want to show, so this is correct, this is almost surely true by renewal reward theorem, I am going to send l to infinity now. Now if I send l to infinity what happens? I have to send this is true for all l greater than or equal to 1, limit l tends to infinity sum over j = 1 to l P j = limit l tends to infinity, the denominator has no l, only the numerator has l, expected R ij l over w bar j.

See if you look at this now you have to show that the limit of the numerator is in fact equal to w bar j. Because you want to show that sum over P j j = 1 to l, l tend to infinity = 1. So, if you look at this, if you just look at maybe I should draw a picture here, this is time 0. So, you are hitting j here, j here, let us say j here and all that. And whenever your state is less than or equal to l you get a reward of 1.

So, if you have l - 1 or 1 or 2 or whatever you get a reward of 1. So, I am saying that you get a, so this reward is 1 whenever your state is less than or equal to l, 0 otherwise. Now as l tends to infinity what happens? If you look at this R ij l R ij l has 2 properties, see it is clearly monotonic in l, R ij l as a random variable is monotonic in l and as l tends to infinity as l becomes larger and larger R ij of l basically approaches w jj which is the width of the interval.

And because of the monotonicity, we can invoke monotone convergence theorem and argue that by monotone convergence theorem, we can argue that see R jj l monotonically increases to w jj, therefore expectation of R ij l approaches expectation of w jj which is w bar of j, monotone convergence theorem, so this is the technical step. So, what is intuitively reasonable is taken care of by the famous monotone convergence theorem is therefore this is w bar of j.

This implies sum over j = 1 to infinity P j = 1, where P j is the time average fraction of time spent in state j. So, you can truly normalize and get that answer that you always wanted. And then you can go back plug it back since P j is known to be, so this P j is now known in this equation, you can get the value of that as being equal to whatever the theorem states right here. And w j bar you can also substitute where is w j bar?

So, you know P j and nu j, so you know P j, you know nu j, so you can get w j bar in terms of the pi's and nu's and you can get that. So, that concludes the theorem, so bottom line is that we have shown the P k the fraction of times spent in state j which is also the steady state probability of state j is given by that equation.

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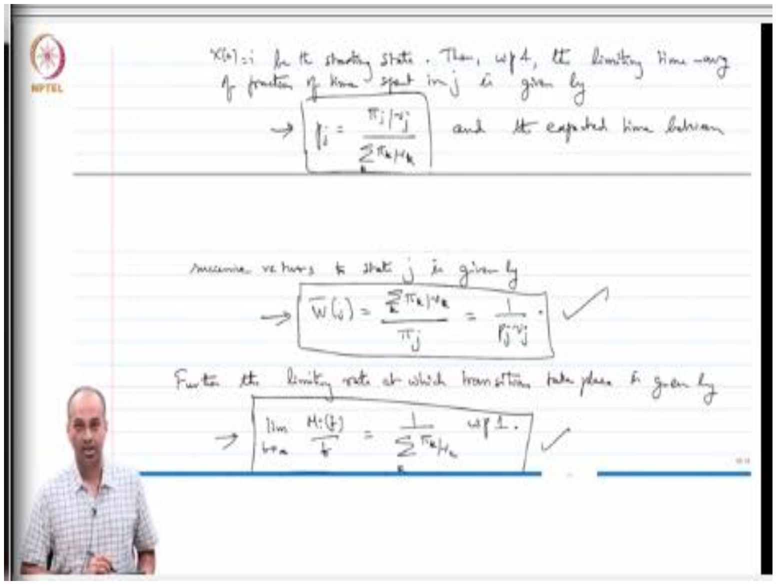

The expected renewal duration is given by that equation and the expected rates of transition is given by this equation, we will stop here.