Stochastic Modeling and the Theory of Queues
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Lecture-78
The Jackson Networks-Part 1

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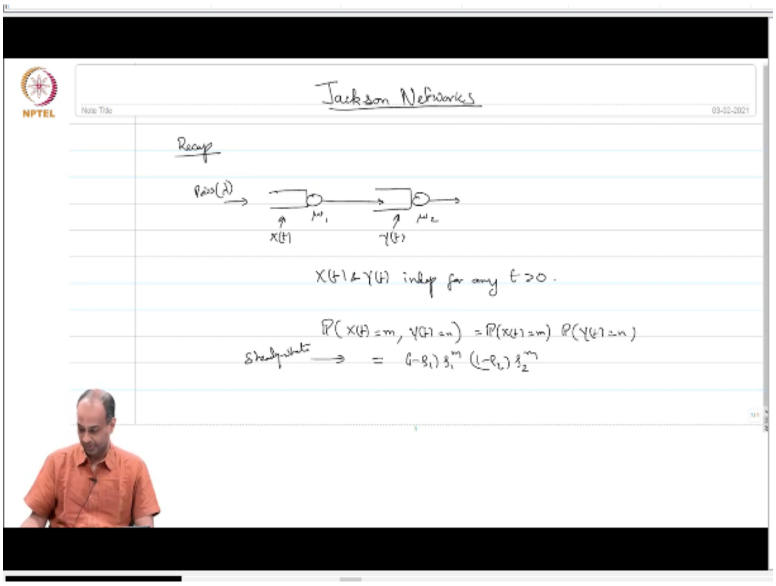

Welcome back, last lecture we discussed tandem queues using reversibility arguments. So, we considered just to give you a recap, you considered a queue in which there is a poisson lambda arrivals on the server of rate mu 1 and the customers that leave queue 1 immediately enter another queue with an exponential server of some rate mu 2. So, this is the tandem system we studied.

We know that the departure process if you look at the departure process getting out of the first queue is independent of the present state of the first queue; this is something from Burke's theorem. So, what happens is that these 2 queues at any given time t they end up having independent states. So, if X t is the number of customers in this queue at time t and Y t is the number of customers in this queue.

Then X t and Y t turned out to be independent for any t and in steady state we said that you will have probability I mean for a very large t or either you start in steady state or wait for a very large amount of time, so you will have probability X t = m, Y t = n just products out into, so this of course is true for all t, X t = m, times probability Y t = n. So, this is of course always true for any t.

And this is true in steady state, you will have the first one being M/M/1 and the second one being in M/M/1, you have 1 - rho 1, rho 1 to the m 1 - rho 2 rho 2 to the m. So, at any given time t these 2 tandem queues behave as though there are independent M/M/1 queues. Of course the queues are not independent. In fact if you look at X t for some t and Y tau for some other tau they are not independent, they will not be independent in general. It is not at all the case that the process X t is independent of the process Y t.

That is not what we are saying, but the random variable X t for a fixed t is independent of the random variable Y t for that t. That is what we are saying in this tandem queues and we also argued that if you have FCFS the time spent in the first queue and the time spent in the second queue by a customer are independent. So, that the total time spent in the system can be added up as independent random variables.

And that is because of part C of Burke's theorem. We argued all of this in the last lecture. Then we also had an example where some fractions of customers are sent back into the queue. It is like a feedback system, in that case what happened is that the arrival process into the queue was not poisson but the entire system could still be seen as an M/M/1 with a service rate mu times queue where queue is the probability of leaving the system. So, these are things that we saw in the previous lecture.

Now this kind of networks where you have either tandem queues or some fraction of customers going back into one of the queues you could have 3 or 4 queues where from queue 3 you could go with some probability to the second queue and the output of second queue could either leave the system or go back to the first queue. All sorts of these things are possible and these kind of with exponential service rates and poisson arrivals. So, these kinds of networks are known as Jackson networks and that is the topic of today's discussion.

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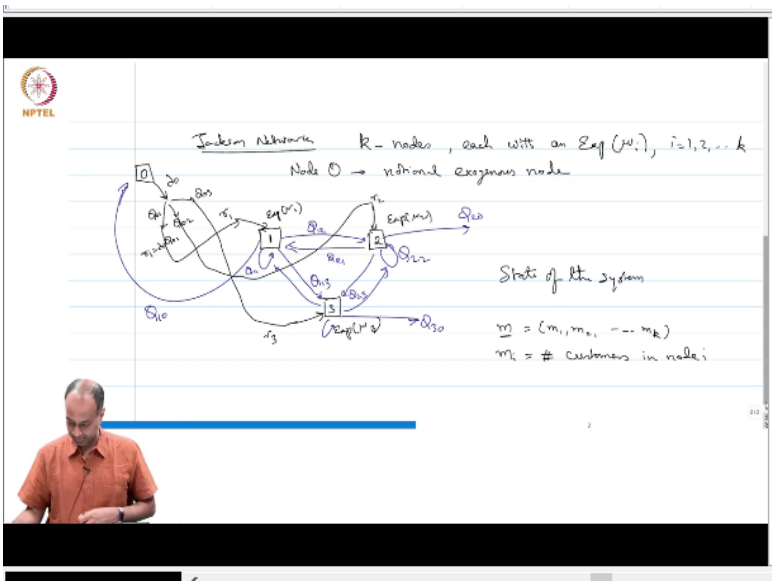

So, what is a Jackson network? It is just a generalization of tandem queues and this feedback queues. So, you have some k nodes, so I am going to say the k nodes of the Jackson network, there are k nodes each with an exponential server X mu I, i = 1, 2, dot, dot, dot k. So, the ith node has a server which serves with service times exponential mu i. There is also a notional node 0 which is a notional exogenous node.

So, node 0 is external to the system, it is better to not think of node 0 as one of the nodes in the system, you can think of it as something outside from which exogenous traffic comes into the Jackson network and once service is completed the traffic that leaves all these queues goes back into node 0. You can view it that way. So, the picture is as follows. So, you have some let us say that is node 1, so maybe I should draw squares.

So, that you do not look at it as some CTMC or some such. So, there is a node 1, that is a node 2, let us say there is a node 3 and so on. So, what happens is there is some node 0 outside which you can view as exogenous node. So, traffic comes to each of these nodes from the external node with some rate. So, let us call this rates exogenous arrival rates as r 1, r 2, r 3 etcetera.

So, there is traffic coming out of this exogenous node, of some rate lambda 0 and so this r 1, r 2, r 3 they are all poisson processors, their poisson processes of rates r 1, r 2 and r 3 they are independent poisson processors. Now since we are looking at a notional node 0 from which all the traffic is coming you can split you can do a Bernoulli split with probability Q 0 1, Q 0 2, Q 0 3 let us say.

And you can say r 1 is equal to lambda 0 Q 0 1 and then that goes here and likewise the split traffic of node 2 goes here, the traffic of node 2 goes here. So, that is how the exogenous, exogenous means arising from outside this network. So, this exogenous traffic comes in at this poisson rates r 1, r 2, r 3 which are simply lambda 0 Q 0 1, lambda 0 Q 0 2, lambda 0 Q 0 3.

And there is exponential mu 1s over here, the exponential mu 2's over here, exponential mu 3's over here. And upon completing service let us say I am a customer who arrived first at node 1; upon completing service at node 1 with probability let me draw it in a slightly different with probability Q 1 2 I go to node 2 and probability Q 1 3 I go to node 3 and probability Q 1 1 I look back to node 1.

So, this is like looking back to the same node is like what we studied in the previous class where some fraction of traffic just went right back into the same queue and with some probability let us say Q 1 0 it goes back, Q 1 0 it leaves the network. So, I am going to say that it goes back to node 0 which is both my origin and sync of all traffic; this node 0 is just notional.

You can just think of it as leaving the network completely. Likewise if I arrived from the exogenous node to this mu 2 server then I can go after completing service for exponential mu 2 amount of time I can go to Q 2, this is Q 2 1, I can go here with probability Q 2 3, Q 2 2 and so on, likewise for node 3, I can do the same thing for node 3 or go back to node 0. So, this Q 2 0, just got back to node 0 Q 3 0 node goes back to node 0, with some probability I go back to where I came from I just leave the network.

So, this is how Jackson network works. So, this picture may look quite messy but really the concept is very simple, you have exogenous traffic of different rates if you do not want to think of this r 1, r 2, r 3 as coming from the same process lambda 0 and then split that is perfectly. You can just think of r 1, r 2, r 3 as being rates of independent position processors coming at exogenously to these nodes.

After completing service with some probability Q ij I go from node i to node j. On probability q i 0 I leave the network, it is as simple as that. So, this picture may look unnecessarily complicated but it is really not that complicated. And so you have this whenever I am in node I the successive choices of going to some other node j this probability Q ij and these choices are independent across successive service completions at node i.

We also assume that the routing is instantaneous, so there is when I leave node i and go to node j the moment I depart node i I have already entered node j, these are all assumptions that we are going to make. So, this is called a Jackson network. Now if you look at this system we want to find out how the system operates. In fact what we can see is that the system corresponds to a Markov chain. So, we are going to put a state of the system at some any time t I am not going to make time t explicit.

So, I am going to call it some vector let us say m 1, m 2 dot, dot, m k which corresponds to the, so m i is the number of customers in node i. This is my state of the system and we are going to argue that the state of the system m which is simply the vector of the number of customers in each node evolves according to a CTMC. And see that is easy enough to see because you have the exogenous arrivals are poisson and then you are splitting the departures in iid fashion.

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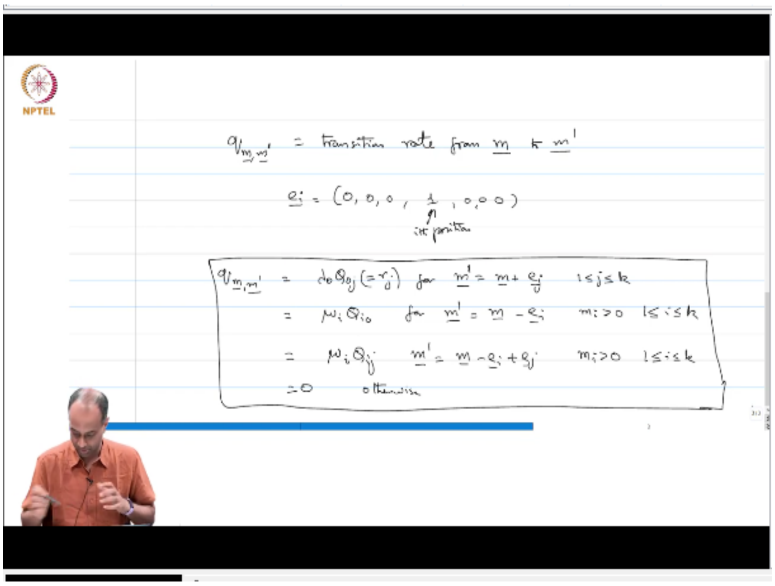

Let me now say how the transition rates work let us talk about q m, m prime. These are transition rates from m to m prime. So, you have 2 different states m and m prime which correspond to the number of customers in each queue and what are the transition rates? Now what you can easily show is that if so what are the 3 things that can happen, there are only 3 possibilities.

So, let us say the system is in some state m, m 1 through m k and so what is possible here. Remember that you cannot have simultaneous arrivals or departures, so we are going to ignore that. So, they could be the first thing that could happen is an arrival to some state. So, if there is an arrival to an ith node this the ith entry m i alone will increase by 1. That is possibility number 1.

The possibility number 2 is that some packet, some customer in some node j complete service and leaves the system in which case m j the jth entry will reduce by 1. The third possibility is that somebody completes service in some node j and goes to node I, in which case the state will go from the jth entry will reduce by 1 and the ith entries will increase by 1. These are the only transitions that are possible.

So, if you look at if you just define ei to be the vector of 0s except there is a 1 at the ith position, otherwise all 0s. Then you can write down the transition rates as follows. You can write q m, m prime as, so let us look at the arrival first, let us say there is an arrival in an exogenous arrival to some node j. So, that will happen with rate lambda 0 Q 0 j which is simply your r j.

This is the rate of the transition whenever m prime is m + ej. So, so you can go from m to m prime or m to m plus ej at this rate or you can leave, you can have mu i at node i you can have a service, so which happens at that rate for m prime = m – ei. So, this is true only for m i greater than 0. I mean you can have a departure only if there is somebody there has to be a non-empty the node i has to be non-empty for this to happen, whereas the first one can happen in any case.

So, this is true for it can happen in any of the k nodes that even can happen in any of the k nodes, the second transition which corresponds to an exogenous departure happens when you have somebody in service at least 1 person in service and third possible transition is that you have a departure from one of the nodes immediately joining some other node. That is also possible. So, that happens with probability mu i Q ij.

So, I complete service in node i but I do not leave the system I instead go to some other node j. This happens at rate for this case m - ei I leave node i but then I end up going to node j.

This is for m i greater than 0 and for this is also for 1 less than or equal to i less than or equal to k. So, these are the possible transitions of the CTMC. So, what are we saying the Jackson network described above the state m which is the vector of the number of customers at any given at some time is evolves according to a CTMC whose transition rates are given like this.

You go from a state vector m to a state vector m prime at these rates and there are only 3 possible transitions that are possible. All other q m m primes are 0. So, maybe I should write that also, maybe I should just erase this and say equal to 0 otherwise. No there is no other state transition. These are the only possible transitions. Now for this kind of a process I want to determine.

So, these are the transition rates, so what are what are the things that are of interest to me? I want to determine the steady state behaviour that is the main goal of this exercise. I want to find out p m, p of m where m is some state vector. What is the probability that my state vector is m? Now the issue is this system reversible? The answer is that the system is not reversible.

So, even if you look at this very simple system which this you know this tandem queues I should have probably mentioned this in the previous lecture itself. Even this tandem queue system is not really reversible see we are making heavy use of reversibility arguments and Burke's theorem for the first queue to infer something about the process in the second queue and concluding certain independence and M/M/1 property and all that.

But if you look at the process X t, Y t even in this tandem queue the process X t, Y t considered as a vector, of course this is a very simple Jackson network, it is not a reversible CTMC at all. If you look at X t, Y t as a state vector because you can have a transition that corresponds to a decrease in X t and an increase in Y t; whenever there is a customer going from one queue to the other but the reverse transition is not possible, you do not have any customers going from Y t to X t.

So, the network is not reversible. So, I mean it is pretty clear that the more general Jackson network that we have put down here this messy network cannot be reversible. So, the Jackson network running in forward time and in reverse time will not look statistically identical at all. But we can still use some guesswork theorem to conclude what its steady state properties look like. So, the key issue is that although the Jackson network is not reversible if you run the time in reverse the hypothesis is that the time reverse system is another Jackson network.

It is not reversible in the sense that it is statistically different from the network we have put down but the reverse system will be a different Jackson network. And as it happens we are able to guess the reverse transition rates of the reverse Jackson network we hypothesize that the reverse network is a reverse Jackson network with a different set of rates. We are able to do some guesswork and guess those reverse transition rates and from there we are able to infer some steady state probability using the guesswork theorem that we studied before.