Integrated Photonic Devices and Circuits
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Lecture - 18
Optical Waveguides: Theory and Design: Orthogonolity of Guided Modes

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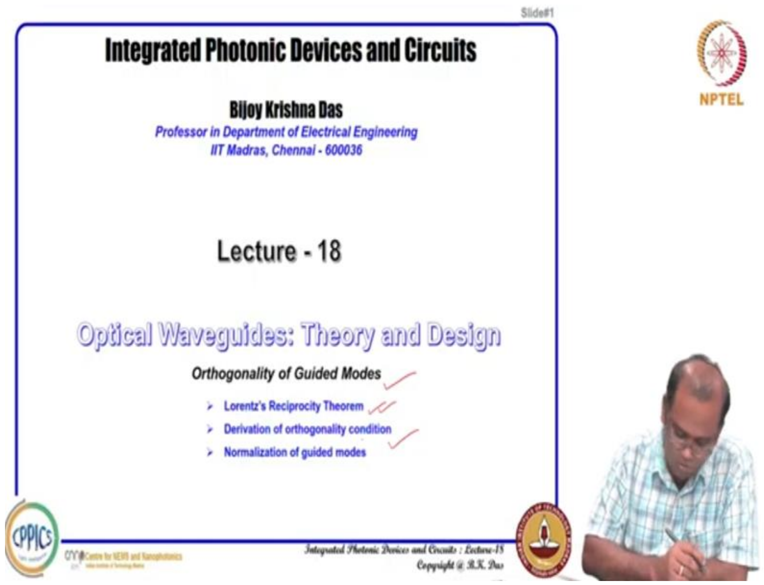

Hello dude, so in this lecture today, I will be discussing about orthogonality of guided modes. In the previous lecture, I have discussed how the guided modes can be solved for a given waveguide structure. And we have also discussed about the dispersion quality, dispersion relation of the guided mode and also we have discussed about the polarization convention. So, there are 3 different types of modes are there TE-light mode, TM-light mode and hybrid mode.

So, all these guided modes for a given waveguide structures, they follow certain rule and then major and important rule is they are basically orthogonal. So, orthogonality any Eigen solutions any Eigen equation if you solve normally whatever the solutions you get typically they are called they are orthogonal and in light electromagnetic wave point of view propagating in a waveguide we have to establish some mathematical framework to utilize this orthogonality condition for designing various type of waveguide structures, devices, circuits etcetera.

So, towards that direction, we need to understand first very important aspect of electromagnetic theory is the Lorentz reciprocity theorem, you have to consider any 2 modes for an electromagnetic wave in a system and we can show that there are certain relations if they follow any arbitrary 2 modes, and that relationship is known as the Lorentz reciprocity theorem. We will be just discussing that thing just a proof the relation, how it is derived?

And that Lorentz reciprocity theorem is used to show that the guided modes in a waveguide are orthogonal. So, we will be then discussing the derivation of orthogonality condition we need some mathematical relation and to prove that and if it is satisfying that particular relation then we can say that, this modes are orthogonal then they are guided modes by the most optimal Eigen values Eigen vectors and then we will just show the normalization of guided modes.

So, in next lectures, we will be discussing about coupled mode theory. So, in a coupled mode theories, we will just try manipulate the guided modes to interact each other then, so, that they can interact each other so, that we get the particular functionality out of the system. So, in that case we need to know how to normalize individual guided modes in the beginning when you are launching. So, then we know that this is the normalized mode and then as it propagates, how they are interacting among themselves, you will be able to quantify that.

So, that is what why we need to do normalization for any guided modes of whatever you are getting solution out of the Eigen mode solver that has to be normalized first.

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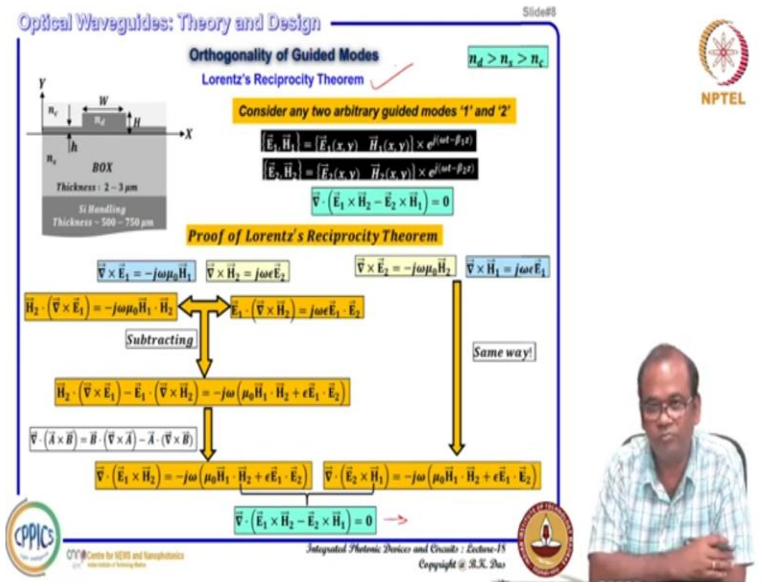

So, let us move on. So, again I take a reference of this type of structure, it is just x direction and y direction and z is the x, y, z that means, out of the screen and that means propagation direction. So, mode will be confined in the xy plane and 2 dimensional waveguide mode and then it will be propagating along z direction. And then let us consider any 2 arbitrary guided modes, let us consider it is say for example, any multimode waveguide structure W is consider such that it is a multimode structure.

Or any other type of waveguides also you can consider where you can have a number of modes can be supported, it can support a number of modes, meaning it has a solution number of guided mode solutions it can happen. So, as I mentioned earlier that whenever you are solving full vectorial all the way vectors and all the Eigen mode, Eigen values you solved some of the modes you will begin seeing as a guided mode and some of them will be leaky under editing modes.

Many of them you will be getting only we are interested we will be focusing only on the solutions which are actually guided modes, because that is what we need for our integrated optics, or photonic integrated circuits. And here also we will be just restricting ourselves any 2 arbitrary guided modes not all other mode solutions. So, first guided mode we designate as mode 1, we just give an index like 1, where associated electric field vector is E 1 and magnetic field vector is H 1.

And you can have this vector field entire vector of the field that can be x, y dependent mode and travelling wave factor that that means omega t - beta 1 z both this one will be multiplied to magnetic field and electric field. Similarly, mode 2 you consider another mode, where the electric field under associated magnetic field is designated E 2 H 2 and x, y distribution for magnetic field and so on.

And then this Lorentz's reciprocity theorem says that any 2 solutions Eigen mode solutions. And in this case, we will be only focusing on guided mode, but Lorentz's reciprocity theorem for all types of modes, but here we will be trying to take it as a restricted guided mode I meant to say so, that is a that divergence of E 1 H 2 E 2 H 1 and if you subtract them and take a divergence, then it will be 0 that is actually Lorentz's reciprocity theorem.

You know that E cross H that is the value is known as pointing vector right pointing vector it is called pointing vector that means energy flows along in the perpendicular to both perpendicular to the E and H, but whenever electromagnetic wave propagates a guided mode for example, at different modes propagates, so, then I obviously, I can say that what is the total field is total energy is flowing, I can get total field of electric field coming out of all the modes total field coming out of all the magnetic fields.

Then I can say that E cross H what is happening that is a total energy, but if I restrict ourselves, what is the energy carried or power carried by individual moves, then we must write E m cross H m star half real part and this is a per unit area you have to integrate that is the energy flow energy carrier or power carried by mth mode. But in this case we are involving 2 modes not 1 mode, 2 modes. If 2 mode is there, then if I try to get E 1 cross H2 not E 1 cross H 1.

I am not considering E 1 cross H 1 that is the power carried by the 1 mode. But, we are considering E 1 and H 2 and vice versa E 2 and H 1 and then if I try to take they have some they will have some value. But if you subtract them and they have taken divergence total value that will be 0. I want to prove that first because that will be helpful for setting up orthogonality condition for the guided modes.

Let us take a curl equation. So, curl of E 1 = - del B del t, j omega mu 0 H 1, minus del B del t instead of that I am writing and curl of same mode, say mode E 1 and curl of H 1, E 1 I am just entering here, H 1 I am entering here H1 is what that means curl H equal to del d del t which is a dielectric medium sigma equal to 0. So, j omega epsilon E 1 so, it is a mode 1 curl equation and just writing for curl E and curl magnetic field that is for mode 1.

Similarly, for mode 2 curl of H 2 j omega curl of H = j omega corresponding E 2 curl of E 2 - j omega mu 0. So, all these all equations. I clad this because here you see E 1 cross H 2 that is what I have just written E 1 and H 2 curl equation I have just written side by side. Starting from here next what do you do this equation you just take H 2 as a dot product both side, H 2 dot product both side so H 2 this one, you are getting this one and here H 1 was there is 2 same thing I am writing repeating just H 2 dot product I am writing.

And from this equation what you are getting you are just taking E 1 dot here because H 2 involved, E 2 involved. So, we are just bringing E 1 here. E 1 this one and then E 2. So, 2 equations I just got here, let us see what is happening here, subtract them, just subtract this thing H 2 curl of E 1 this comes here and this comes here, subtracting left hand side comes here, right hand side comes here, minus this minus a.

So, if I take minus j omega is coming like this. So, I am subtracting this 2 whatever starting from the curl equation the straight forward, then what do you do, you use this vector identity, you see, if you take a cross of 2, any 2 vectors A and B and take divergence, they can be just written as like this. So, it is like A dot B cross C this is a vector, this is a vector. So, vector identity you can write that actually, you we borrowed from the vector algebra, so, we can write like this.

So, it is something right hand side if you see it is like this, this left hand side of this one this vector identity right hand side it is looking like that, what is that that A is E 1 and B is H2. So, if we are writing A is E 1 and B equal to H2, so, then divergence of A cross B equal to this one. So, divergence of that means, we can write this thing, this thing together left hand side as a divergence of E 1 H 2 E 1 A cross B divergence of A B equal to equivalent to E 1 B equivalent to H 2.

So, from the vector identity we are just after subtracting left hand side can be compacted as like this form. And what about right hand side as it is I have written so, I get 1 equation here, this is 1 equation. Same way, if I start from here, same procedure, for example, here, we have just taken took a dot product of H 2 here I will be taking a dot product of same H 1 and here I will be taking a dot product of E 2 then follow the same process same way, then you get another equation.

Another equation there if you just compare they are identical basically E 1 is converted into E 2, H 2 is converted into H2, otherwise same you are trying so, but they are different basically. So, in this 2 equation, if you see first term E 1 H 2 of Lorentz's that expression we have achieved and second one, E 2 is 2 that means, we have to subtract these 2 if you subtract the right hand side is identically fixing. So, if we subtract, then it is 0 proved. So, that is what we have written. If you just subtract them, then divergence of E 1 H 2, E 2 cross H 1 = 0, this is actually called Lorentz's reciprocity theorem. We will be using that next.

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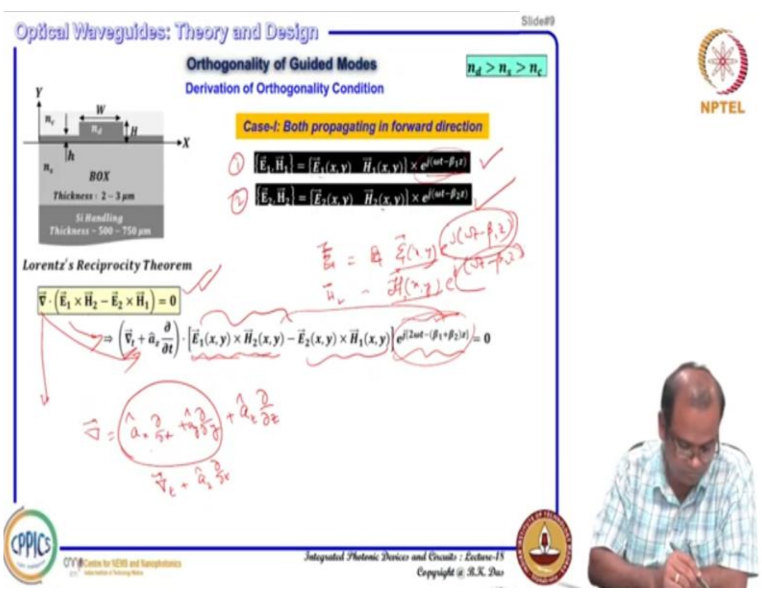

We have written Lorentz's reciprocity theorem, we have considered 2 modes, one mode 1, mode 2 and start from Lorentz's reciprocity theorem, you know this one is Nabla is a vector we represent like this. So, I just decompose this is x, y because our waveguide cross section is x, y cross section I just combined them together as a one vector called del t, transverse Nabla. So, this is a 3 dimensional Nabla where having x, y, z all the components are there.

But if I just put transverse, transverse means transferred that cross section and transverse section of the waveguide, that is del t, so, I can write del t + a z del z. So, that is what this Nabla is decomposed into this one and dot product. And then this E 1 H 2 that means this E 1, whatever value here E 1 multiplied by this one and then H 2 is multiplied by this one. So, you are just putting E 1 x, y H 2 x, y and then this one.

So, E 1 if I write E 1 that means, this means non italic. And when I am writing here E 1 x, y e to the power j omega t - beta 1 z and H 2 means I will be writing H 2 x, y e to the power j omega t - beta 2 z. So, E 1 H 2 means, E 1 x, y, H 2 x, y E 1 x, y, H 2 x, y and this one and this one if you multiply e to the power j omega t beta 1 + beta 2 z. So, that is nothing so that means the is one multiplied by this one that is taken factor.

Similarly E 2 H 1, E 2 x, y, H 1 x, y and then if you multiply E 2 and H 1 this phase factor will be the same that is common to both the term I have written this one so, that this way I am writing so, that I can take this as a factor this can be removed.

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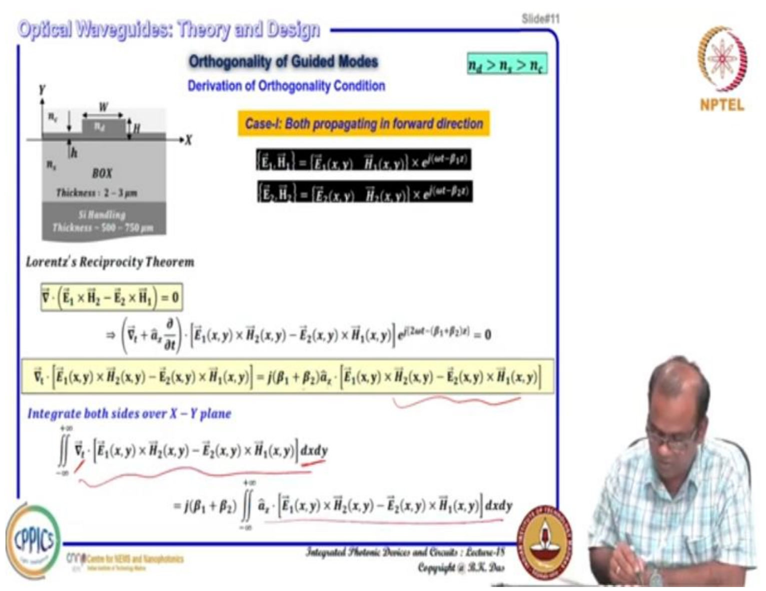

So, this is actually equal to 0. What do I do then I just did this Nabla t transverse dot I have taken this one, this I have taken x, y and then these one if you just do operate here all these are x this should be del z not t, this would be del z and then del del z if you operate then what you will be getting you say beta 1 + beta 2 z will be factor. So, minus j beta 1 + beta 2 you will be getting right. So, this will be here minus and you are taking that right hand side.

So, this may be minus, minus into minus plus. So, that will be equal to 0. So, this side this omega minus sign omega. So, that means, what I am writing, I am just operating on this one first term I keep as it is because it is a transverse del del x, y I do not know the whatever the value is there that derivative will be there I just kept as it is and this one if you operate on that that means you have to operate on that this is the z dependent function is there.

So, minus j beta will be there, minus j beta if you just operate on this one we will be getting beta 1 + beta z in both cases here and here. So, that is the value we will be getting this side. So, I think this is a correct sign, no minus sign up there, because here you will be there and there will be there. So, that will be taken to that side. So, one term this one, another term you will be getting this one this term will come contributes here as well as here. So, you can write like this.

So, this term will be constant this side this side will be there along with that is z dependent function will be there just simple derivative is just to listen not rocket science anything. So, now, next step important step you have to integrate both sides over xy plane, xy plane is the wave guide plane cross sectional plane, if I just integrate both side over x, y, so, this is almost divergence transverse 2 dimensional divergence.

I would say of this one and you are just integrating what dx and dy and right hand side j beta 1 + beta 2 and then is a z is there, a z thing is their dot product, you are just good putting that product here. So, you are writing minus a beta / t and that is will be going this is the thing is there. So, right hand side, again integrating minus infinity plus infinity will be getting like this. So, left hand side, right hand side.

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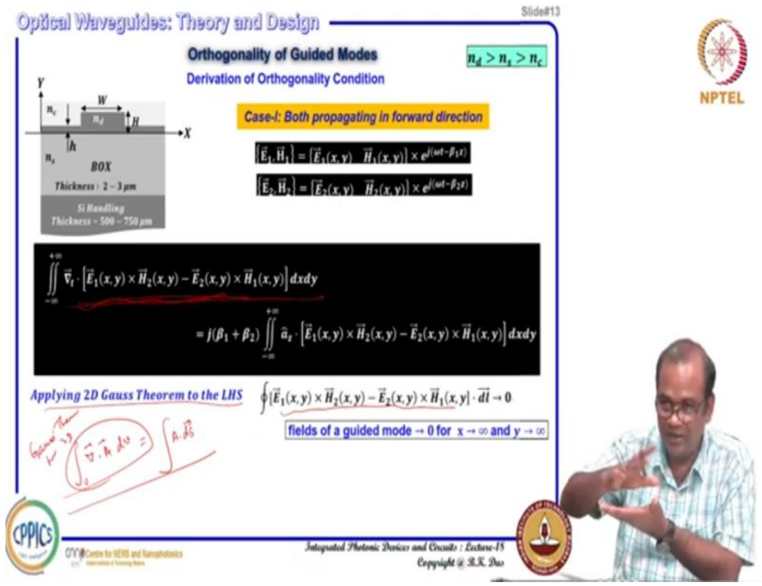

Now, let us focus on left hand side. they are I have just highlighted here left hand side and right hand side again just to proceed, delta t Nabla t, and here right inside this one. Started Lorentz's reciprocity theorem again. Now, applying 2D gauss theorem to the left hand side, this one. What is gauss theorem? Gauss theorem says that if you are taking a volume integral divergence of any vector that will be actually equal to A ds surface integral.

So, if you are taking a divergence of a vector and then volume integral that will be exactly equal to surface integral of that volume, the volume surface whatever the value is there you take the dot product to the surface element and then integrate over that will equate to the divergence of the make that vector that is actually called gauss theorem of divergence gauss theorem, for 3D. Gauss theorem for 3D, but we want to apply this gauss theorem in 2D case.

Here yet also divergence but 2D only the xy plane your z plane you are forgetting. So, when you are considering 3 dimensional volume and applying gauss theorem, you are reducing it to surface but when it 2 dimensional problem divergence theorem, if you are reducing then it will be reduced to the perimeter line. So, 3 dimensional will reduce to a plane and you are getting a surface integral, 2 dimensional divergences if you use gauss theorem you will be reduced to a line perimeter only 1 line you will be getting.

So, that is why the left hand side if you just apply divergence theorem, the entire vector and writing dot dl in the perimeter So, If you have a surface waveguide surface, you are taking divergence and then integration progression over the surface instead of that ultimately if you do the line integral of that surface, whatever the cross section you are considering only the perimeter region if your value you are just taking and then integrating that will be the same.

That is that divergence it is a vector same way divergence theorem I am using. But problem is that once you do that, and I mentioned that while starting from the power calculation everything in the previous lecture, we have seen that you can do E cross H minus infinity to plus infinity, x can be minus infinity plus infinity, y can be also minus infinity to plus infinity. So, this can be done. So, here also when you have cross section you are taking a large area.

But your waveguide core is very small area, large area cross section if you are integrating and then you are reducing to the line integral and guided mode when x tends to infinity plus minus y tends to plus minus infinity that particular region when it is extended to infinity. The field E 1 is to all if they are from guided modes their actual this should be 0 because guided mode is confined within the core.

So, outside it will be 0 outside the core far away from the core x equal to plus minus infinity that will be 0. So, that means this left hand side if you apply gauss theorem, you are applying gauss theorem that is actually directly you can just use. You do not need to use gauss theorem left hand side similarly gauss theorem right hand side because left hand side goes there actually helps to simplify the right hand side as it is am keeping.

So, left hand side if I just extend minus infinity to infinity obviously, this is 0 no doubt about, it is for any cross sectional for large cross section if you are doing that is true, but if you are just concentrating your cross section lower than the waveguide mode size that is not correct. So, we are applying in your integration to extend large cross sectional area. So, now, let us move on.

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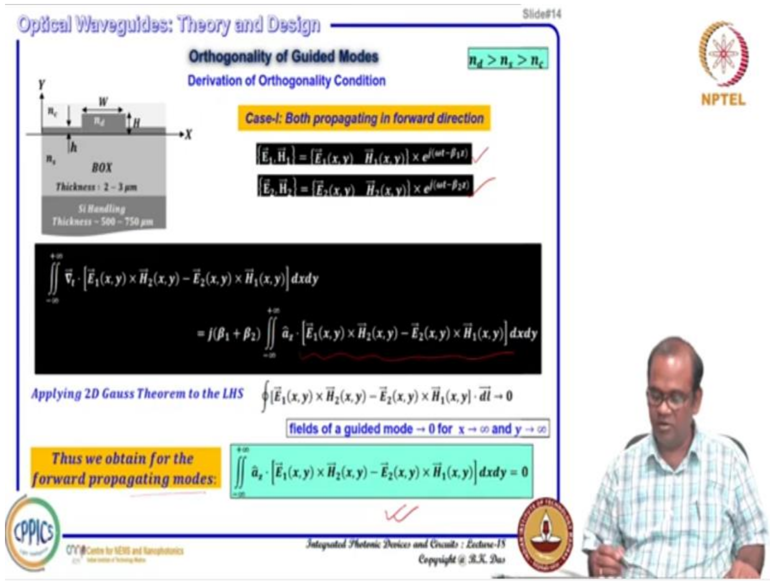

So, we can write we obtained forward r propagating modes like this only this one, we are considering 2 modes left forward propagating because minus, minus we are considering right for forward propagating minus omega t - beta z that is meant for forward propagating wave. So, the left hand side, right hand side and left hand side is 0 means right hand side equal to also 0 must be equal to 0 we have written that so that is good.

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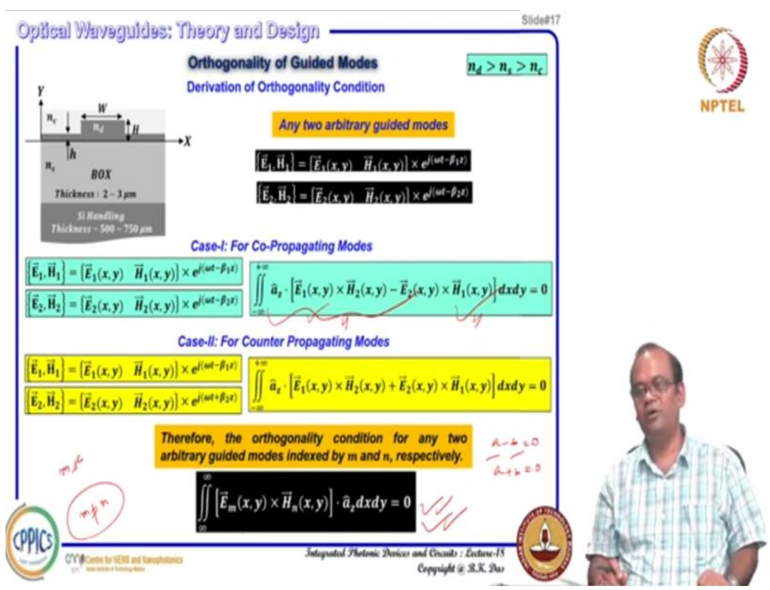

Now, I just consider for co-propagating modes whatever we have derived this one, starting from the Lorentz's reciprocity theorem. Now, you see this mode, mode 1 propagating in the forward directions mode 2, propagating in the forward direction. Now if you just consider this, both the modes, they are free to because if a mode is allowed to propagate in the plus z direction, it can also identically propagate in the minus z direction as well, mode 2 also case will be same.

Now, we have tried to see that 2 modes identical mode distribution, they are sending energy in the forward direction. Now, one of them if it is in negative direction, then also we can try to use that is also more solution that also obey Lorentz's reciprocity theorem. So, one of them is the positive direction another is the negative direction counter propagating. So, in that case, if you treat same way, then what are you supposed to get a look at this thing.

So, one is beta 1 z – beta 1 forward propagating and other is negative direction, but the same mode, mode field distributions are same, H 2 I am considering the same only their direction changed, field distributions everything is same direction. So, normally, you know one electric wave field if wave propagating in this direction and if it is E filed here this direction, then H field will be out of the screen that will be proportion direction this is H.

Now, we propagation if you are changing, then this E field sign will be changed. So, not only the sign in the phase factor will be changing also the transverse direction this magnetic field or electric field one of them will be directionally changed minus sign will be using. So, in the user of here Lorentz's reciprocity theorem, that particular fact you have to keep in mind. If you do that, then you are getting similar equation almost like co-propagating mode same, but except minus and you are getting plus here. That is the difference.

So, for 2 modes, propagating in the opposite direction, you can use reciprocity theorem, you are getting the same divergence, gauss theorem divergence theorem you can use and then integrate over the entire cross section and then you are getting this thing. Now you compare these 2 they are same mode just direction is different, mode means normally we will consider beta 1, beta 2, that is the solutions that actual pattern I am just considering.

So, that pattern field distribution x, y distribution and whatever the field distribution here whatever field distribution here they are same, whatever field distribution here whatever field distribution here they are same. So, in this 2 equation, you do not have any use of beta plus or minus nothing is there that is factor already rather, you have the mode same x, y cross sectional distribution is there, if you compare them It looks like that, it is like a - b = 0 and a + b also equal to 0, a - b = 0 and a + b = 0.

What does it mean? It does means a = b. So, that thing this should be equals them. So, if that is the case, then what I can write a this one should be equal to this one, we have written that. So, if I write that equal to this one, then for any 2 modes s sorry, not that equal to i totally, if you have a - b = 0 and a + b = 0, that means each of them equal to 0, a also will be equal to 0, b = 0 otherwise, a - b = 0, a + b = 0 will not be there.

So, in that case, what you can write this one equal to 0 now, this one equal to 0, I just take first term, instead of 1 and 2, now, I just generalizing this thing, I can consider mth mode, one should not misunderstood whether one means the first mode and 2 means second mode it can be any, if you have a 10 different modes, you can choose any one of them n if it is different modes, you can choose n is one of them obviously, while deriving this thing.

We have considered m not equal to n because if they are same mode, then E 1 and cross H 1, it will be H 1 right that means power carried by their same mode. So, whatever I have solved here that is you have to consider that the both the mode separate modes that can be forward direction also backward direction one of them you can choose you are just cladding one mode and another mode forward direction.

And or that same would reverse direction these 2 situations are considered, but there actually different mode m not equal to n that is what you have considered and you end up with this one and that is actually called orthogonality condition. If 2 modes you are just considering and mth mode, you are taking electric field and nth mode you are taking magnetic field, if you are taking a cross product, just cross product alone it cannot be 0, but if you integrate over the cross section, then only it will be 0.

So, E cross H, if I take E field from one mode and H field from another mode in a particular point and take E cross H I may get a value. But the situation here is saying that if you are taking

E field and H field at a particular point and you integrate that value E cross H, E 1 cross H 2 over the cross section, then only you will he will be getting 0, one particular point it may get value, but integration that will be 0 that is actual orthogonality condition.

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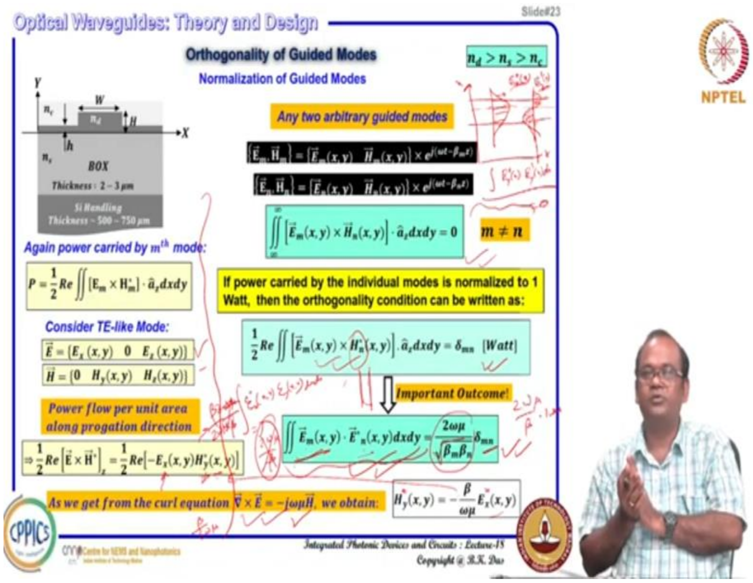

So, that can be a little bit proceeding forward. So, we have written this is the so called orthogonality condition; we are just trying to see that, this orthogonality condition how we are using for our other application like normalization of a guided motion. So, let us see okay. So, now we see I know that power carried by a mth mode is just power half real part E m cross H n m, n same that means is this n become now m same mode and z direction.

And this is again and again we have discussed that the total power carried by a mode mood can be expressed like that, now move on. The power carried by the individual mode is normalized to 1 watt. Then the orthogonality condition can be written as this one, you will be writing like this half E m H n = del mn watt, if you see if this is equal to 1 watt, we know that power carried by a mth mode.

Now here m and nth mode you are considering when you are putting m = n that means that is becoming this equation. So, m = n that means del mn and this should be 1 watt m not equal to n that will be 0. So, this thing in general just considering the power carried by a mode and whatever the Lorentz's reciprocity theorem we have used it cladded them together you can write orthogonality condition in this form.

Now, you will see, let us consider a TE like mode, I have discussed earlier that a guided mode can be TE like, can be TM like, can be hybrid also, if I just consider a TE like mode that means, your electric field will be in the xz plane and magnetic field in yz plane, y component, z component and obviously, I can always consider that E x probably x component, z component in E x normally is higher.

So, that transverse electric field is there considering propagation direction and here H y can be larger that type of specialization can constrain may not consider also that is fine. Power flow per unit area along a propagation direction we can represent like this this thing actually we are writing again E cross a star just any given mode we are considering and considering these components associated with a mode you just take E cross H considering these are the components.

E cross H that means you can consider like this a x, a y you need vector a z and then you have E x, 0, E z and here you will have 0, H y star, H z star. Now, power flow along z direction if I just consider that means, I have to consider z component of the E cross H that means E x multiplied by H y. So, this one multiplied by the E x H y like that I think this would be plus sign probably yes E x, 0, 0 and H y star. So, this would be plus saying that.

So, power flow per unit area along the propagation direction is a plus this component E x H y component if it is known then we can write like that fine. Now, if we just tried this work this one this one is a little bit modification I want to do how we are doing because instead of electric field, magnetic field component we know that once we have electric field component for a given mode, I can find out what is the magnetic field component.

From this curl equation we have seen earlier already using the curl equation and this field component I can find out what is the value of H y like this. So, if this H y I am getting I think this minus and minus sign will be cancelled somewhere. So, then whatever you will be getting this here if you are putting you can write this equation you can modify it here if you are just writing this one then what we are getting beta / 2 omega mu, beta / 2 omega mu times E x times E x star.

This H y star means H y star if you write it will be E x star, E x / E x star that will be 2 omega it will come like this power plug will be 2 omega mu / beta times E x, x, y and E y x, y dx dy that will be power flow to omega if that is actually 1 word like this whatever you consider, so, this equation we can now modify in terms of only electric field associated to the nth mode also. So, electric field with a mth mode and this comes electric field with the nth mode of the magnet this magnetic field converted into nth mode.

Then this will be multiplied by 2 omega mu / beta the side that should be equal to 1 if I just simply write this, this thing, should go that side there so, that, this would be beta / 2, this would be a beta / 2 omega mu that is why it is coming this side 2 omega mu this one. So, instead of beta since we are involving m and n, that is why we are writing here just simply because m = n means it is beta m basically, it will come otherwise it will be 0 del m will be equal to 0.

That is why in generally write beta m times beta n times square root. So, if they had their ultimate 2 omega / beta it will be there because here if you just put this one, this one you are putting then it will be beta / 2 omega mu this power will become. So, just use your pointing theorem, what is the power carried by a mode and starting from Lorentz's reciprocity theorem, where you are getting m not equal to n clubbed them together, then you get a very nice equation involving only electric field.

So, you do not need to bother about magnetic field. So, this thing actually, we can use wherever necessary for orthogonality condition, we can check that if electric wave coming from mth mode and electric wave coming from nth mode, if you are taking their if they are vector you can take a dot product and the integration over dx dy and if they are identical mode, then you will get 2 omega mu / beta times 1 watt.

Because 1 watt is always there other is there to be 0. So, electric field of 2 different modes, if you are having then just integrate over them, then they will be 0 you just take an example, suppose you have a 1D waveguide if you see fundamental mode will be like this. So, this is you are say E y, 0, x this is the x axis, this is the z axis, this is your x axis and then higher order mode if you see then it will be like this.

So, this is for example, electric field the positive direction oscillating and 1 half higher order mode E y, 1 x higher order mode, 1 half positive and another half negative. So, if you just try to see, see this equation that means you have to see E y, 0, x and E y, 1 x, dx what would be the value because you know if we just every point you have to find out what is the value of E y x E y 0 x and what is the value of the E y 1 x.

So, if you just see upper half, if you just multiply them and integrate you get a positive value and lower up you just multiply them and you get an you get integrate them you are getting a negative value identical amount. So, if you integrate them you are supposed to be 0 that means any 2 mode, it can be 10 modes out of 10 modes, any 2 modes you consider see the electric field distribution, this would be identical polarization identical type of situation consider if not you have to take a dot product of course.

So, any 2 modes, there are electric field distribution if you take and if you integrate over the cross section, there must be equal to 0 and if they are coming from save mode, if you integrate they will be normalized to 2 omega mu / beta propagation constant. So this part, I will be using to describe coupled mode theorem which is the basic building block for the integrated optics or integrated photonic integrated circuits that will be the subject of the our new lecture, the next lecture. Thank you very much.