Course Name: Optimization Theory and Algorithms
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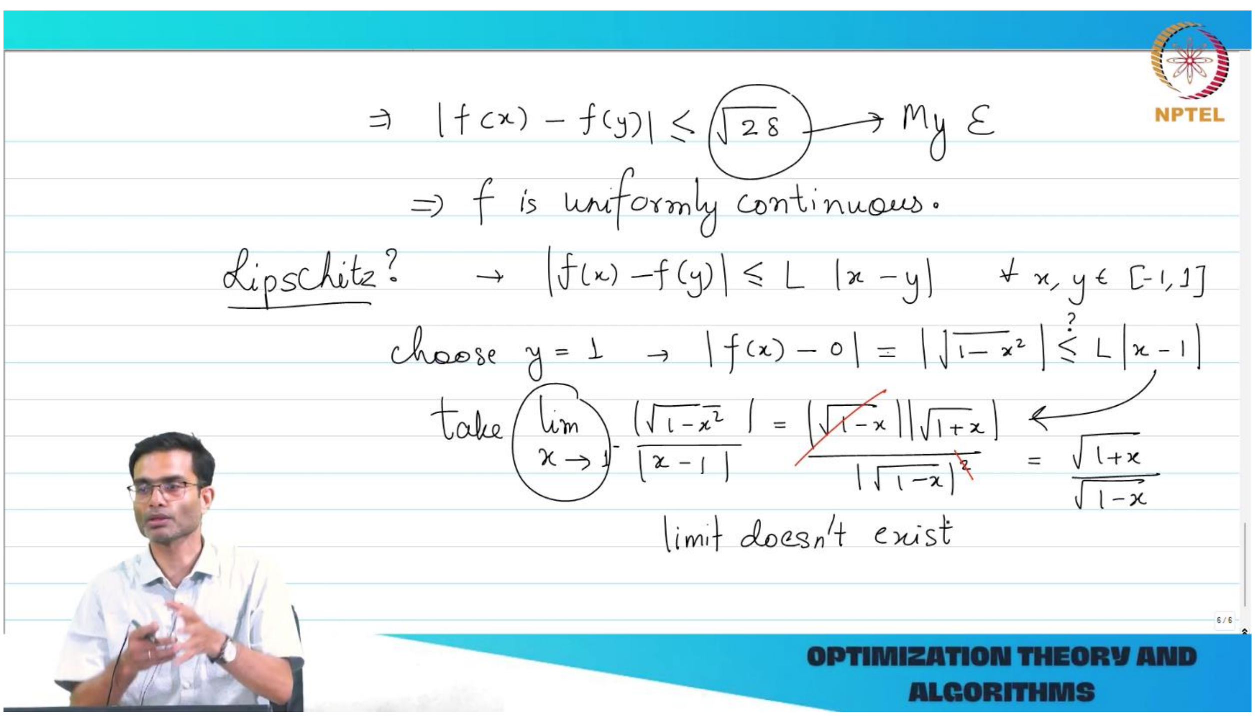
Week - 02
Lecture - 10

Summary of Background Material - Calculus 2

Any intuitive guesses? You do not have to justify, just intuitively. I hear one no. So, I will give you the intuition before we come to the answer, and this is strictly intuition only, which is I mean it is not rigorously correct, but just to drive intuition. Look at the definition of Lipschitz continuity. I have $|f(x) - f(y)|$. If I take the modulus $|x - y|$ on the other side, what does that expression kind of remind you of? Derivative, right?

Now, that means that the definition is saying the derivative somehow should be bounded; it should not explode. That is the intuitive feel for Lipschitz continuity. Now, when you look at this function $\sqrt{1 - x^2}$, is it differentiable, or are there places where the derivative can blow up? At the endpoints, it can blow up because when I take this derivative, I am going to get a $\sqrt{1 - x^2}$ in the denominator. So, that is going to be my derivative. Hint: try to prove this.

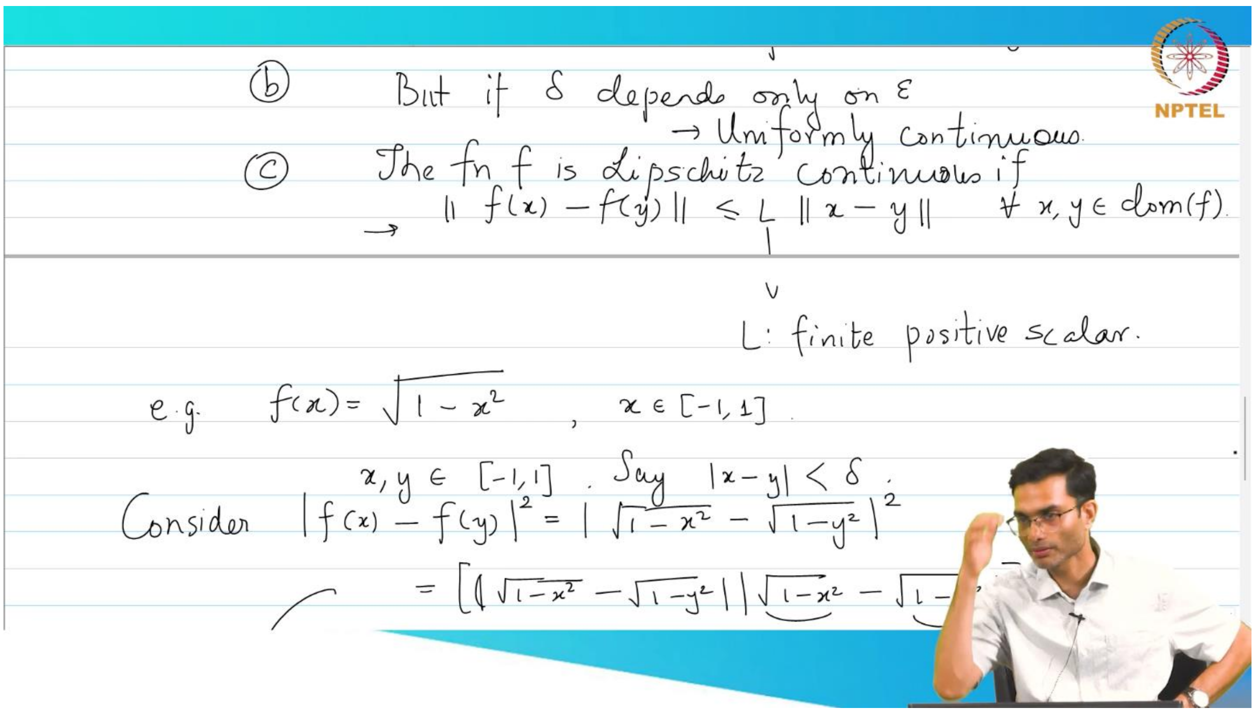
For this, what do I need? I need $|f(x) - f(y)| \leq L|x - y|$ for all x, y belonging to $(-1,1)$. This is what I need to do. Now, as we have already got the hint, endpoints are where some mischief is going to happen. If I prove that this is not true even at one point, it is over. So, let us choose one of the endpoints; let us choose $f(y) = 1$.



So, what happens over here? This becomes $|f(x)|$, what is $f(y)$? 0. So, that becomes $|1 - x^2|$, right? And this will have to be what? Less than or equal to I am going to put a question mark over here, less than or equal to what? $L|x-1|$, okay? So, now let us take the limit as x tends to 1 from the left-hand side of 1 because the other side is not in the domain, right? And what expression I want to take? Basically, this $\frac{1-x^2}{x-1}$. What is this limit? I have just taken this guy onto the other side, and I am asking you what is the limit over here?

Can I open up the denominator somehow? What is going to happen? We can split $1-x^2$ and I can also write this as $\sqrt{1-x^2}$, square root, and then square it.

So, same thing. What do I get? Is it clear? It is $|1-x|$, which is the same as $|x-1|$ and $1-x$ is square of square root. So, can I cancel something? I can cancel one square root; I can cancel this guy, I can cancel this guy with this guy. So, what am I left with?



Numerator: $\sqrt{1+x}$. Denominator: $\sqrt{1+x}\sqrt{1-x}$. And when I take this limit in mind, if x is tending to 1, does this limit exist? The limit does not exist, right? It blows up.

Implies what? I cannot find a positive number L such that this expression is always true. The closer I take x to 1, the more this expression blows up. Therefore, I cannot find an L. So, our greed was not rewarded; this function is not Lipschitz continuous. So, these are very subtle points for every function to evaluate what type of continuity it is, right?

This is a little bit of a generalization from what you saw in high school. Any doubts on this? I mean this algebra is very simple, right? This is dealing with what are the expressions we use? $a-b$ into $a+b=a^2-b^2$. I mean this is the level of stuff that we did. There is nothing very complicated.

Just applying it intelligently to finding out these constants, do they exist? So, with that, I think we have run out of time also. So, we can end over here. Yes, question?

Yeah, but you cannot stay at that constant because the moment I take it closer, that constant gets exceeded.

So, for Lipschitz continuity, I need to find an L and that L has to be fixed for all x and y in the domain. Oh, sorry, the uniform continuity, okay? So, say that again.

Correct. Well, not $51\delta_2$ into 51, right? The geometric sense is that if I am in a domain where x lives and the range where f lives, right? The mapping is taking x to $f(x)$. So, geometrically, what is happening is if I am squeezing the two points over here, I am also squeezing by a proportionality constant over here; that is what the math is telling me.

And that is what is happening: if I am squeezing over here, the proportionality of squeezing in the range is related to the point at which I am squeezing. Then it is continuous. If it is not related to the point where I am squeezing, it is uniformly continuous, right?

So, if I take two points whose distance is 51, their map will give me f which is 51 multiplied by some constant, alright, that is fine. Now, can you make it? Is that going to be true for all pairs of points I can take? If yes, then you have one of these two notions of continuity. Does that satisfy the geometric intuition? Is it Lipschitz?

Can you define the derivative at the kink? So, that may still not, but the piecewise derivatives are finite. So, you may be able to get by. Yeah, but δ^2, I can just define a new constant κ.

But that you are generalizing from one example, you know it may not. Yeah, I am not sure; I mean we can look more. So, by the way, the rest of you, if you are done, you can give your feedback chip to either me or Sai Sanjay and head out.