Course Name: Optimization Theory and Algorithms
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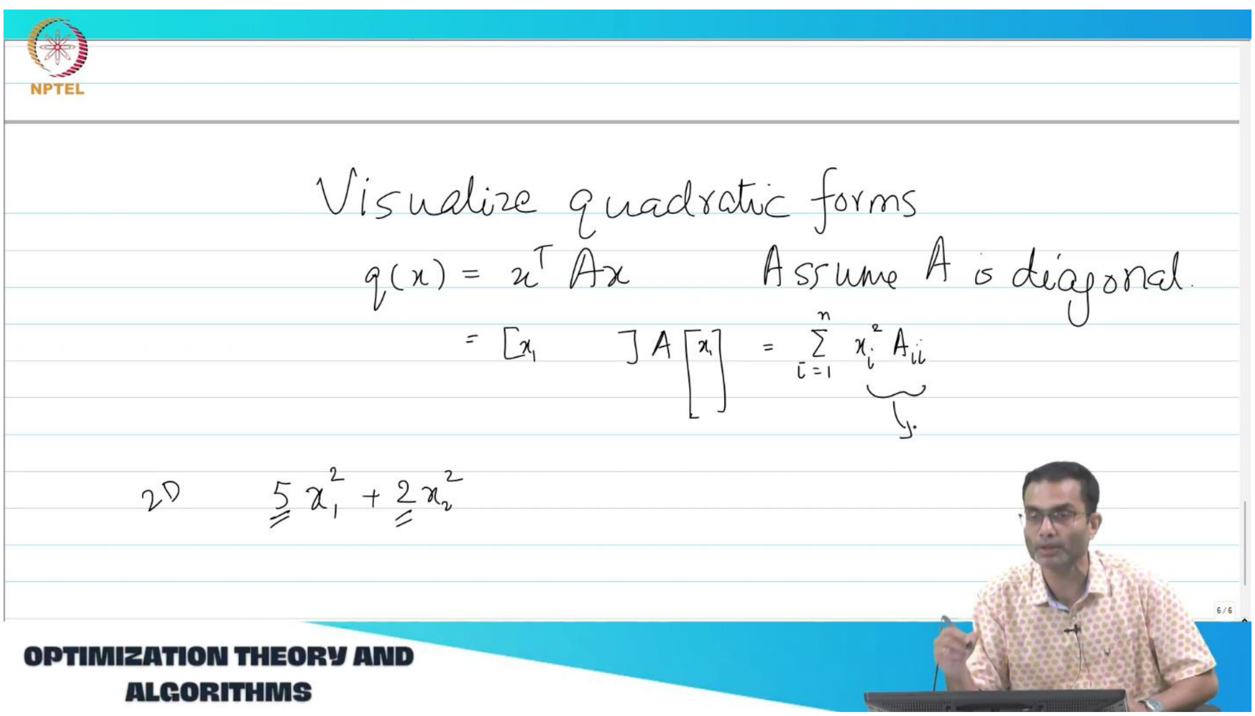
Lecture - 34

Visualizing Quadratic Forms

Now let us having got these basic properties of U in mind, now let us go to something that builds a lot of intuition. Let us try to visualize these quadratic forms, right. We have written down this quadratic form over here, right over here. We have already done a little bit of visualization of this. Now let us look at it a little bit more carefully, ok. So, let us visualize So, I am going to take I am going to write it like this and to visualize of course, it is always good to start with the simplest possible case.

So, the simplest possible A would be a diagonal matrix, right. Now, if so, let us assume A is diagonal. What is our running assumption about A? Symmetric positive definite. right.

So, therefore, can I say anything about A now if I if I am taking the special case that A is diagonal, can I say anything about the entries? They all have to be positive because they are itself the eigenvalues, right. So, if I visualize this like this, right and now I insert A to be a diagonal matrix, if I open up this x^TAx, what do you think I am going to get? I am going to get a summation. So, x_1 here, x_1 here. So, x_1 will multiply by what? λ_1 or a_{11} multiplied by x_1 again. So, what do I have left? $x_1^2a_{11}$, then I have $x_2^2a_{22}$, right.



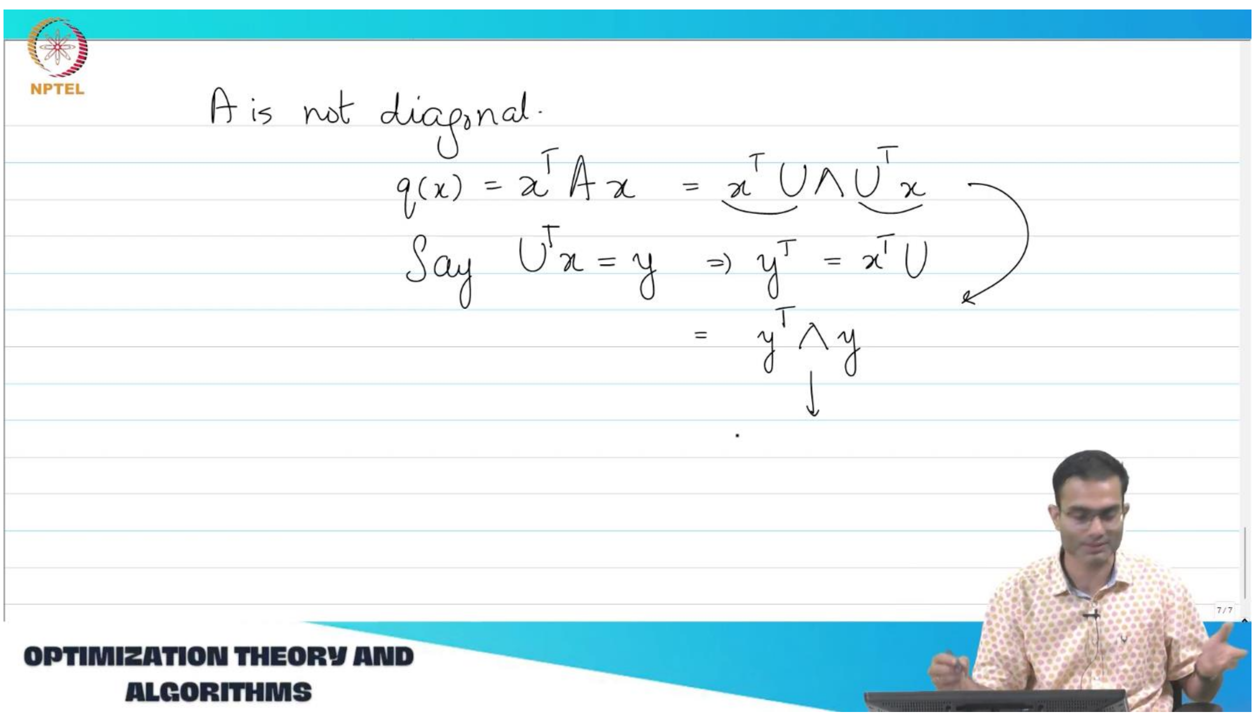
So, this is nothing but

$$\sum_{i=1}^{n} x_i^2 \, a_{ii}.$$

Now, if this were just two variables supposing $n = 2$, right, this would be what x_1^2 something plus x_2^2 something. What is this the locus of? Supposing in 2D this would be let us say $x_1^2 \times 5 + 2 \times x_2^2$. If you want to graph this what will it look like? An ellipse right and these numbers are always positive because the diagonals are eigenvalues which are positive right. So, in general this is going to correspond to what? Ellipsoid in n dimensions ok. So, we can easily visualize this in 2D, I can draw ellipses on a piece of paper.

If I want to visualize it in 3D, you can visualize a like a chocolate eclair right, it is a 3D ellipsoid and then it right. Beyond that tough luck right, at least normal human beings cannot do it. So, this was the case where A is diagonal ok. So, it is a ellipsoid. In which coordinate space? x_1 for example, if I take the 2D case in the x-space I am going to draw ellipsoids right.

If I go on this ellipsoid what is the meaning of this ellipsoid? If I walk on this ellipsoid what is going to happen to the value of ϕ? It is going to remain constant that is why these are called the contours of the function where contour means the function value does not change as I go along it like the map of a trekking hill. So, now let us having understood this, this is really obvious you could have done this in high school, right. The next thing is let us relax this assumption that A is diagonal. So, let us say that A is not diagonal, ok. What should I do then? If I try to write this summation will I get something as neat and tidy as this? No right, because A is a full dense matrix and I am just going to get everything everywhere.



I can write it out, I will get some expression which gives me no intuition. So, is there something to help me out? I can use the eigen decomposition, I have not used that so far right. So, if I if I try that out $q(x)$ which was $x^T Ax$ and $x^T A = U\Lambda U^T$, then

$$U \Lambda U^T x.$$

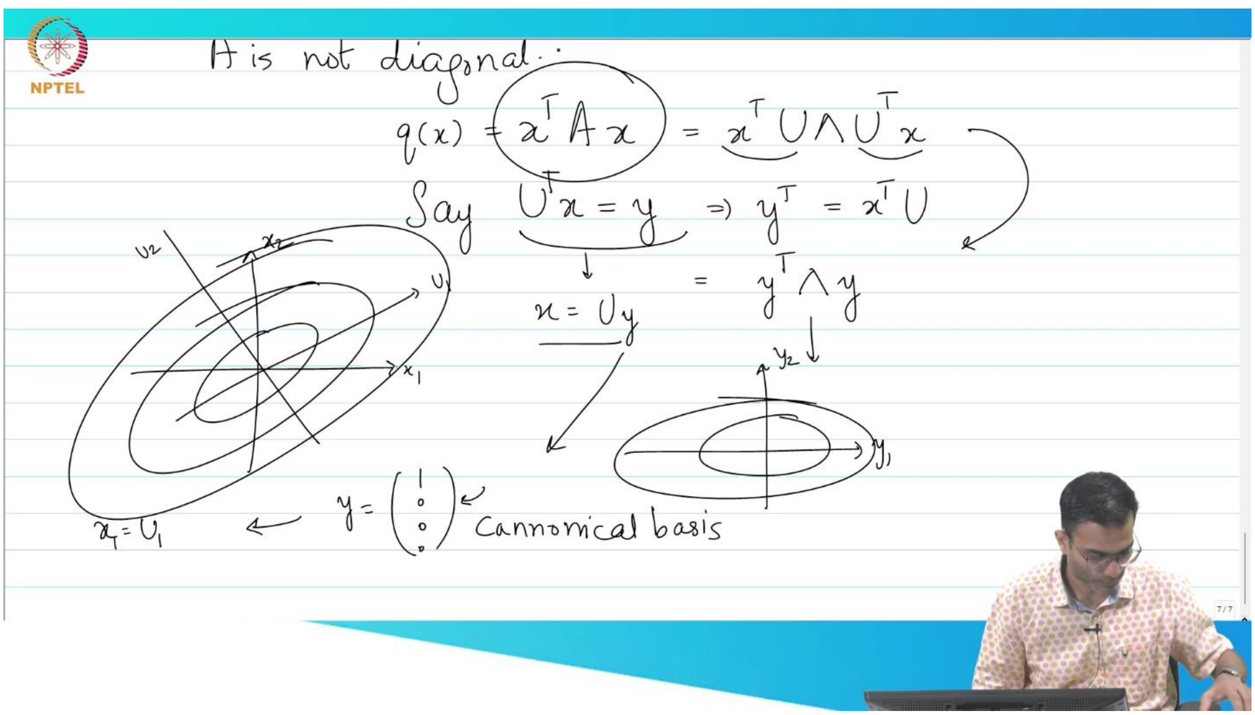
Does this look like the previous case? In some way if we can bunch terms together right, if I can group this and this I have something similar right, I have a diagonal matrix sitting in between. So, supposing $U^T x = y$, right.

So, implies what is y^T? $x^T U$. So, what happens to this expression? This becomes equal to

$$y^T \Lambda y.$$

So, in the space of how will this look in the space of in the y-axis space? So, y_1, y_2, y_3 if those are my coordinate axes what is it? It is an n-dimensional ellipsoid in the y-axis space. So, if this is my y_1, y_2 it is a regular ellipsoid over here. How does it look back in the x-space? To get the answer to that what is my key? My key is over here, right. So, if U, so we wrote that $x = Uy$.

So, let us look at it like this and this is $x_1 x_2$, ok. So, let us say that one of the eigenvectors let us say U, let us say this is U_1 and I know that the eigenvectors are orthogonal to each other. So, U_2 therefore must be like this, ok. Now, looking at this expression I can say that $x = Uy$. If I if I left multiply by U^T from here I get this, right.



So, can I interpret this now? Supposing I give $y = (1,0,0,0)$, like this. ok. So, 1 in the first entry, 0 everywhere else. What will pop out of this expression? The first eigenvector use a matrix I am multiplying it by the what is what is the name for this kind of a vector? You will hear you will read this word often in the literature what is it called? Basis is one basis is a more general word something more specific than that canonical basis. So, when I stick this over here what I am left with is $x = U_1$, right.

That means if I am the y-axis y_1-axis corresponds to in the x-space which axis? The U_1-axis, right. That means if there were an ant walking along the y_1-axis in the x-axis space where is the ant walking? It is walking along the U_1-axis right. So, similarly if an ant is walking along the x-axis along the y_2-axis it means in the x-space the ant is walking along the U_2-axis right. So, all that means is that my that is how my contours look like So, the simple idea of rotation was very helpful because if I directly took this objective function there is no way can you know I can group the terms together to realize it is actually just a rotated ellipsoid ok. But this angle is always going to be 90 degrees from this comes from the property that the eigenvectors of a positive definite matrix are orthogonal. Any questions so far? Alright, good.