Course Name: Optimization Theory and Algorithms
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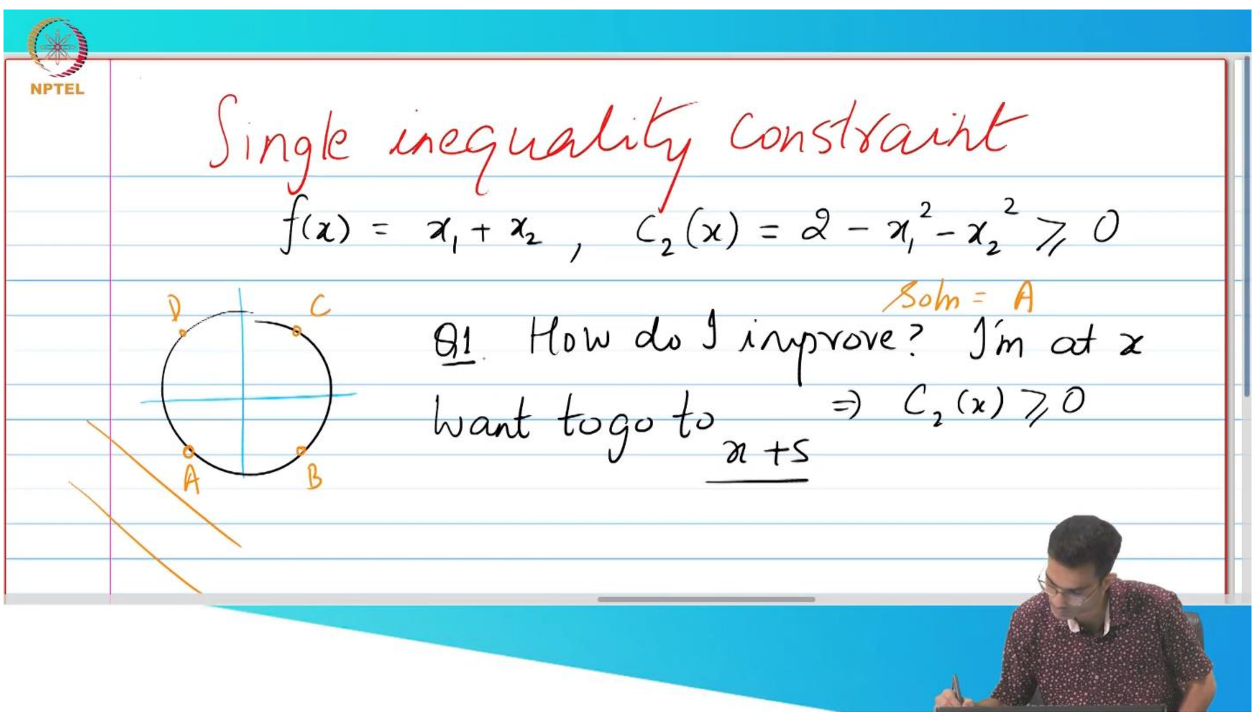
Week - 09
Lecture - 62

Single inequality constraint

So let us start. Equality was simple, but that is only half of the story. The other half of the story is in constraint optimization. You will not always have equality constraints, you will have inequality constraints. So, we have done a single equality constraint. Now, let us do a single inequality constraint. As you will see, this picture is not as simple as the other picture. Let us have a look at it, okay. I am going to keep things almost the same as before. So, we are going to keep the same objective function, okay.

So, $f(x) = x_1 + x_2$, okay, and I am going to number my constraint now as C_2 and I am going to write it as $C_2(x) \leq 0$. What is the convention for denoting an inequality? Greater than or less than? Greater than. Greater than. That is our convention, but as I have already mentioned, other references may have it the other way.



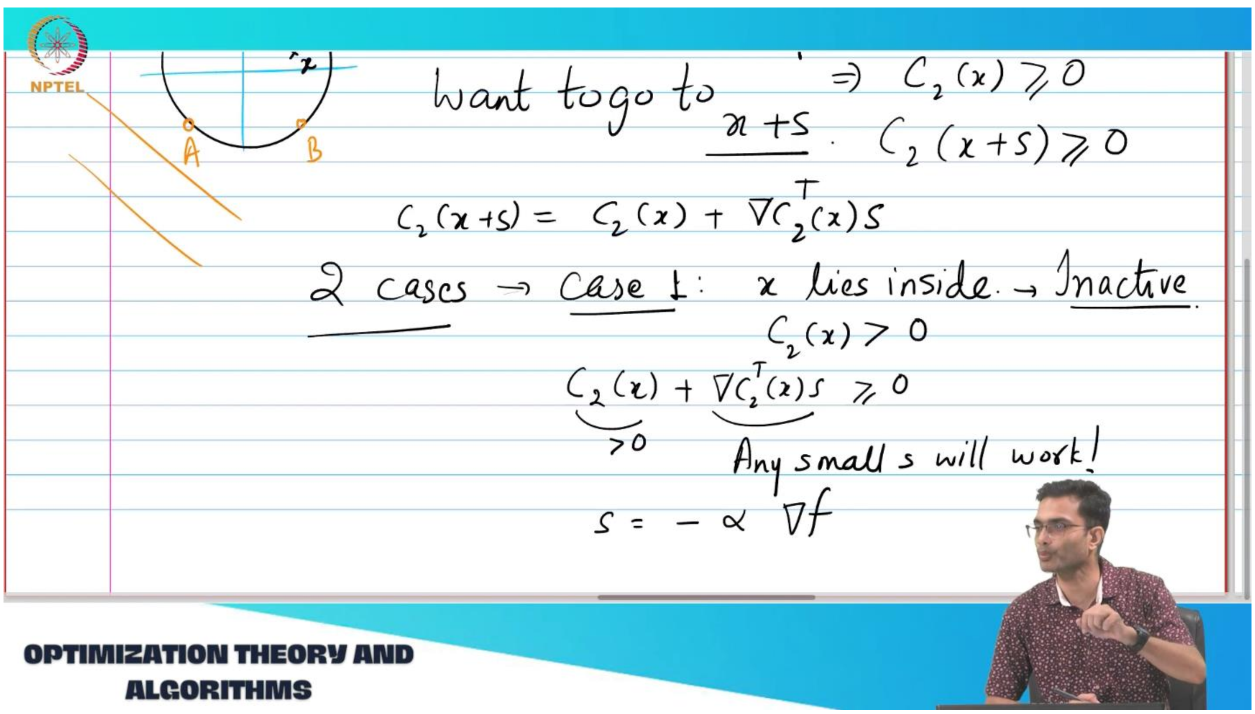
Okay. Good. So, if I were to simply sketch this, what am I saying here? What is my constraint? Stay inside the circle, inside or on the boundary. That is the meaning of this constraint. I am not sketching it, it is clear where we are.

Now, you already know the solution to this. Before we start solving it, we can figure out what the solution to it is. These are the contours of constant f. I am forced to be inside this. What will be the correct, what will be the solution out of those four points? This point once again over here because the value of f is the least at A and the value of f is most at C. Obviously, the solution is A.

Now again two things, now that you have got the hang of it, there are two questions that we have to ask. First is, if I am at x, how do I improve my position? The second question we will ask is how do I know when to stop? Those are the two basic questions I asked in the case of equality constraint. I have to ask it in this case. Let's ask that question. So question one, how do I improve? So, I am at some point x.

And as I said, we are always going to be in the feasible set, right? So it implies that $C_2(x) \geq 0$, no compromise allowed on that. I want to go to $x + s$, that is what I want to do. Obviously, if I am moving to this new point, the same feasibility condition should hold. This should also hold over there. Can I proceed the same way that I proceeded earlier in the case of the equality constraint? There is a little bit of an issue over here, right? What is that issue? We do not know which one is? You mean between $C_2(x)$ and $C_2(x + s)$, okay. So, what does it depend on?

So, let us write this down first: $C_2(x + s) = C_2(x) + \nabla C_2(x)^T s$.



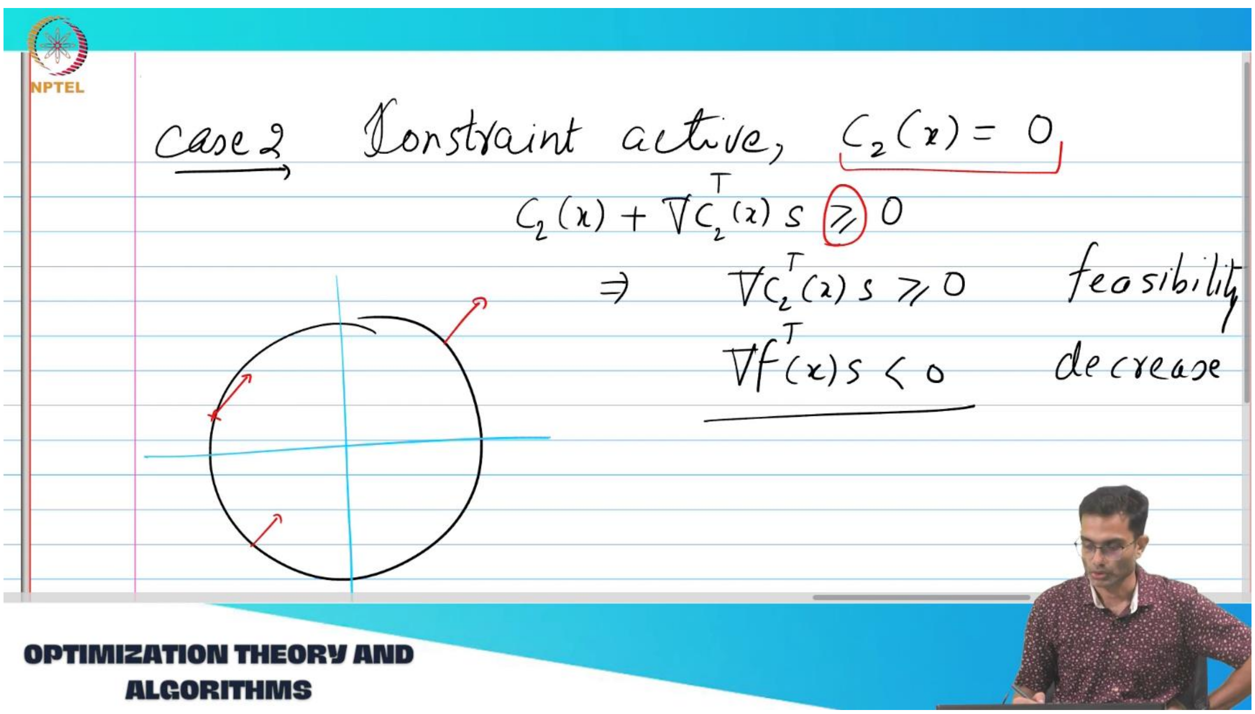
Now, both of these terms, the first term on the left-hand side and the first term on the right-hand side, both have to be greater than or equal to 0. So, it is not very useful to tell me what should be the sign of $\nabla C_2(x)^T s$. So, does anyone have an idea of how we can simplify this analysis? Can you think of cases? We can think of two cases. What would those two cases be? Right, in okay, so whether $C_2(x)$ is equal to 0 or if it is strictly greater than 0. In the language of optimization, what does it, how will we say this? Is the inequality active or inactive? Those are the two cases.

And that you can see is going to help me because if a constraint is active, what would be the value of $C_2(x)$? Zero, right? So that will help me to analyze both cases. So we get two cases over here. So, case 1, I am just going to take that x lies inside the feasible region, so this is my point over here, this is x. So, x lies inside and this is, what did you say? This is inactive constraint. So, if it is inactive, what is the value of $C_2(x)$? Strictly greater than 0.

Okay. Very good. So, my condition became $C_2(x) + \nabla C_2(x)^T s \geq 0$. This term is greater than 0. So, what does this tell us about the second term? I mean, there is no trick question here. What does it tell us about the second term? Can it be positive? Can it be negative? Must it be positive? Must it be negative? It must be? It must be negative.

Is it necessary? I mean not necessary, right? Supposing s were a very, very small negative number. Is it okay? It may be okay because the net total may still be greater than or equal to 0. So actually any small s will work. So we will just make a note.

Okay. What is a good example of a small s that will work? Again just use your intuition from unconstrained optimization. $s = -\alpha \nabla f$. I am anticipating that if I go in the negative gradient, I will also accomplish the decrease of the function and this α I am putting down there so that I can make sure that this inequality is satisfied. If it is, if I make α small enough I can make sure that this inequality stays greater than or equal to 0, okay. But in general, any s I can take small enough will work. And we have only spoken so far about the constraint, we have not yet spoken about the function value, okay.



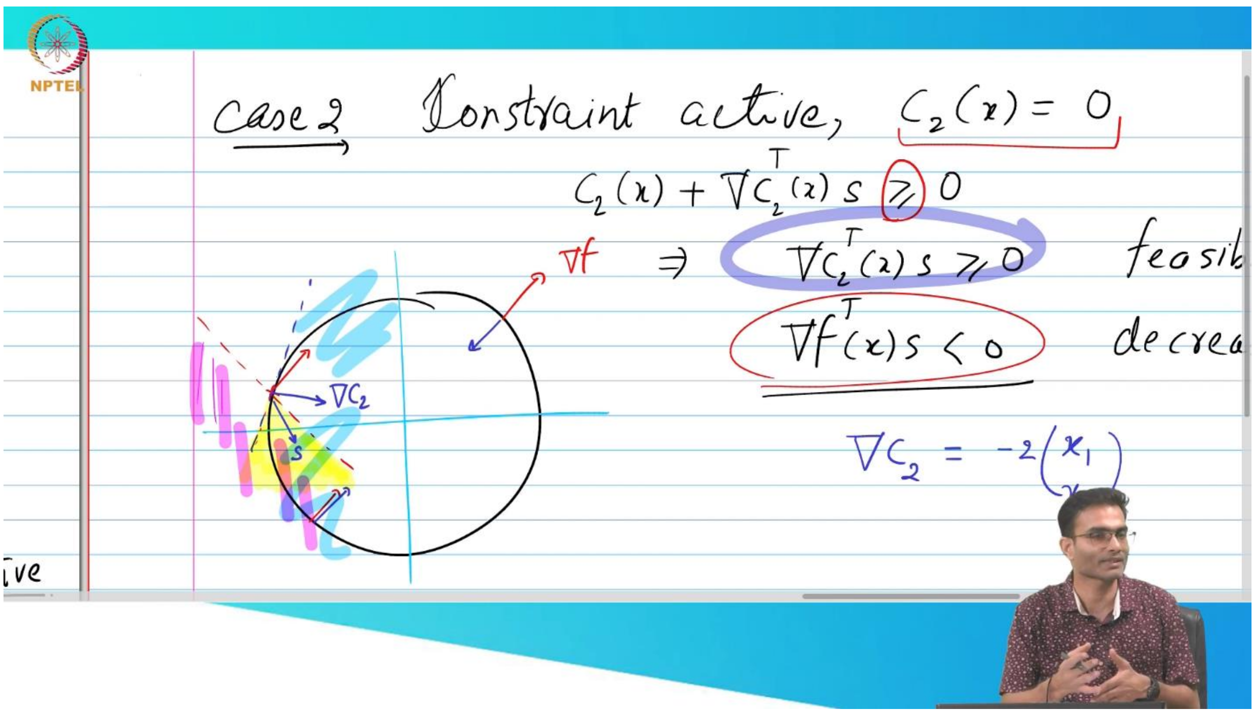
So now, this is case 1. Next, let us look at case 2. Yeah. Say that again, ∇C could be. ∇C is not in my hand, ∇C if you give me x, ∇C pops out as it is, so it is not in my hand. So, if you want, you could normalize α by $\| \nabla C \|$ or whatever, in terms of talking. s is the guy which is in my hand. So, our case 2 was when the inequality was active, or let us say the constraint was active.

So, that meant $C_2(x) = 0$, that is the meaning of an active constraint. So, what does that tell me? So, I had $C_2(x) + \nabla C_2(x)^T s \geq 0$.

So, this simply became what? $C_2(x) = 0$. So, this became $\nabla C_2(x)^T s \geq 0$, okay. For decreasing the function value, what did I need? Same logic that we applied in the case of the equality constraint, I will get for feasibility and for improving the function value.

By the way, in case 1 also, this will hold true, right? So, if you want, you can just quickly make that note over here. When we spoke about case 1, what this condition that we derived over here—any small s will work—that gave me feasibility. For decreasing the function value, the same logic applied, and it would give me $\nabla f(x)^T s < 0$.

From here, I came up with this guy. Okay, if I choose $s = -\alpha \nabla f$, I am satisfying both. α is the knob that keeps me feasible, and $-\nabla f$ is the knob that gives me a decrease in the function value. So, that is the same for both cases.



So, let us look at this graphically. Which way is my ∇f? It is always the same vector: $(1,1)$. So, I am here, and let us take some point over here. So, ∇f is still over here.

Yeah? What thing? No. Okay. That is a good question. I am at a point x. So in case 2, I am saying that the constraint is active. That means $C_2(x) = 0$. The new point that I move to need not be a place where the constraint is still active.

It is like saying if I am at the border, I can move inside. Then that inequality will flip too. So, to keep both options open, I am keeping this as ≥ 0.

So, this was my ∇f. Now, which way is—sorry, this is, yeah, this is ∇f. Which, what is ∇C? We actually did not write ∇C_2.

What is ∇C_2? So, this was your C_2 over here. So, $-2x_1, x_2$, which graphically means what? Radially. So, it is a little different in the same direction. So, this is $-2x_1, x_2$. So, it is pointing this way over here, this way over here, and this way over here.

Now that we have got this, let us look at the function value related to the constraint.

$\nabla f^T s < 0$. So, I would have to draw something like this, and which side of this dashed line am I? Should I be in? Bottom, right. So, it is like this. This is where I need to be in order to have $\nabla f^T s < 0$, alright.

Where should I be now for the second condition? The second condition is this guy: $\nabla C_2^T s \geq 0$. So, for that, what do I need to do? Again, consider this, something like this.

This line, this dashed line is perpendicular to what? What is this over here? This is ∇C_2, and I have to have $\nabla C_2^T x \cdot s \geq 0$. So, which half of the plane should I be in? The half that includes ∇C_2, right. So, if I were to take a highlighter now, it would be this part, right.

So, therefore, what am I left with? Is there any intersection here? There is an intersection.

There is this part over here. It is, in terms of a geometrical figure, what does it resemble? It is a cone. The intersection of these two half-spaces is a cone, right? So if I pick, if I pick any s like this, is it legitimate? Yes, it is legitimate because it satisfies the pink highlighted part, which is $\nabla f^T s < 0$, and it also satisfies $\nabla C_2^T s \geq 0$, right? So you notice that case 1 and case 2 were a little different. Case 1 was simple—pick any small s in the negative gradient, it will work.

Case 2, I have to look at this intersection of two half-spaces, right? So you can imagine that the answer to our second question, "When should I stop?" will be related to, for example, when will two half-spaces not have any non-zero intersection, right? So we will deal with that in the next session. So just keep these two situations in mind: case 1 and case 2. That's why inequality constraints need a little more care.

You have to, now supposing I had 10 inequality constraints, I would have to go through each of those cases to make sure which inequality is active, which is inactive, and then work out the conditions for each of them. So we'll formalize this in the subsequent session. Anything that's not clear here, it is very important that you get the geometry absolutely clear as to what is going on. If you only work with algebra, this is going to be confusing, but when you look at the geometry, you know, okay, there are half-spaces that are intersecting.

So, case 1 was that I am inside the circle, right? So $C_2(x)$ is strictly greater than 0. So $C_2(x) + s$, by definition, I want to continue to be feasible, that is why I can write this condition. Now, $C_2(x)$ I have said is greater than 0. So that means we do not know how much greater than 0 it is. So the second term can even be a small negative or it can be as big as possible positive number, and I will still be feasible.

So in some sense, that is why I am saying that any small s will work to maintain feasibility. The second thing that I have to make sure is that the function value is actually decreasing. So for the decrease, we have worked out what is the condition? That $\nabla f^T s$ should be less than 0. So if I take an s of this form and substitute it over here, I will get $-\alpha \parallel \nabla f \parallel^2$, which is always less than 0.

So it works. So this is a, you can call this a candidate. Is this candidate unique? It is not at all unique. I can choose any legitimate descent direction.

It will work. So this is not unique. Just an example. Fairly straightforward, right? Okay, so we will stop over here. Yeah, question. I did not find that it was exactly a cone.

I drew it and it looks like a cone. But you can, if you know the definition of a cone, you can generate a cone by the intersection of two half-spaces. So you have to start with the formal definition of a cone and show that the intersection of two half-spaces gives the cone. So this cone is, I mean this is not your ice cream cone kind of thing where it is always coming to a point, it is a geometric cone.