Game Theory and Economics

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Module No. # 02

Strategic Games and Nash Equilibrium

Lecture No. # 09

Application of weak domination: Voting

Welcome to the 9th lecture of module 2, of the course called game theory and economics. Before we start this lecture, let me take you through what we have discussed in the previous lecture.

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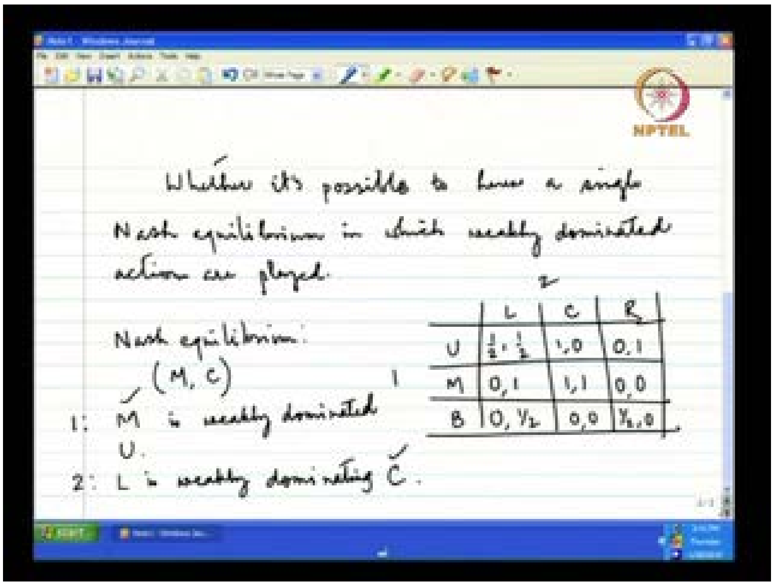

What I have done in the previous lecture is that we have defined what are known as weakly dominated action and strongly dominated actions. We have discussed some properties of this weak domination and strong domination.

In particular, we have said that if we have Nash equilibrium, then in the Nash equilibrium, strongly dominated actions are not played. As far as weakly dominated actions are concerned, in Nash equilibrium, which are not strict Nash equilibrium that is non-strict Nash equilibrium, it is possible that weakly dominated actions are played; we have shown an example of that. In Nash equilibrium, which is a strict Nash equilibrium, we have shown that weakly dominated actions are not built; so these are the properties.

Today, we shall start from there and we shall be discussing some of the exercises related to weakly dominated actions. So, one exercise that we shall do today is, whether it is possible to have a single Nash equilibrium, in which weakly dominated actions are played. Notice, we have already seen that if weakly dominated actions are ever played in Nash equilibrium, it has to be a non-strict Nash equilibrium. So, this Nash equilibrium that we are going to construct here is going to be Nash equilibrium, which is not a strict Nash equilibrium.

But, let us see, if that can be constructed. I have constructed the following game; there can be other cases also in which there is a single Nash equilibrium. In that single Nash equilibrium the actions of the players are weakly dominated. So, let us suppose, there are two players, 1 and 2; 1 has 3 actions U, M, suppose B and 2 has 3 actions, L, C, R. Following are the payoffs (Refer Slide Time: 03:50).

Now, if this is the game, then notice that the Nash equilibrium in this game is just one, there is only one Nash equilibrium which is at M, C, this is Nash equilibrium, because given that player 2 is playing C, if player 1 plays M, he gets 1. If he deviates, he gets either 1 by playing U or he gets 0 by playing B, so he cannot be doing better than what he is doing by playing M from player 2's point of view. Similarly, if he plays C, he gets 1, if he plays L, he gets again 1, if he plays R, then he gets 0, so he cannot be better off.

We can check other pair of actions; we shall see that in each of the players of actions, the actions are such that they do not constitute Nash equilibrium, so this is the unique Nash equilibrium. At the same time, let us also see that what is the action M for 1? For 1 action M is such that M is weakly dominated by action U, how do I say that? Because, if player 2 plays L, U is better than M, if player 2 plays C, U and M give him the same payoff. If 2 plays R, again U and M giving player 1 the same payoff, so U is weakly dominating M.

Similarly, for player 2, if I consider L and C, I shall see that L is weakly dominating C. So, C is a weekly dominated action for player 2, weakly dominated by L. Which means that both MC are weakly dominated, whereas in only Nash equilibrium of the game, both of them are like being played.

So, here, we have an example, where players have weakly dominated actions and they are being played in the unique Nash equilibrium that the game has.

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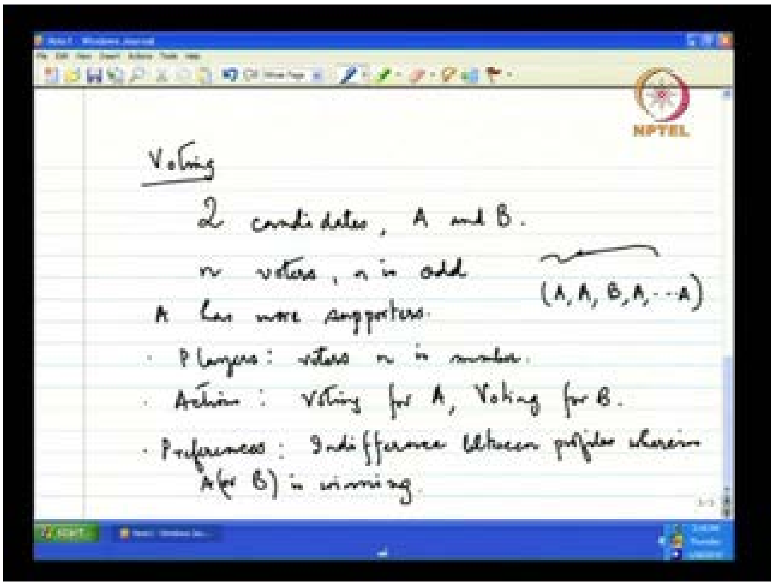

That was one example of weakly dominated actions and how they can be played. One illustration or one example that we shall do today, related to weakly dominated action is the case of voting. Suppose here, the game is the following that there are two candidates in the field, suppose A and B. How many voters there are? Suppose there are n voters, who can vote either for A or for B, n is odd. Extension is not an option, so everybody must vote. It is so happens that number of people who would like A over B is more than the number of people who like B over A, so A has more supporters.

Now, if there is no extension, then obviously there is a clear winner, because number of the voters is odd. So, either A is going to win or B is going to win. It may see that A is always going to win, but that is not the case. It may happen that even the supporters of A are voting for B, consequently B will win in that case.

Are you going to set this up as a game theoretic problem? First, the player's - voters are the players here, n in number; two actions voting for A or voting for B, these are the only two actions. Preferences, now what they define? This is quite logical and intuitive; in all the profiles, where A is winning, the voters are indifferent between them. If you remember, the preferences are defined over the action profiles, so action profile here will be n element vector. Each element representing the action of 1 player, so it will look like the following.

Suppose A, A, B, A, like that so there will be n such elements. Now, from this, it may happen that the number of A’s is less than the number of B’s. For all such profiles the voters are indifferent, because in all those cases, B is winning. So, if B is winning, it does not matter for a particular voter whether he himself is voting for B or someone else is voting for B or by how much margin B is winning. Those things are immaterial for any voter, as long as a particular candidate is winning players are indifferent between those profiles.

So, it does not matter by what method or by what margin a particular candidate is winning. It so happens that the players who like to see A win or more, than the players would like to see B win that is the main story.

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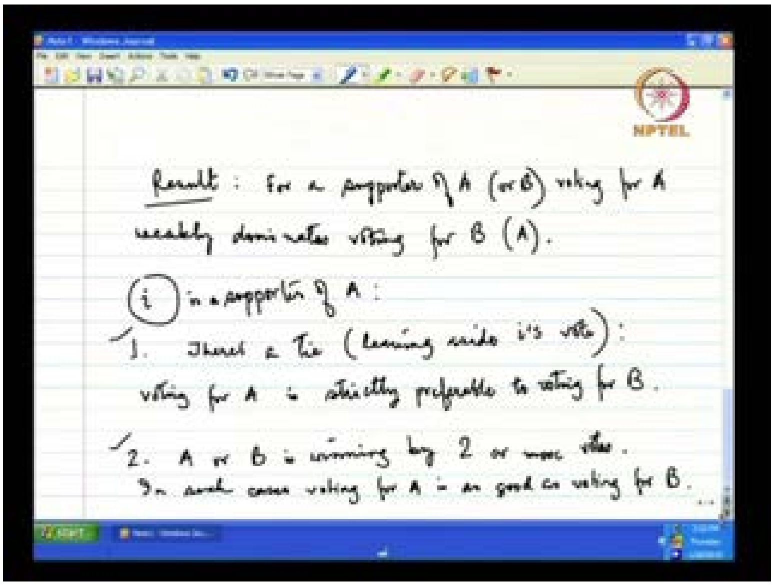

What is the result is that for a supporter of A or suppose B, voting for A weakly dominates voting for B. If I am a supporter of A, it is a weakly dominant action for me to vote for A, than to vote for B, because there are only two actions to consider and vice versa.

If I am a voter for B, it is weakly dominating to vote for B, than to vote for A, what is the reason for this? Look at this game from any individual’s point of view. Suppose, i is an individual a player, a voter, is supporter of A. Now, if i supports A, then what can be the profiles of actions of other players except i? If I leave i out, then it may so happen that in the rest of the players' actions, there might be a tie. Now, if there is a tie, when leaving aside i's vote, this tie is crucially dependent on the fact that n is odd. If n is odd, then if I take one person out, now I have even number of voters. If I have even number of voters, there are two candidates, there can be tie.

Now, if there is a tie, then what is best for i. In that case, voting for A is strictly preferable to voting for B. Now, if I have to prove that voting for A is weakly dominating over voting for B, then I have to show that in rest of the cases also voting for A is better than voting for B or the person is indifferent between voting for A and voting for B.

Now, how can I show that? What can be the other cases? Again, if I leave i out and if I consider the voting pattern of other voters, what can be the other cases? The other cases could be that one of the candidates A or B is winning by two or more votes. In particular, the margin between these two candidates A and B can be either two, it can be four, it can be six, like that it can never be an odd number, because we have even number of voters.

Now, whatever be the case, if one of the candidates is winning by 2 or more votes, notice that vote of this player i is becoming immaterial. For example, it may happen that A is winning by 2 votes; in that case, if he votes for B, A is still winning, because now only the margin between A and B is reducing by 1 vote and A is still winning. If i votes for A, then the margin rises, but A still wins.

As far as the outcome is concerned, i's vote does not make any difference, if the margin between A and B is 2 or 4 or 6 like that and vice versa, in the sense that if B is winning by 2 or 4 or 6 votes, then i does not like that case, but he cannot change the outcome of either. If he votes for A, B still wins by, may be a margin of 1 or 3 or 4 like that.

In such cases, voting for A is as good as voting for B, so he is indifferent between these two actions, voting for A or voting for B. Basically, this case 1 and case 2 exhaust all the possibilities for i to vote.

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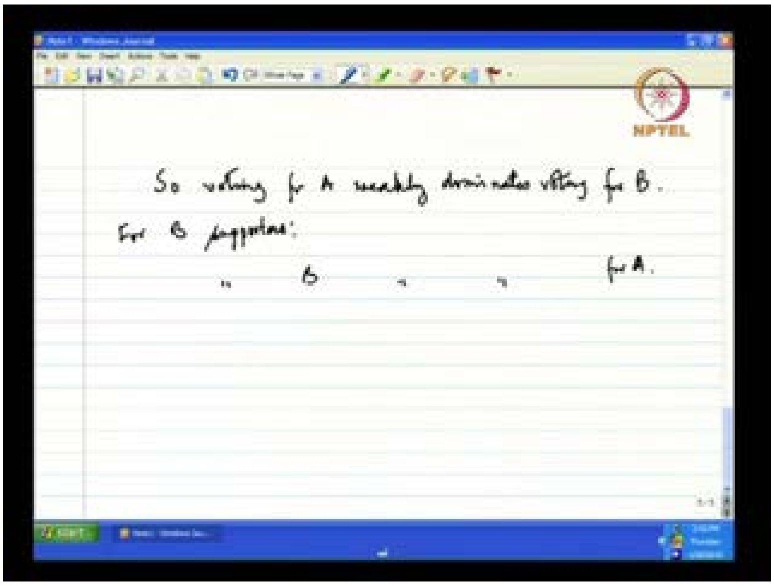

If that is the case, then we find definitely that action A that is voting for A is weakly dominating. For A weakly dominates voting for B, so that is one result that we have; this is for the supporter of A. For B supporters, voting for B will be such that it weakly dominates voting for A, so that is that.

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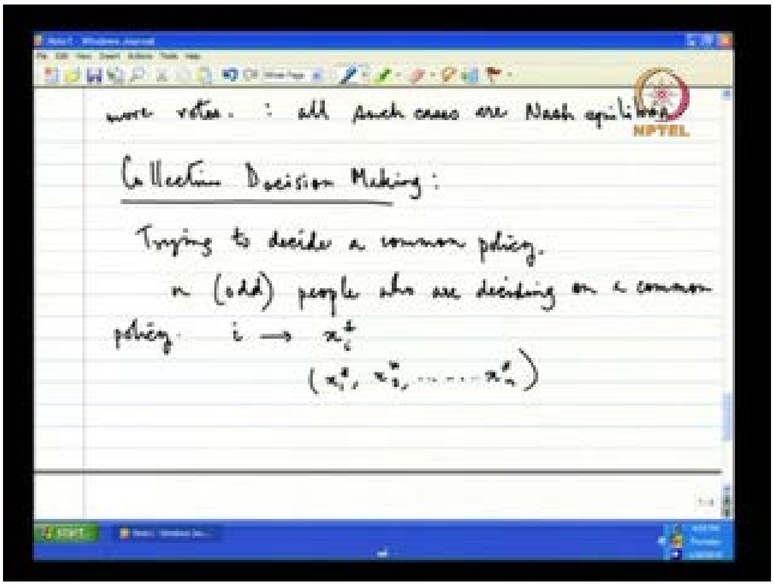

Another sort of voting, which we shall call collective decision making, is something which you will seek to discuss today. Here, it is not that people are voting for candidates, people are trying to decide on a common policy. So, this is trying to decide common policy. Who are these people? Suppose there are n people, n is again odd people who are deciding on a common policy. This policy is such that it can be represented by a number for the ease of visualization. Let us suppose, this number is a positive or minimum, it can be 0.

For example, let us take the case of defense expenditure by a government. Now, people have their own ideas about what this value could ideally be. For example, someone who is more nationalistic, more rightist kind of person, he may like to have a higher value of this variable that is the defense expenditure, than someone who is a more liberal, more lift oriented person.

Every person in this story has an ideal value of this number, which he likes to see as the common policy. Suppose there is an individual i, i belongs to this set of n people and his favorite is given by x i star. So, x i star is his favorite number, he likes that the common policy evolved, decided by the entire set of people be closest to x i star and this happens for everyone.

Every person 1, 2, etcetera, n person, every one of them has an ideal or what he likes to see as the common policy, these are given by the stars. Now, obviously, these numbers can differ, if these numbers differ, then how these communities of n people get to decide what will be the common policy? Because that common policy can be a single number, in that case, what it does? The mechanism that it involves is that everyone of this n people is asked to announce what is the number that he likes to see.

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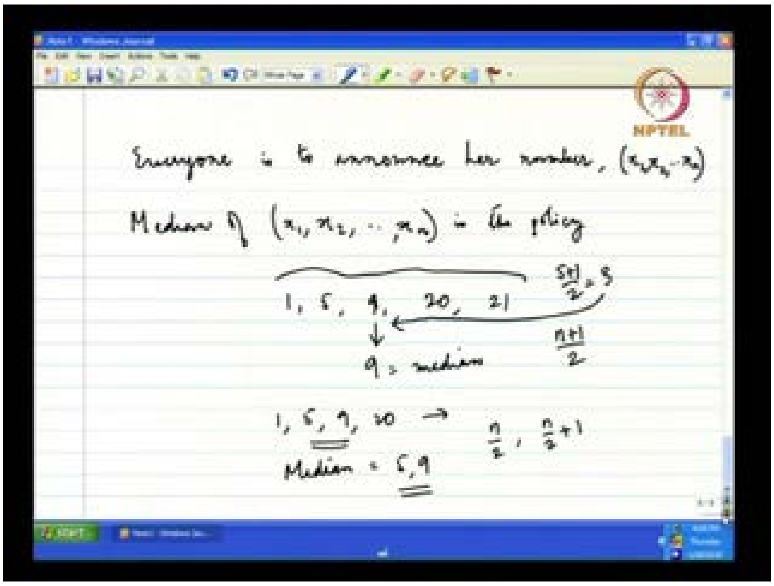

Everyone is to announce his or her number and let us call this number as x 1, x 2, dot, dot, dot, x n. So, these are the announcements made by each player, based on these announcements made by each player, the common policy is decided, how is it decided? The median of x 1, x 2, dot, dot, dot, x n is the policy. So, people announce their numbers and then the median of those numbers is taken to be the policy of the group.

Now, just to clarify what a median is, median is a middle most number. Suppose I have a set of numbers, suppose 1, 5, 9, 20 and 21; so these are the 5 numbers, then after I have got this numbers, I arrange them in an ascending order like I have done here. Then, I pick up the middle most; here the middle most is the third number, so 9 is the median.

Middle most number is how I pick the number. If there are n numbers, n is odd, then how do I decide which is the middle most? I take n plus 1 divided by 2, because here there are 5 numbers, so 5 plus 1 divided by 2 is 3, so third number is 9. If there are even numbers like 1, 5, 9 and 20, then there is not a single middle most number, there are two middle most numbers 5 and 9.

In this case, median is both, so there are two medians here, both are medians. So, here, the formula is what? Suppose, this is n, then I take n divided by 2 or n divided by 2 plus 1 that will give me the serial number of the medians. Here, it is 4, 4 divided by 2 is 2, 4 divided by 2 is 2 plus 1 that is 3; 2 and 3, second and third numbers are the medians, so that is the definition, the idea of median.

Now, if this is the set up that there are n people, they want to decide on a common policy, each of them has a favorite policy and each of them will like to have the common policy of the group to be closest to his favorite policy, then they will announce some numbers. The median of those numbers will be taken to be the common policy, so that is the setup.

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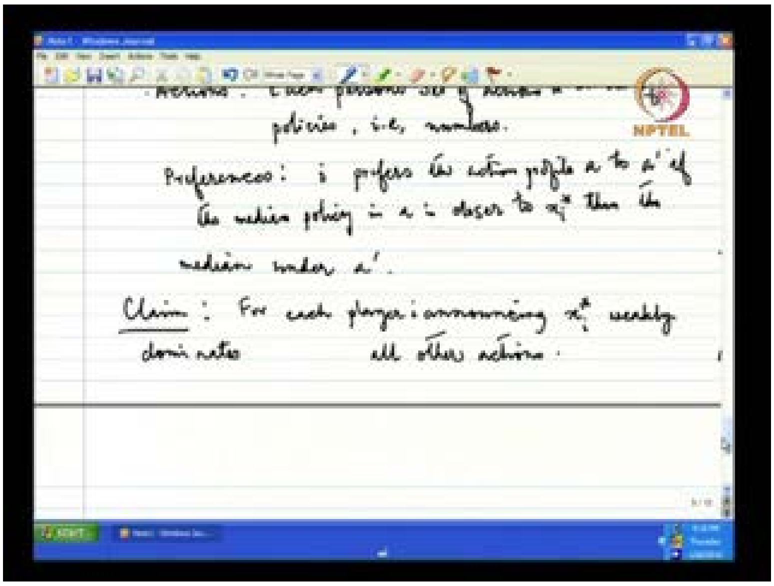

So, here, basically if I have to setup in game theoretic terms, n people and actions, anyone can announce a particular number that number can be any number. It can be any positive or 0 non-negative number basically. It can be negative also, but I am just keeping the structure simple. What is the preference? i prefers the action profile a to a dashed, if the median policy in a is closer to x i star, then the median under a dashed.

Basically, preferences as always depend on the action profiles. If i is given two action profiles a and a dashed, then how does you compare? Which one does he like better than the other? The answer is the following that i likes a to a dash that is he likes a better than a dash.

If in a - the median that is chosen in a, is closer to x i star, x i star is his favorite number, his favorite policy than the median policy, which will be decided a dashed is the profile. So, if a dash is the profile, the median is going to be further away from x i star than the median under a, in that case, i is prefer a to a dash. So, this is the setup in game theoretic terms. The claim is that again it is related to the idea weak domination that for each player, i suppose announcing x i star weakly dominates all other actions. Now, notice this is a more strong result than what we have seen before. In the previous exercise, there were two actions and we saw one action was weakly dominating the other.

Here, one is saying that for each player, for example player i, he has this action x i star that weakly dominates over all other possible actions that he can take. So, it is weakly better for him to take this action x i star, x i star is the favorite policy that the true policy that he likes to see than all other actions that he can possibly take. This result basically is suggesting that in such setup, where people are asked to tell their preferences and we take the median of these preferences.

There is an incentive for people to reveal their true types, if there is an intensive for them to tell the truth, in many cases, it may happen. As we shall see later on in many other setups, it may happen that people do not tell the truth. They want to influence the ultimate decision of the community and that is why they tell something, which are not their true types before, so as to make the final decision closest to their true decision.

But here, we see that that is not the case that if people are asked to tell their types, there is an intensive. There can be cases, where people will tell their true types and what is the proof for this? The way we approach this problem is the following.

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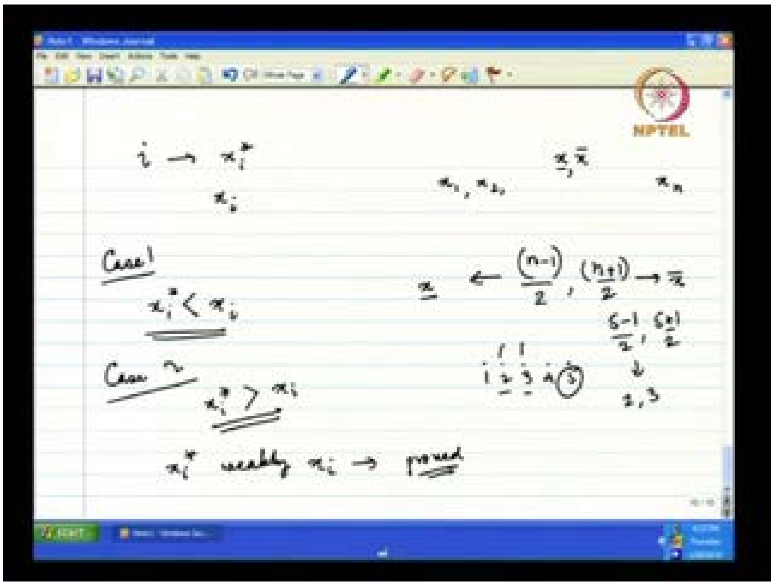

Suppose, I take i out, any player - i is any player, I am taking him out, I know that he has this favorite policy x i star. The announcement he makes is generically given by x i. Now, if I take i out, I take the announcements made by other players and I arrange them in ascending order. So, this is how they are arranged x 1, x 3, dot, dot, dot, x n, so there are n minus 1 elements here, in this entire series.

Now, in this vector or in this series of elements x 1, x 2, x n, etcetera, there are two middle most elements. Because here, since there are n minus 1 elements here, n is odd, so that means n minus 1 is even, so there are two middle most elements here. What will be their serial number? That I can find out by dividing n minus 1 by 2, because let me take an example. Suppose, I have five elements 1, 2, 3, 4, 5, now if I take 1 out, then the middle most will be 2 and 3. Now, how do I find out 2 and 3 by some general formula from the number 5? What I do is I take 5 minus 1 divided by 2 and 5 plus 1 divided by 2, so I get 2 and 3 - these numbers. So that is the same case we are doing here.

Since to begin with, there are n players, I am taking 1 out, so there are n minus 1 player, the middle most one will be the serial number n minus 1 divided by 2 and n plus 1 divided by 2. Suppose the announcement made by with these two players are x under bar and x over bar.

There were these people, are somewhere here, now what I am going to do is that I am going to consider all possible cases of location of x i star. i has a favorite policy which is x i star, but x i star can be any value. It can be less than x 1, it can be more than x n and it can lie anywhere in between that is one thing. Secondly, I am going to consider first case 1, is that suppose x i star is less than x i, which means that he is considering - player i is considering that he is going to make an announcement which is x i, which is strictly greater than x i star.

Then, I am going to show that this kind of consideration, whether I shall make an announcement greater than x i star, which is my favorite policy, is going to be weakly dominated by x i star. Basically, it means that if I consider making an announcement, which is more than my favorite policy that announcement is either going to give me less payoff than announcing my true favorite policy or that announcement is going to give me the same payoff that I will get. If I announce my favorite policy, this is case 1, then will be a case 2, where x i star is greater than x i.

If I can prove, in both these cases x i star weakly dominates x i, then I am done. This means that no matter what other announcement I take, announcing my true type that is x i star is the weakly dominant action, it weakly dominates over other actions.

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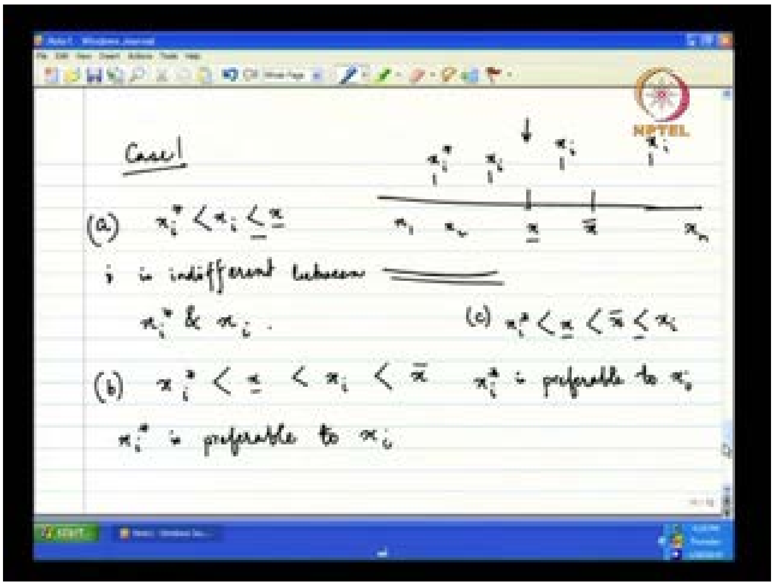

Now, let us first take case 1. Now, remember, there are all these x 1, x 2, etcetera. Somewhere here there is x under bar, there is some x over bar, here there is suppose x n, they are in ascending order. Here is x under bar, here is x over bar. First take the case that x i star, which I know is less than x i, is less than x under bar, so they are somewhere here, to the left of x under bar. Let us consider case of equal to also, it does not matter. Now, in this case, what are the payoffs of player 1, if he announces x i star and if he announces x i?

If he announces x i star, which is suppose here, the alternative is he announces something here. In either of the cases x under bar, x over bar where the middle most, before he made his announcement. Now, when he is making announcements, his announcements are to the left of x under bar, which means that x under bar becomes the median. It does not matter really whether he announces x i star or x i, because in both the cases, x under bar remains the middle most. So, in this case, where x i star is strictly less than x i, which is less than equal to x under bar, i is indifferent between this action and x i, so this is what I get from here.

This is just a, what is b? It can very happen that x i star is less than x i, but this x i is somewhere here, so that is x i is greater than x under bar and is less than x over bar. Now what happens? If i makes the announcement x i star, so he is making some announcement here to the left of x under bar, then x under bar becomes the median of the entire population.

If i makes the announcement x i somewhere here, then x i itself becomes the median of the entire population. Which is better? Having a median x under bar is better or having a median at x i which is greater than x under bar is better? If you remember, the preference was such that any player likes that action profile, where the median is closer to his favorite policy. So that means that in the action profile, where x under bar is being chosen as the median that is preferable to him. Which means that in this case, where x i is greater than x under bar and less than x over bar for player i, announcing x i star is preferable, because that is giving him x under bar as the policy than announcing x i.

So, here, I had a case where announcing your true type is strictly better, let us take case c. Case c is the case, where suppose x i star is here, suppose x i the announcement that he is planning to make can be greater than x over bar also, this is the very high value or it can be equal to x over bar, does not matter. In that case, which is the better action, announcing x i star or announcing x i?

Here also, we shall see that announcing x i star is strictly preferable than announcing x i, why? Because, if he announces x i star like before, x under bar becomes the median. If he announces x i, which is either x equal to x over bar or greater than x over bar, then the median becomes x over bar.

He has to choose whether it is better to have x over bar as the final policy of the community or x under bar as the final policy of the community. Since, the difference between x i star and x under bar is less, than the difference between x i star and x over bar, it is better for him to announce x i star. So, here, x i star is preferable to announce x i, which is greater than x over bar.

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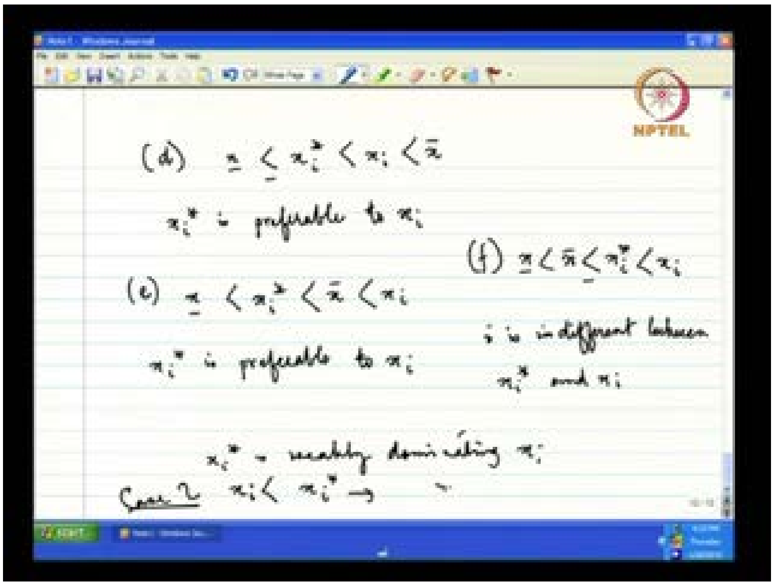

Now, in all these three cases, what we have done is that we have considered that the value of x i star is such that it is less than x under bar, but that is not the only possible case to have, it may have happen that I can take other cases of x i star. For example, x under bar, this is another case d, in this case, notice if player i announces x i star, then that itself becomes the median, because then this is the middle most. On the other hand, if he announces x i, again x i will fall between x under bar and x over bar, so that itself becomes the median.

If the choice is between x i star and x i, obviously x i star is better, because the final policy is just your own favorite policy that is the best that you can do. Here, x i star is preferable to x i, so here I am taking the other case where x i can be greater than x over bar also. Here also it is similar, if player i announces x i star, x i star itself becomes the policy of the group, whereas if player i announces x i, x over bar becomes the middle most and that becomes the policy of the group. So, obviously, player i will like to have x i star, is the policy of the group, so x i star in this case is preferable to x i.

We are more or less at the end of this discussion. The last case I need to consider is this one. Both of them are greater than x over bar, in this case, it does not really matter, whatever i announces whether this is x i star or x i, x over bar becomes the median, which means that i is indifferent.

I have considered all the cases, where x i is greater than x i star that is player i is considering some announcement, which is strictly greater than what his true type is. I see that x i star is weakly dominating over x i, he can do either worse by announcing x i or he can do as much as good as she is doing after x i star. From this, x i star is weakly dominating x i, similarly, the case of x i being less than x i star, this is case 2, will be similar. There also I shall get the same thing that x i star weakly dominates x i.

From this, both of them combining together, we shall reach the conclusion that announcing was two type, is weakly dominating over announcing some other policy for each of the players. So that is more or less the end of the lecture, what we have done in this class, in this lecture is that we have analyzed some of the exercises, which use the case of weak domination. We have taken the case of voting, taken the case of corrective decision making that is the end of the lecture; thank you.

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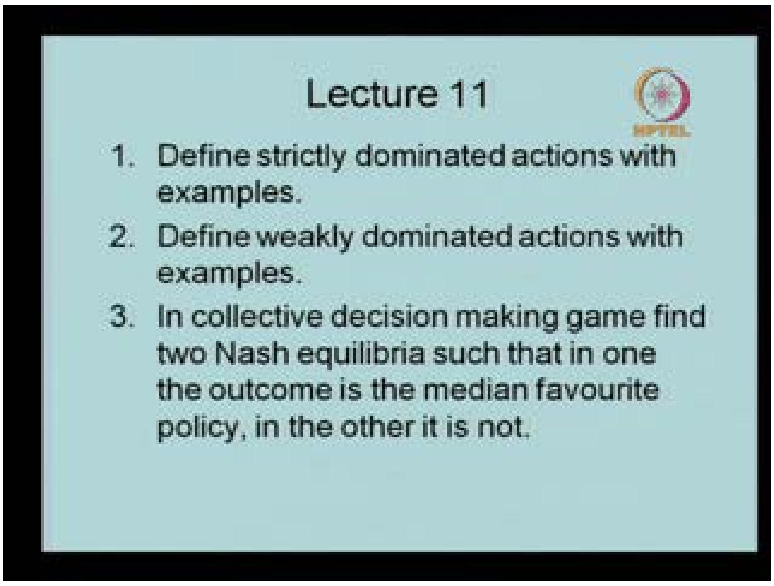

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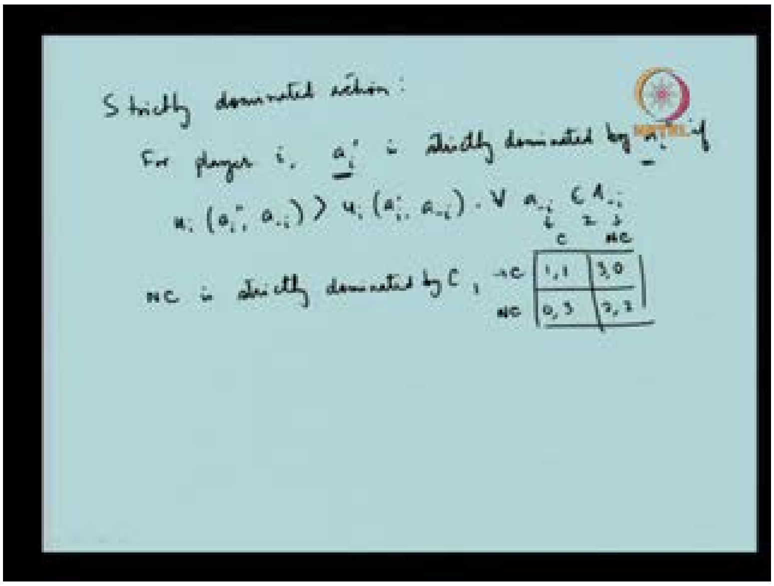

Define strictly dominated actions with examples. Strictly dominated action for player i, a i dashed is strictly dominated by another action a i double dash, if the following condition has to be satisfied. So, this is the definition of strictly dominated action symbolically. What basically is being said is that a i dashed is worse than a i double dashed, for player i, no matter what actions the other players are taking. The other players could be taking any random actions, it does not matter. For every possible vector of actions of other players, a i dashed gives this particular individual lower payoff compare to a i double dashed. For examples, we can think of first game that we discussed that is Prisoner’s dilemma game.

This is the Prisoner's dilemma game, these are the payoffs. Now, here, there is one weak, one strictly dominated action and this strictly dominated action is NC. Which action is strictly dominating? It is C, you can see this very clearly. If player 2 plays C, then for player 1, playing C is better, because C is getting in 1 and C is getting in 0. If player 2 plays NC, then again for player 1 playing C is better than playing NC, so that is why NC is worse, no matter what player two plays.

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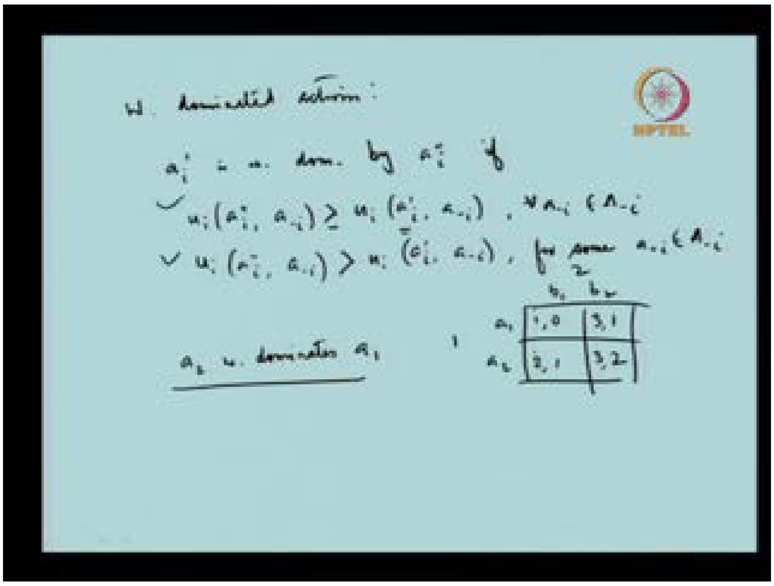

Define weakly dominated actions with examples. Weakly dominated actions; so I am not writing it in detail, as I did in last time, is weakly dominated by a i double dashed, if this is for all a naught i belonging a naught i; this is for some a naught i, belonging to a naught i. So, at least for some vector of actions by other players, playing a double dashed i is strictly better. For all other vectors, it could beat least as good as playing a i dashed. An example, here for player 1, a 2 weakly dominates a 1, 2 is greater than 1, 3 is equal to 3.

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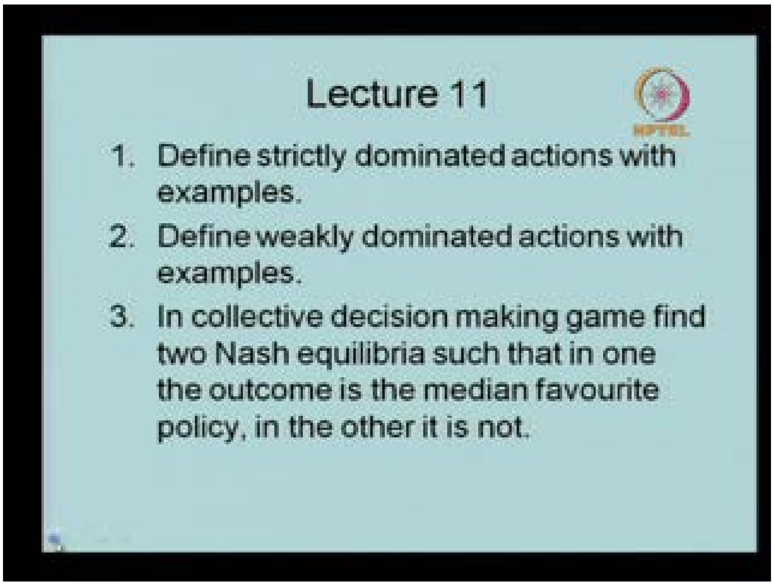

Last question, in the collective decision making game find two Nash equilibria such that in one the outcome is the median favorite policy, in the other it is not.

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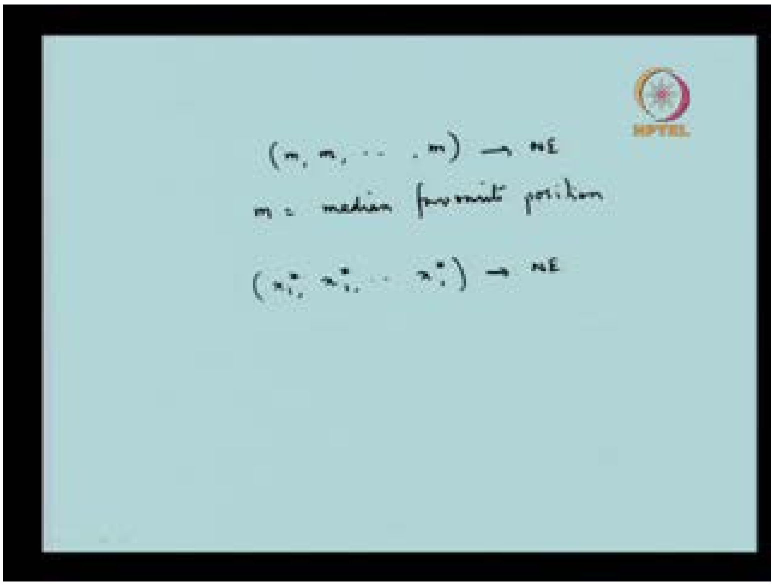

So, collective decision making of model if you remember, we have to find out one in which median is the Nash equilibrium. Let us suppose everyone announces the median favorite position, in this case, this is Nash equilibrium, because no one can deviate and change the outcome, because the choice then becomes m. So, this is Nash equilibrium and that is what we had to provide.

Another example, when it is not, suppose every one announces x 1 star which is the favorite policy of the first player, in this case, again this is Nash equilibrium, because I cannot change my action and change the outcome. But, here, what is the outcome? Outcome is x 1 star, which is not equal to m, so this is an example; that is it.