Game Theory and Economics
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Module No. # 03
Illustrations of Nash Equilibrium
Lecture No. # 05
Different Aspects of Bertrand Model

Welcome to the third module lecture 5 of the course called game theory and economics. Before we start today's lecture let me take you through what we have been discussing in the previous lectures. So, we have been discussing the various applications of Nash equilibrium in game theory, and the model that we have been discussing so far is the Bertrand model of oligopoly.

So, this is the model about markets, here there are some firms who are trying to sell their goods to the customers, and what they are aiming at is, to maximize their profits. The important thing about Bertrand competition is that the firms they sell the similar goods, you cannot distinguish between the product of firm 1 and firm 2, and while selling their goods they decide the price of the good, so it is the price of my product on which I have control, how much what will be the level of output is not something which is directly determined by me, given the price that I quote for my product there will be some demand for my products in the market, and whatever be the demand for my products I have to meet that demand.

So, the level of production in the market is something which is not directly determined by me, it is determined by the demand side of the market, I only decide what will be the price of my product.

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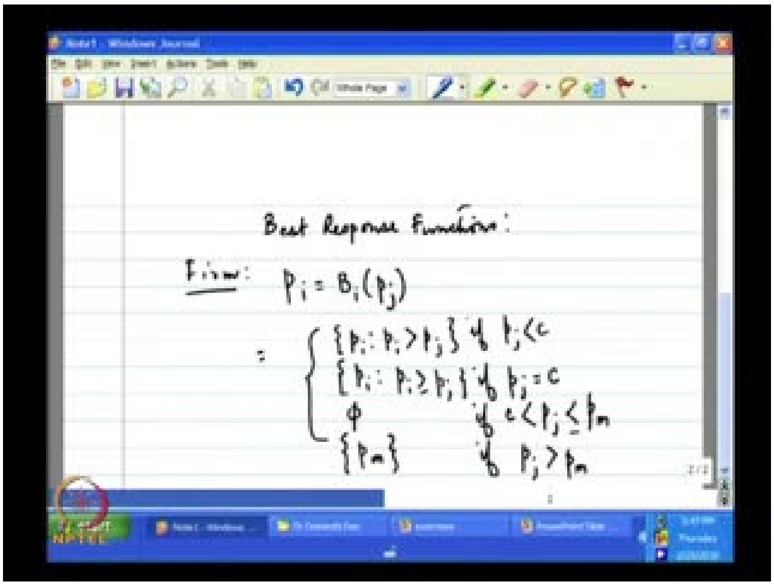

So, this is the setting, and so far we have seen that in this market if you simplify the story a little bit, and suppose there are only two firms, we are going to generalize this module later on, so for the time being suppose there are only two firms - firm 1 and firm 2, and their unit cost of production is the same - small c, and suppose the demand that deface is a very simple kind of demand function given by linear demand function; then we have seen that in this setting we can find out the Nash equilibrium, and we were in the process of finding that Nash equilibrium; and how we try to find that Nash equilibrium is to construct the best response functions.

So, we have basically so far constructed the best response functions of firm 1 and firm 2; we have seen that the for firm 1 the best response function is given by q 1 which is a function of q 2, and this is given by a set of 3 ranges, so this is the best response function of q firm i as a function, it is the price that they are determining.

So, the price that firm i decides is given by p I, it is a function of p j, the price that is determine by the other firm; and we have seen in the previous lecture that there can be three different values or range of values of p j which we have to consider separately; and for each case the best response of firm i will be different and given by the set of values.

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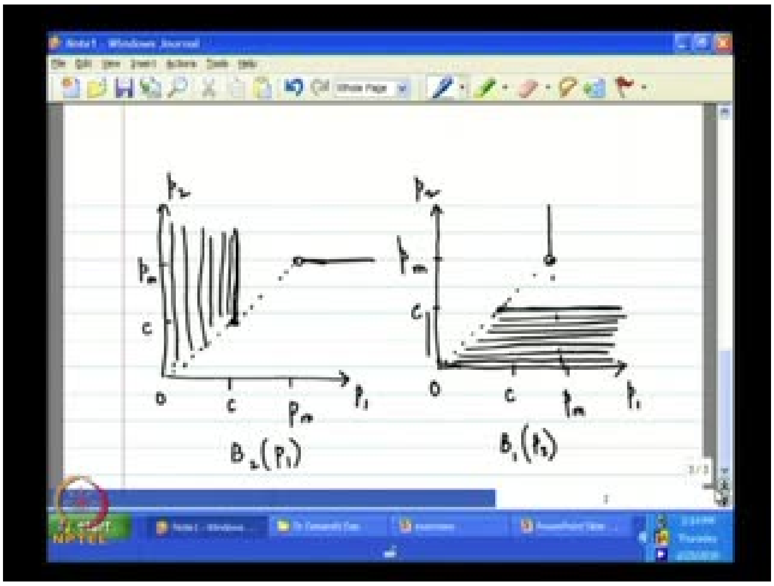

Now, given this, what we can do is to plot this best response function for firm 1 as well as firm 2 and look at what is the equilibrium; so, suppose, this is c, and suppose this is p m, what we know is that if p 2 is less than c then p 1 will have to be greater than p 2. So, here is this point c c, and suppose this is the point p m p m, and this is the 45 degree line; if p 2 is less than c, that is we are talking about this range, if p 2 is less than c firm 1 best response is to charge a price which is strictly greater than p 2, so all this points will be covered by the best response functions, but not this points on the 45 degree line.

If price of firm 2 is exactly c then of course this point is going to be covered, so this I am writing as a dot, a black dot, dark dot, and this points to the right of this dot are also covered under the B 1 - that is best response function of firm. If firm two’s price is greater than c and it can go all the way up to p m, at the value p m the firms monopoly profit is being on, so at p m the profit of each firm is maximized.

So if firm 2 is charging any price between c, not including c, more than c, and it can charge a price equal to p m; the best response for firm 2 will be charge a price which is just a little less than p 2, but the point is that there is no one best response function here; so, whatever be the price of firm 2, I can never defined p 1 which is closest enough to p 2, because p 1 is a continuous variable, no matter what closest p 1 1 propose is I can purpose some other p 1 which is even much closer than that p 1 to p 2, which means here there is no best response function up till p m; if firm two's price is greater than p m, then firm one's best response exits it is given by p m.

So, I have this vertical line here, and this is p m; here I have this circle which is an empty circle, which means that this point p m p m point is not on the best response function of firm 1, so this is how B 1 p 2 looks like; and similarly, I can define or I can plot B 2 p 1, suppose c is here, p m is here, c is here p m is here.

Now, here what will happen is that, if p 1 is any value less than c, if p 1 this is the points c c, if p 1 is any value less than c can go all the way up to 0, firm 2 will charge a price higher than that price; so, if this is the 45 degree line all this points in this shared region will come under the best response function of firm 2; at c if firm 1 is charging the price c not only those points which are on the vertical line will be covered, but c c point will also be covered, so I have a black dot from c to p m.

So, this is the point p m p m from c to p m, there does not exist any best response for firm 2, the reason is as before; and at p m also there does not exist any best response function, so I have an empty dot here; if firm 1 price is more than p m then firm 2 will go on charging the price p m, so I have this horizontal line here.

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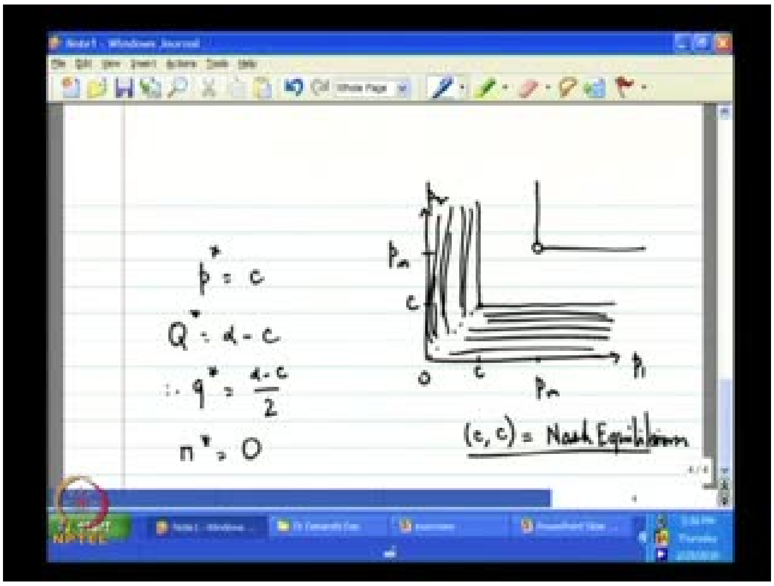

So, this is how two firms best response functions look like; and as we have just discussed before how to find out the Nash equilibrium, Nash equilibrium is situated at the intersection point of this two sets of functions, and so we just examining this two and superimposing; on the other I can figure out that this is c c this is p m p m that if I superimpose one on the other I have this sort of a diagram at p m p m I have an empty circle, and I have this two horizontal vertical lines.

So, this is how super imposition of two best response functions B 1 and B 2 will look like, and we see that there is only 1 point at which this two sets of curves are intersecting, are having some intersection, and that point is given by c c, so c c is the Nash equilibrium. So, this is the equilibrium in Bertrand model that there are two firms, they will in the equilibrium charge a price which price is just equal to their unit cost of production, which means that the price in the market is c which means Q is what, Q is alpha minus c because price is c, and which means that individual output level is going to be half of this alpha minus c divided by 2; and mind you profit is going to be 0 here, because there is no difference between the price and the cost, cost is just seating up all the price that the firm earns from the market.

So, there is no profit, the firms are producing some output, and this is happening irrespective of the number of firms; if you remember in Cournot model, this situation happened only if the number of firms became infinite; here even if there are 2 firms, the competition ensures that there is no profit earned by any firm, they are earning their normal profit. So, this is the story then in Bertrand equilibrium, it is much more different from the Cournot model that we have studied before.

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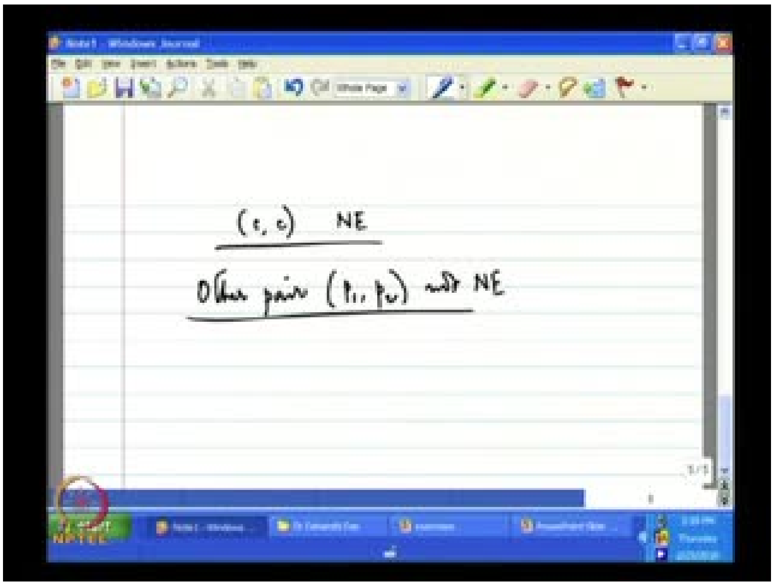

Now, this is a systematic exposition of the equilibrium that we have this equilibrium at c c, but we could have used a more direct argument to say that this is the equilibrium, and this is the unique equilibrium mind you, because there is no other point at which the best response functions are intersecting with each other; why, this is the unique equilibrium we can show that in some steps; first we have to logically argue why c c is the equilibrium, this is Nash equilibrium because of the reason that if the other firm is charging c can I do better by charge not charge c; the answer is no, because when I am charging c I am getting 0 profit the market is getting divided and though I am getting half of the demand in the market.

Since the profit margin is 0, so my total profit is also 0; if I charge a price little less than c, the price that the other firm is charging, I get the entire market that is true, but then my profit will turn to be negative because then the profit margin is negative. If I charge a price higher than c, then obviously no customer is going to come to me, because my rival is charging a lesser price,

So, in that case my profit remains 0 which means this is a Nash equilibrium; however, this not a strict Nash equilibrium is there any other pair P 1 P 2 which is a Nash equilibrium, so we have to show that c c is unique, that no other pair exits which is a Nash equilibrium; the reason is the following - suppose this other pair is such that the prices are different, and both the prices are higher than c.

If both the prices are higher than c then there is profitable deviation for at least 1 player; the firm that is charging the lower price it is earning profit, because there is profit margin, but it can earn better if it raises its price just a little bit, it can earn higher profit because it is still retains the market, because the other firm is charging a higher price; so, by charging a little bit more price I still retain the market provided of course that this P 1 and P 2 that I am considering a less than p m; so, even if they are greater than p m, then any firm can earn higher profit by charging p m rather than their prices P 1 and P 2.

So, if P 1 and P 2 are different, and both of them are higher than c, then that cannot be a Nash equilibrium, that we have just argued; if they are equal but higher than c can that be a Nash equilibrium, the answer is no, because in this case when the prices are equal the market is getting divided equally; so, any firm can just lower its price a little bit, and can capture the entire market; and if it can capture the market then it is getting higher profit compared to the case where the profit is getting divided.

So, no pair of equal prices where the level of prices is greater than c is a Nash equilibrium; and thirdly, if the prices are less than c, then can that be a Nash equilibrium; if the prices are lesser than…, suppose the prices are equal, if the prices are equal then the firms are getting negative profit, so each of them will do better by charging higher price, so that is not a Nash equilibrium; if there unequal again the firm which is having a lesser price lower price can do better by charging a price at least equal to c, then it is getting 0 profit.

So, we are basically ruling out all these possibilities; another possibility we have not consider so far is that one of the prices is c, and suppose the other price is greater than c, but that also is not a Nash equilibrium, because the firm which is charging the price c can do better by charging a price just a little higher than c. So, in that case it will earn some positive profit by charging seats getting 0, so this is more or less the demonstration we did not have to depend on construction of this best response functions to argue that c c is the Nash equilibrium in this case.

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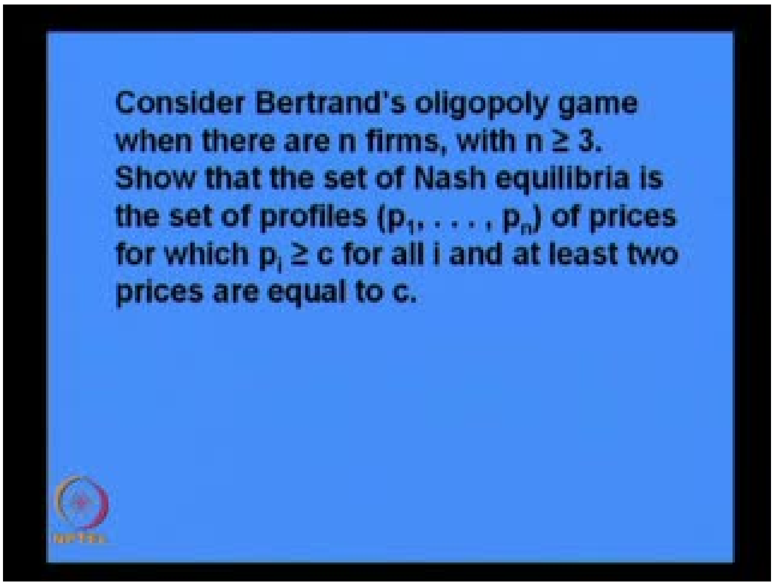

Next what we are going to do is to take some exercises and try to solve them. Consider Bertrand's oligopoly game when there are n firms, with n greater than equal to 3. Show that the set of Nash equilibria is the set of profiles p 1, p 2, dot dot dot p n of prices for which p i is greater than equal to c for all i and at least two prices are equal to c.

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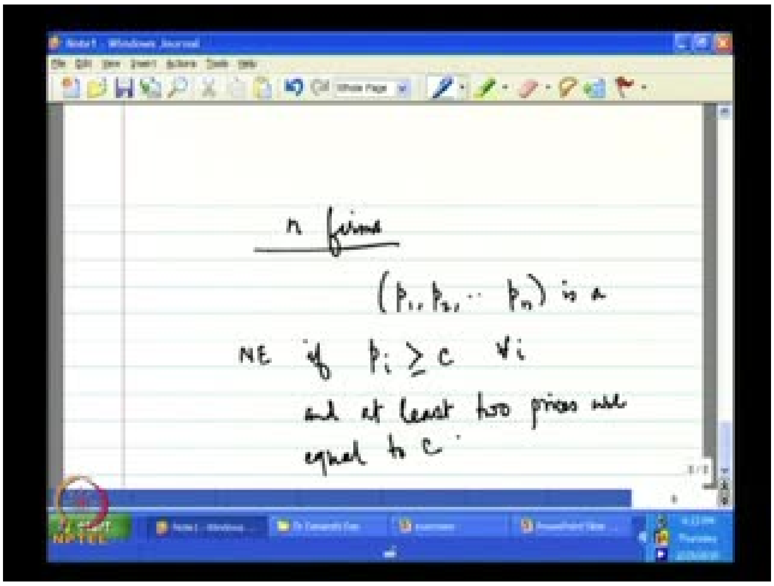

So, here we have a general case n firms are there, and what we need to show is that p 1 p 2 p n is a Nash equilibrium, if all the p i(s) are either equal to c or more than c for all i and at least two prices are equal to c. First let us show that such a profile is a Nash equilibria, and then we shall show that no other profile of different characteristics can be a Nash equilibrium.

Now, if two prices are equal to c at least two and other prices can be more than c; then notice that if my price is equal to c I am charging the lowest price then I am earning 0 profit, the firms which are charging more than c there also earning 0 profit; so, in this case nobody is earning any positive profit. Question is, can someone do better and earn some positive profit the firms which are charging c, and there are two firms at least two firms; if they charge less than c negative profit, if they charge more than c since there is at least one other firm which is charging c I am not going to get any market, so I am going to earn 0 profit, so I am not better off.

Similarly, for firms which are charging more than c, if they charge a price different what different price they can charge they can charge a price less than c; if they charge a price less than c, they are going to get the market but again negative profit, so this is indeed a Nash equilibrium.

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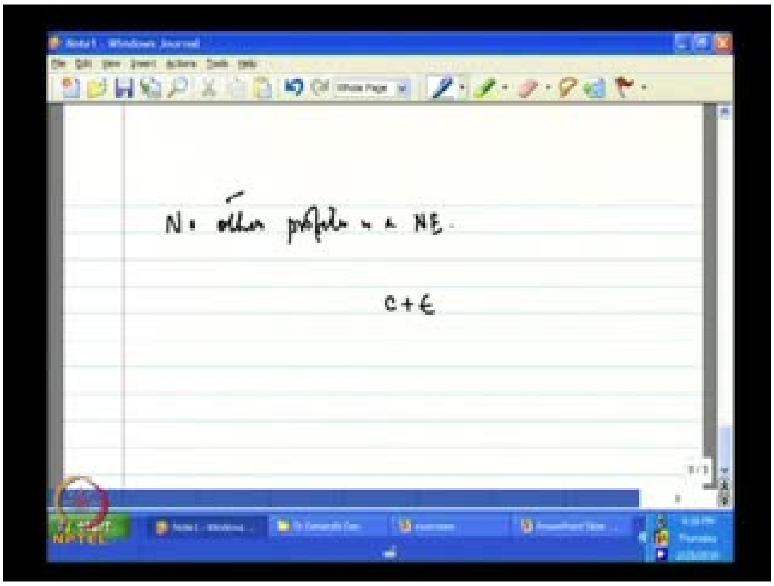

So this is a Nash equilibrium that is true, we also need to argue that no other is Nash equilibrium. Now, by saying that no other profile what could be the other profiles, one could be that all the prices are greater than c, but at one just one price is equal to c, that is not a Nash equilibrium, because the firm which is charging that c price can do better by charging a price little more than c; if it charges a price more than c then it is going to un-positive profit, and since P is a continuous variable here we are back to the old assumption of continuous prices.

I can always find a price which is c plus epsilon, for example, epsilon can be any small positive number, I can always find some epsilon and such that my price still remains the lowest price, and in that case I earned some positive profit rather than earning 0 profits. So, no profile can be a Nash equilibrium where there is only one price lowest price which is equal to c; it is obvious that any profile of prices, but the lowest price is less than c is Nash equilibrium, that cannot be a Nash equilibrium, because in that case the price which is less than c, that firm which charging that price is earning negative profit.

So, it can do better by earn by charging a price which is equal to c, either it is in that case it is earning 0 profit, what could be the other possibilities? Other possibilities could be that all the prices are such that the lowest of the prices is greater than c; if lowest of the prices is greater than c, and suppose there is just 1 lowest, the unique lowest, then like the argument that we have given before, that unique firm is going to raise the price and earn better profit.

If there are two lowest prices, and both the lowest prices are greater than c, can that be a Nash equilibrium, well, the answer is no, because if there are two firms or more than two firms which are charging the lowest price then they are dividing the market, and if they dividing the market then one of them is going to undercut its rivals, and charge a price little less than that lowest price and get the entire market.

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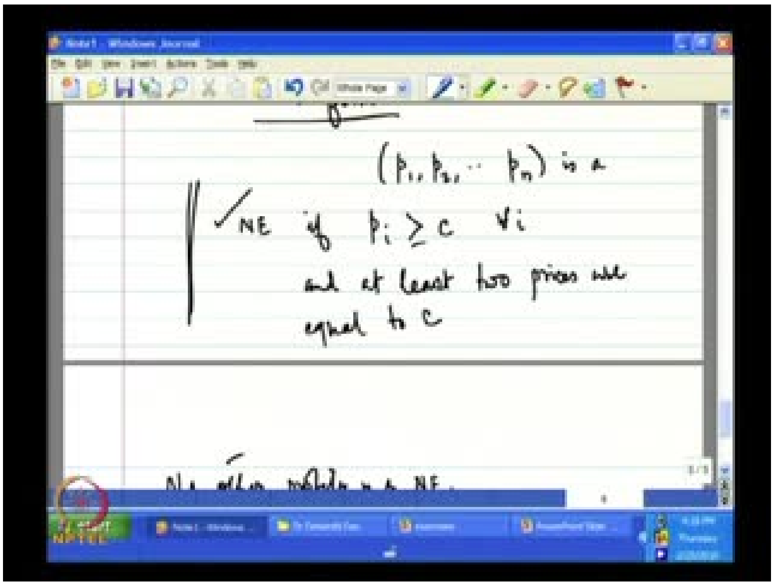

So, basically, we have ruled out all the other possibilities of price profiles, and we have seen that none of those profiles is Nash equilibrium, so this thing that you have left with profile that you have left with is this profile which is indeed Nash equilibrium profile. Let us look at some other problems.

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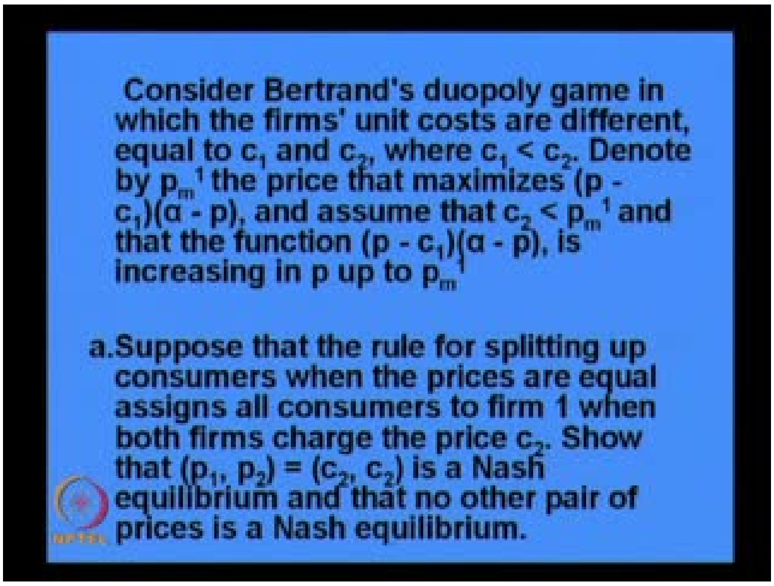

This is another exercise, consider a Bertrand's duopoly game in which the firms unit costs are different, equal to c 1 and c 2, where c 1 is less than c 2. Denote by p m 1 the price that maximizes p minus c 1 multiplied by alpha minus p, and assume that c 2 is strictly less than p m 1 and that the function p minus c 1 multiplied by alpha minus p, is increasing in p up to p m 1.

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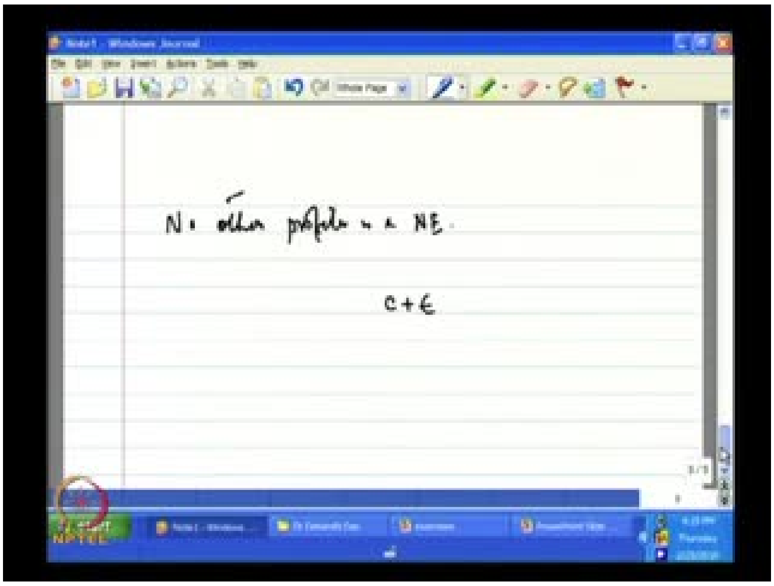

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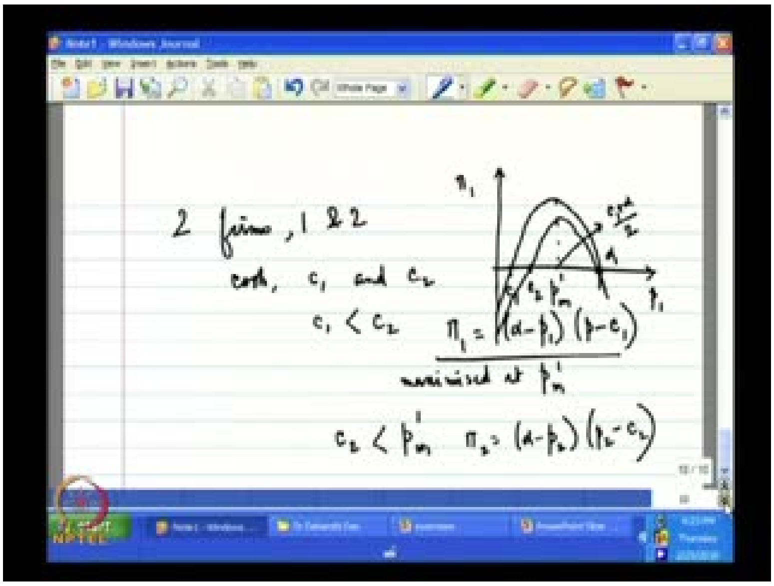

So, if I have to visualize this what is happening? What we have is that two firms are there 1 and 2 with costs c 1 and c 2, c 1 is less than c 2, which means c firm 1 is more efficient than firm 2, it can produce goods at a lesser cost; what we further know is that this alpha minus p p minus c 1, this is basically the profit function of firm 1, this is maximized at p m 1, and we know that c 2 is less than this p m 1, this is given.

We further know that this function is increasing in p up to p m 1 which is true, which you have seen before. Basically, the diagram that is relevant here is this one, we have drawn the diagram earlier also, now it will be for only firm 1; if I have to draw this, then this is how the profit function looks like, this is c 1, this is alpha in the middle, there is this p m 1 at which the profit is getting maximize; this is nothing but c 1 plus alpha divided by 2, what we have been informed is that there is this c 2 somewhere in between c 1 and p m 1, that is what we know.

Notice that if I have to draw firm two's profit function then that profit function is going to intersect this horizontal axis, and it is going to intersect, this horizontal axis once again at the point alpha, because what is pi 2 it is alpha minus p 2 p 2 minus c 2, so it also has an intersection point at alpha, so this is the common intersection point, and c 2 is somewhere between c 1 and p m 1, so this is the story in terms of the diagram, the question is the following:

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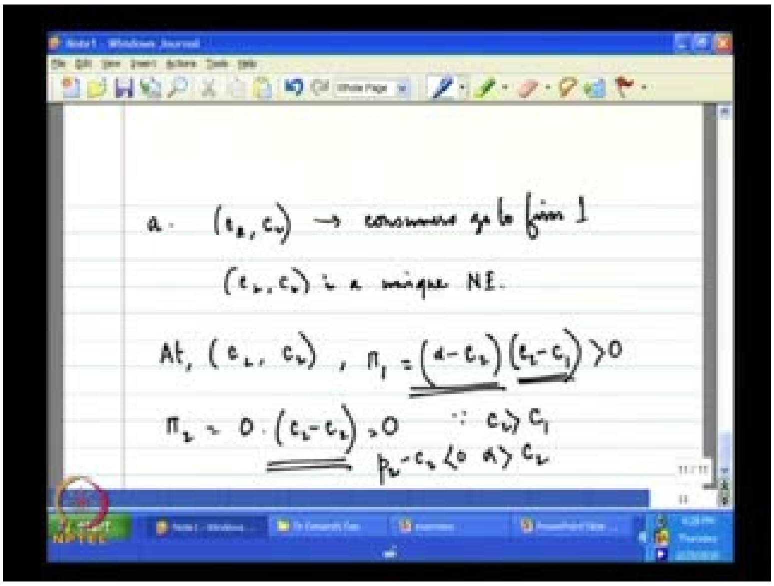

Suppose, that the rule for splitting up the consumers when the prices are equal, assigns all consumers to firm 1 when both firms charge the price c 2. Show that p 1 p 2 is c 2 c 2 is Nash equilibrium and that no other pair of price is Nash equilibrium. So, what is being said is that, suppose c 2 is c 2 c 2 are the prices common price that is charge by the two firms, then all consumers go to firm 1, so firm 2 does not get any part of the market; then if this is the rule of splitting the market, if c 2 c 2 is the price, then we have to show that c 2 c 2 is a unique Nash equilibrium, this is what we have to show.

Now, to clarify the question notice if the prices are equal, if p 1 is equal to p 2, but that level of equality is not equal to c 2, some other level for example, then the old assumption still holds the old assumption is that the market is evenly split between the two firms only when the prices are equal and that equal price is equal to c 2, then the market is not equally split, it goes only to firm 1, that is one clarification.

So, we have to show that c 2 c 2 is Nash equilibrium, and it is the unique Nash equilibrium. Now, c 2 c 2 at c 2 c 2 what is the profit for of firm 1, for example, firm 1 is getting the entire market, so alpha minus c 2 is the demand that it is getting multiplied by c 2 minus c 1, this is the profit, and this is positive because c 2 is greater than c 1 that is given, and alpha is also greater than c 2, that is implicit that we have seen before, that the cost is always less than this alpha the coefficient term of the demand function; so, this both the terms are positive so pi 1 is positive, pi 2 what about pi 2 pi 2 is 0 because firm 2 is not getting any part of the market, so all the consumers are going to firm 1, and so this is equal to 0 0 multiplied by 0, it is 0.

So, these are the profit levels at c 2 c 2 can someone do better; if firm 1 charges a price less than c 2 that if I charge a price a little less than my profit margin is going down, if my profit margin goes down then obviously my profit is less than I was earning before; so, by charging a priceless I am not better off. If I charge price a little more than all the consumers will go to firm 2, so I was earning some positive profit, now I will earn 0 profit, so deviation by firm 1 is not profitable; can firm 2 deviate and be better off, if firm 2 deviates and price charges a price less than c 2 it is going to earn negative profit margin, this part p 2 minus c 2 will turn to negative and if p 2 is less than c 2. If it charges a price more than c 2 can that be a Nash equilibrium, can that be a something which is more than 0, the answer is no, because if it charges more than c 2, then c 2 is the lowest price in the market.

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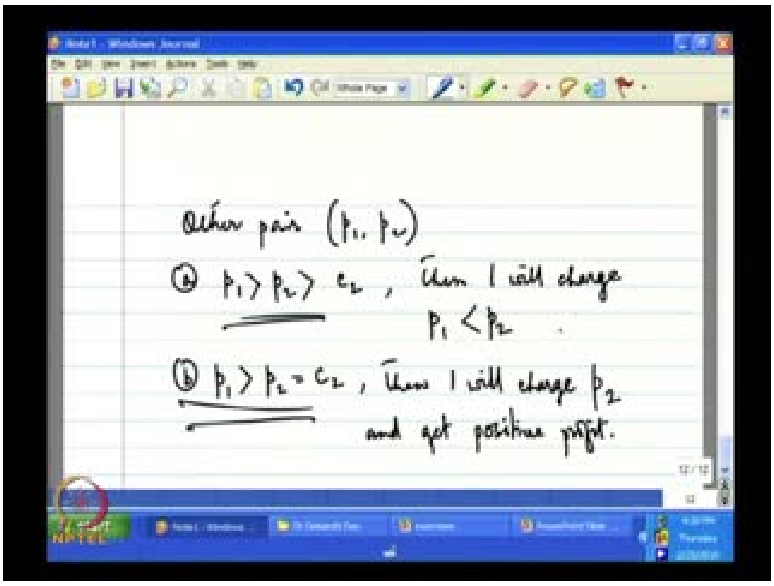

So it is not going to get any customers, so for from the point of view of both the firms deviation is not profitable, and so the firms are going to stay at Nash equilibrium; question is, is there any other pair which could be a Nash equilibrium, so other pair let us call the other pair p 1 p 2.

Now, there will be many cases of this other pair; suppose, p 1 is greater than p 2 greater than suppose c 2, and in that case is that a Nash equilibrium, the answer is no because then firm 1 will charge p 1 strictly less than p 2, so deviation is profitable b; suppose, firm 1 is charging a higher price, firm 2 is charging a price equal to c 2, then firm 1 will charge p 2 and get positive profit.

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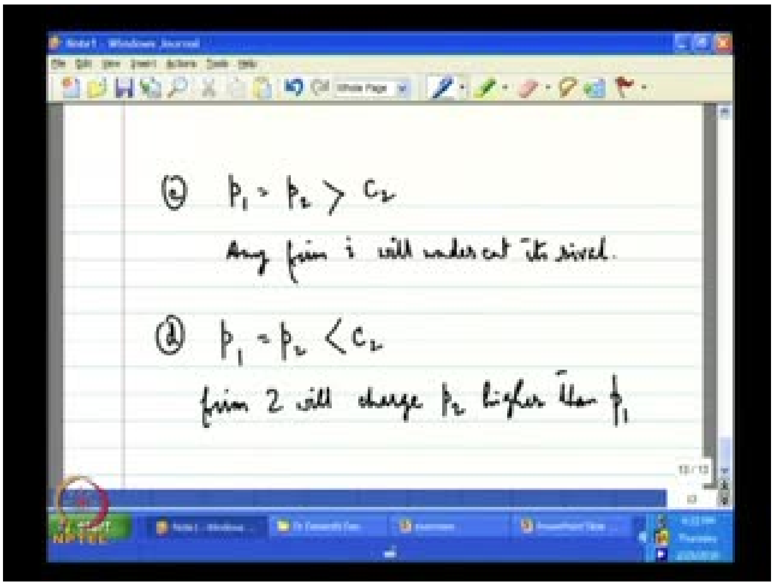

Here firm 1 in this case firm is getting 0 profit here also firm 1 is getting 0 profit if it charges a price less than p 2, it is going to earn positive profit. Case c, can it happen that p 1 is equal to p 2 and both are greater than c 2; well, in this case both the firms are getting some positive profit and the market is split, but if in that case suppose any firm I will undercut its rival, that is charge a price little less than this p 1 is equal to p 2 and get the entire market and earn some positive profit which is which will be greater than this positive profit, so that is there.

What could be the other possibility? Suppose one of the prices is greater than suppose p 1 and p 2 are such that both are of them are less than c 2, this is not a Nash equilibrium because what is happening is that the market is again getting split; and if the market is getting split firm 2 is earning negative profit, because p 2 minus c 2 is negative.

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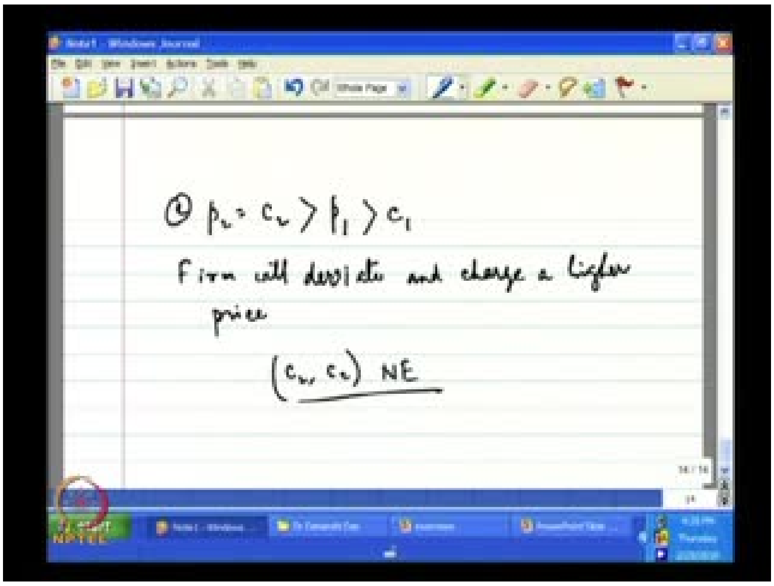

So, firm 2 will do better it will deviate then p 1; in fact we can show that there cannot be a Nash equilibrium where p 2 is less than c 2, but can it happen that p 2 is equal to c 2, but that p 2 is greater than p 1, and suppose this p 1 is greater than c 1, because there cannot be in a Nash equilibrium where p 1 is less than c 1.

Now, this is also not Nash equilibrium, because in this case one firm 1 will deviate and charge a higher price. So, basically, what we have argued as is that no other pair can be a Nash equilibrium; if the other pair prices are such that both the prices are more than c more than c 2 then any firm will have a tendency to undercut its rival; if both the firms are charging price where p 2 is equal to c 2, and firm 1 is getting charging a price greater than c 2, then firm 1 is going to charge a price equal to c 2 and get the entire market around positive profit.

If the prices are less than c 2 but greater than c 1, that cannot be a Nash equilibrium, because firm 2 is earning negative profit if firm 2 is earning a charging a lesser price, it can do better by charging equal to c 2; if firm 1 is charging the lesser price firm 1 can do better by charging not equal to charging a price a little more than what it is charging, but keeping that price less than p 2. So, by exhausting all this possibilities basically we are left with this c 2 c 2 cases which is Nash equilibrium.

Let us now come to the second part of the question. Show that no Nash equilibrium exits if the rule first splitting up consumers when the prices are equal assigns more consumers to firm 2, some consumers to firm 2 when both the firms charge c 2, so this is part b.

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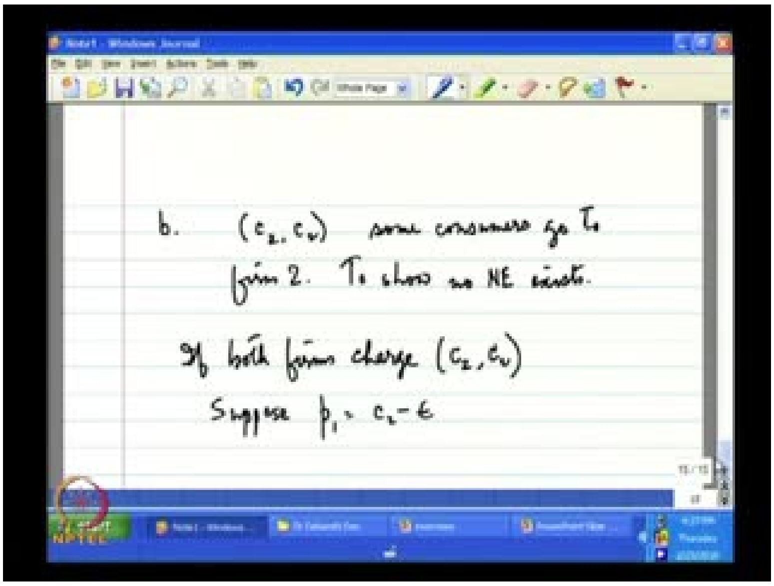

So, if c 2 c 2 is the price pair goes to firm 2, so we have to show that no Nash equilibrium exists, this is not a very difficult proof; if both the firms charge c 2 c 2, and if some customers go to firm 2, notice that this is a case which is different from the case that we had before in a, in a some none of the customers were going to firm 2.

Here some at least may be one or two customers are going to firm 2, in this case since at least one person one customer is going to firm 3, then this is not a Nash equilibrium for the reason that in this case firm one can do better by charging a price a little less than c 2, because if it is charging a price a little less than c 2, suppose p 1 is equal to c 2 minus epsilon.

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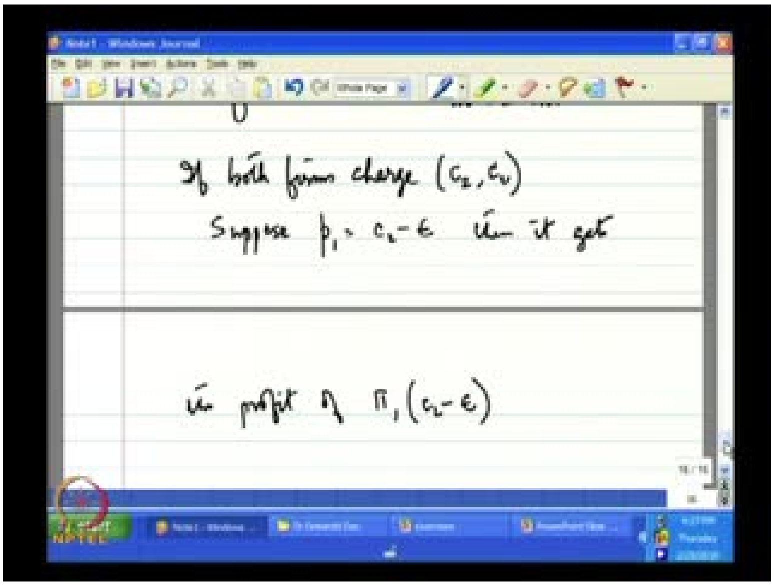

Now, if firm 1 is charging a price little less than c 2 then it gets the profit of pi 1 which is a function of c 2 minus epsilon.

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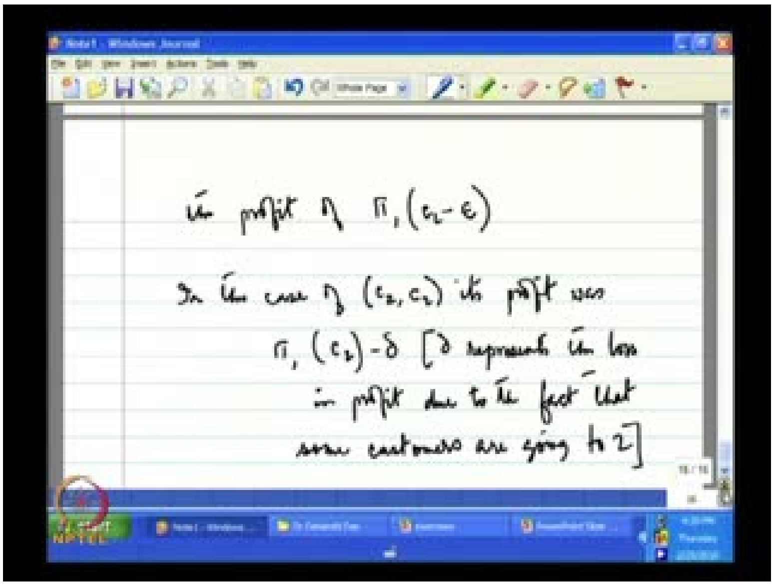

In the case of c 2 c 2, it profit was this minus suppose delta, delta represents the loss in profit due to the fact that some customers are going to two; so, delta is arising because some customers are going to firm 2, this delta was 0 in case a.

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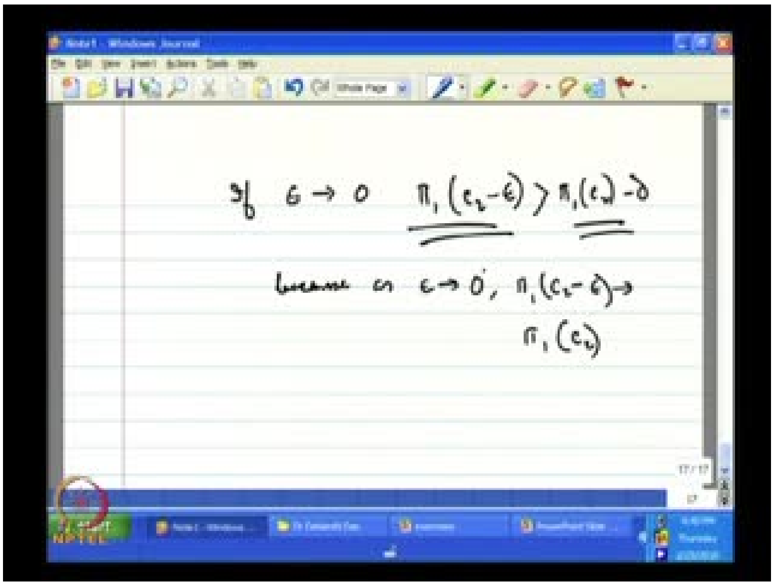

Now so if epsilon goes to 0, c 2 minus epsilon can become more than minus delta, because as epsilon goes to 0 pi 1 c 2 minus approaches c 2; so, this part the left hand side becomes higher than the right hand side, this is epsilon goes to 0, which means that by just undercutting the rival firm 1 can get the entire market and not share any customer with firm 2; and therefore, it this situation of c 2 c 2 does not remain a Nash equilibrium, and by the demonstration that we have given before no other pair will be a Nash equilibrium.

So, this other pairs are not Nash equilibrium and this pair also does not remain a Nash equilibrium, which means that Nash equilibrium does not exists in this case. So, that is the end of the lecture, we shall start with some new exercises in the next class. Thank you.

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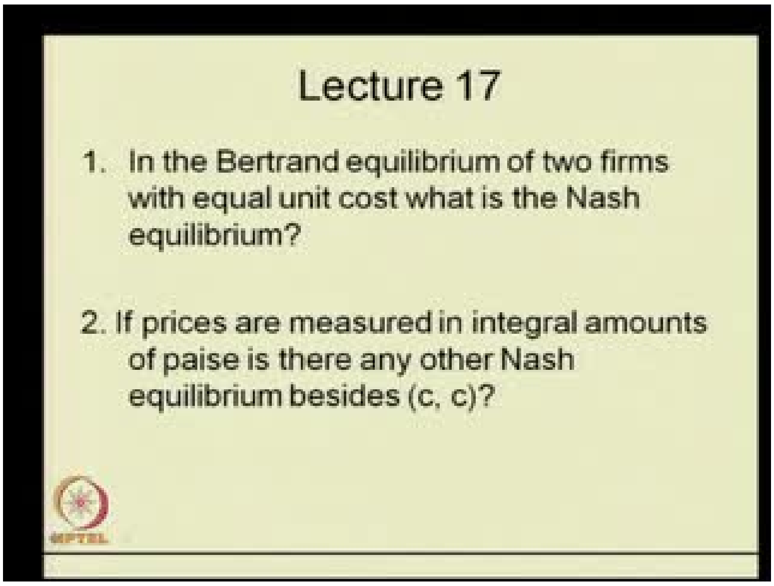

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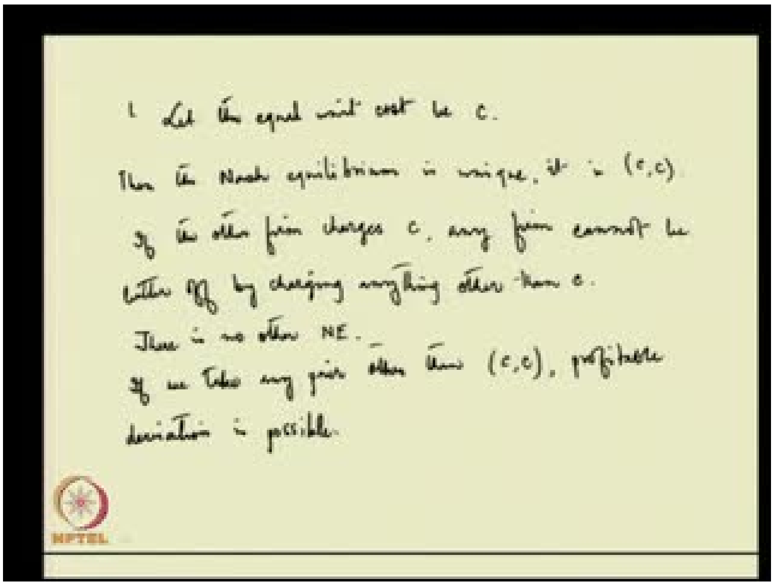

In the Bertrand equilibrium of two firms with equal unit cost what is the Nash equilibrium? So, let the equal unit cost be c small c, then the Nash equilibrium is unique, it is c c, then both the firms will charge the same price which is equal to small c, what is the proof, I will go very quickly over the proof. If the other firm charges c any firm cannot be better off by charging anything other than c, this is because if you charge more than c you are not getting any markets, so your profit is 0. If you charge less than c based on the price of the other firm you are going to get some market, some demand, but infact the entire demand, but then the profit is negative which is worse than earning 0 profit which this firm will earn if it charges c.

So, c c is an equilibrium; what is the proof that there is no other equilibrium, this is because if we take any pair other than c c if the prices are different, if we take any pair other than c c, profitable deviation is profit possible, this is because if the prices are different and both the prices are more than c then the firm which is charging the higher price will undercut the other firm.

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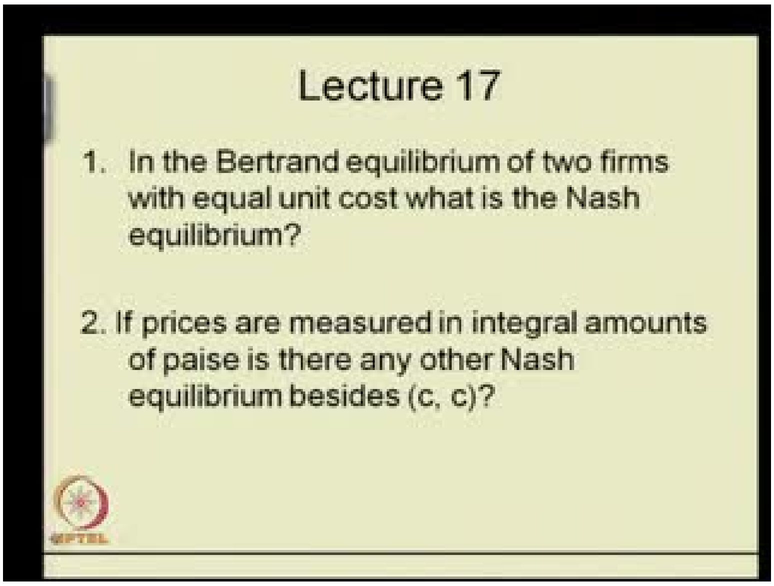

So, there is a profitable deviation; if the prices are same then each firm will undercut the other firm; if the prices are less than c then either one or both of them are earning negative profits, so they will deviate and charge more at least a equal to c which will give them 0 profit, so this is the demonstration that c c is the only equilibrium.

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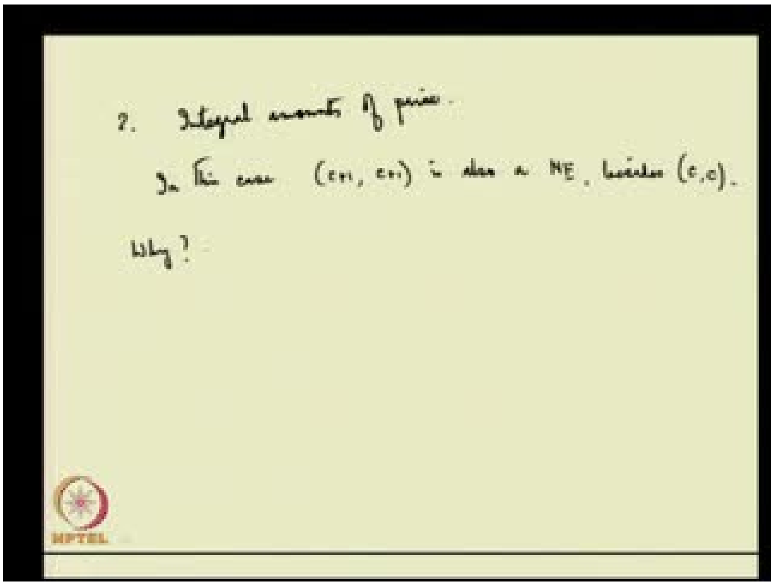

If prices are measured in integral amounts of paise, is there any other Nash equilibrium besides c c, so prices could be either 1 paise, 2 paise, like that, it cannot be one and half paise, in that case c plus 1 c plus 1 is also a Nash equilibrium besides c c, so the c c remains an equilibrium, but c plus 1 c plus 1 is also a Nash equilibrium, why? Because if the other firm is charging c plus 1 and I am charging c plus 1 I am getting some profit; if I reduce my price I charge c I will get 0 profit, so deviation downwards this un profitable; if I charge more than c plus 1 obviously again the profit is 0, so c plus 1 c plus 1 is a Nash equilibrium. Thank you.