Game Theory and Economics

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Module No. # 03

Illustrations of Nash Equilibrium

Lecture No. # 07

Different Aspects of Hotelling Model

Before we start this lecture, let me take you through what we have discussed so far in this particular module; especially in the last lecture. What we have been discussing is that we are trying to see in the situations of elections, where the candidates are fighting with each other to get elected. What could be the choice of agenda of the candidates? How the voters will vote? Who will win in equilibrium? To understand these questions and to give the answer to these questions, we are basically applying this framework of game theory; in particular, Nash equilibrium. This particular framework has a name; it is called the Hotelling's model

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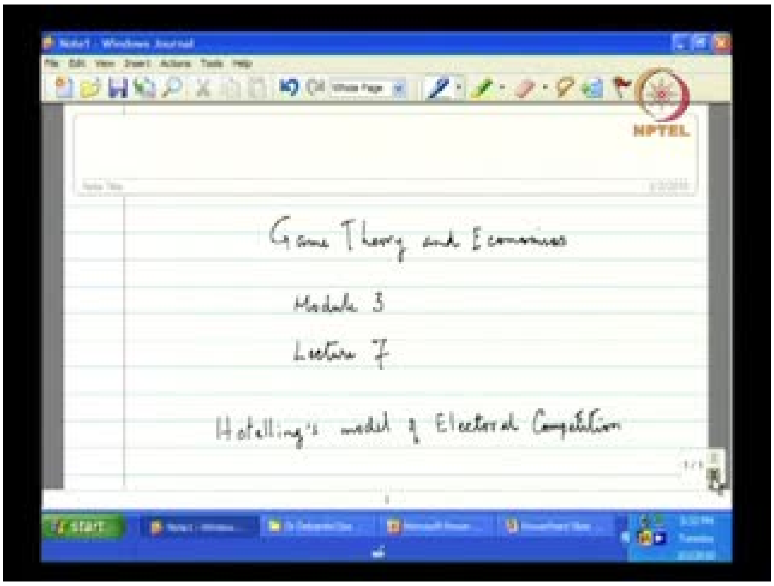

This is Hotelling's Model of Electoral Competition. What we have discussed so far is that to make it simple, there are two candidates who were trying to win in the election.

There is a continuum of voters. Each voter has a favorite position in the sense that each voter has a particular policy, which he or she will like to see implemented. What the candidates want is that they just want to win.

Taken with 3 kind of situations. One is a candidate is winning out rightly. Second is that he is not winning out rightly; he is having a tie with the other candidate in the first position. So, this is the second preferred option for a candidate. This is not the best option to have won to a candidate. The third alternative, which is the worst alternative for a candidate is that he loses. So, that is the worst thing that can happen. This is the preference ordering - you win out rightly that is the best; second is that you are in the first position, but you are having a tie; third is that you lose.

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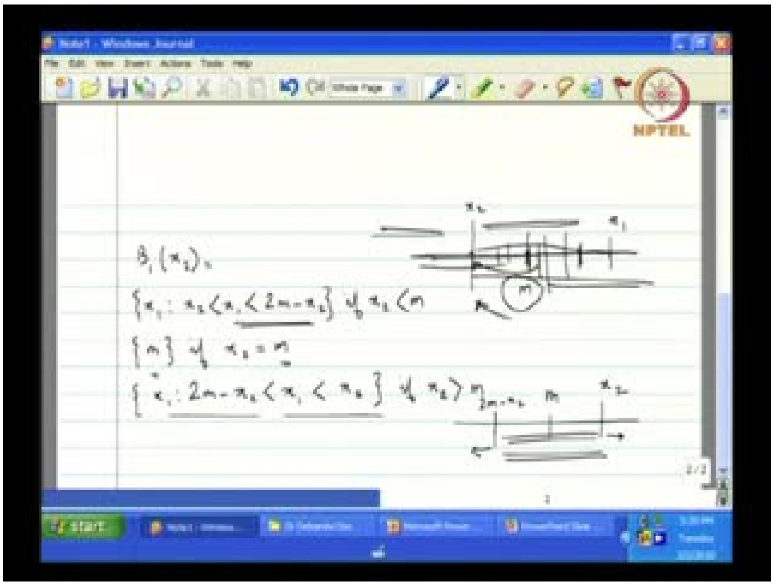

As I said, every voter has a favorite position. These favorite positions are distributed in a continuous manner over this line. Every number in this line will represent the favorite position of at least one voter and M is the median of all these numbers.

Now, since it is a game between the candidates, each candidate will announce some policy and try to get as much vote as possible. If he gets more than half of the votes, he wins. Based on that, we have seen that we can construct the best response functions for the candidates. This is (Refer Slide Time: 03:41) the best response function of suppose player 1. We have seen that this can be represented as… (Refer Slide Time: 03:50) So, this is the best response function of player 1.

Basically, what is happening is that if player 2 announces something here (Refer Slide Time: 04:44) to the left of M. Player 1 has to tend. If one wants to win, has to announce something here on this range. This is because if he announces something in this range less than x 2; this is x 2, then all these votes will go to x 2, which is greater than half because m is here. So, to win, player 1 must announce something more than x 2. However, merely announcing more than x 2 will not do if player 2 announces something here (Refer Slide Time: 05:20).

Suppose player 1 announces something here, then middle point between x 1 and x 2 is falling in this case here to the right of M. This means all these votes will go to player 2. Since this point is to the right of M, player 2 will get more than half of the votes and player 2 will win. For player 1 to win, player 1 should ensure that x 1 plus x 2 divided by 2 is less than m. From that we get this condition (Refer Slide Time: 05:58) that x 1 should be less than 2m minus x 2. So, if this is M, this is 2m minus x 2. x 1 should be less than that and greater than x 2. So, x 1 should be in this range (Refer Slide Time: 06:17) and you know that only in that case player 1 will win.

If player 2 is announcing a position just equal to m, then in no circumstance can player 1 win this election. This is because if he announces something to the left, he is getting these votes; (Refer Slide Time: 06:41) less than half. If he announces something here, he is getting these votes; again less than half. By announcing something other than the m, he will lose. So, if a player 1 announces m in this case, then he can at least tie with player 2. That is the best thing that he can do in these circumstances. So, that is why if player 2 is announcing m, player 1 is also announcing m.

The third situation is like the first situation. Here (Refer Slide Time: 07:19) x 2 is greater than m, then like before x 1 should fall in this range. This point is given by 2 m minus x 2. If player 1 announces something here, (Refer Slide Time: 07:40) then he will lose. If he announces something here, again he will lose. So, he will announce something in this range. That is why we have got this condition. So, this is the best response function of player 1. Similarly, we can find out the best response function of player 2, which we can write as follows:

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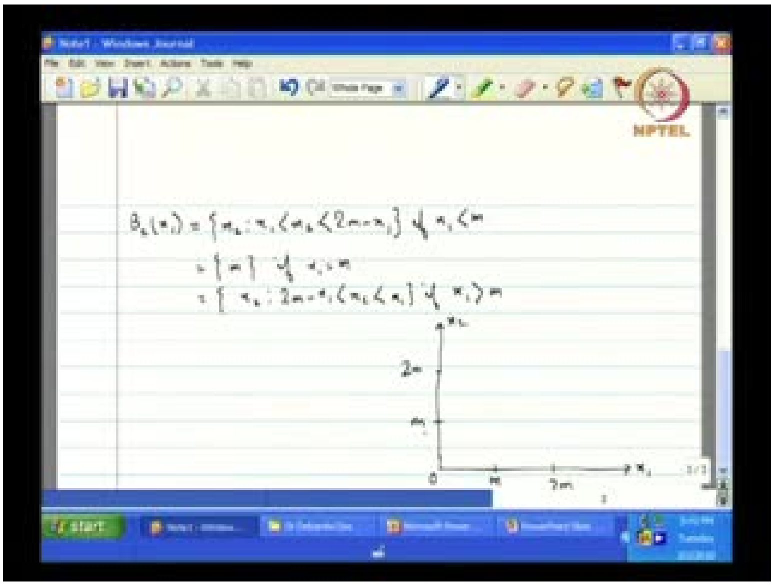

So, this is just replacing x 1 by x 2 and x 2 by x 1.

Now, the next task is that we try to plot these two best response functions and try to see what is the point of intersection or points of intersection. Those point or points will be the points of Nash equilibrium.

Suppose this is the x 1 axis and x 2 axis and we want to plot this best response functions (Refer Slide Time: 09:45). First is this that x 2 is less than m. Then, what we are having here is that x 1 should be greater than x 2.

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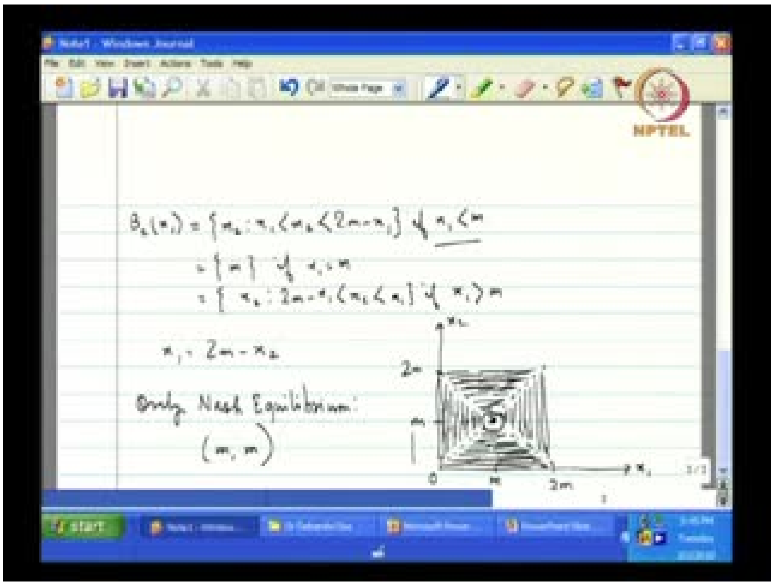

x 2 is less than m. So, you are talking about this range. Suppose this is the point (m,m); this is the point (2m,2m). What we are having is that x 1 should be greater than x 2. So, I can imagine this line, which is the 45 degree line. x 1 should be greater than x 2 means we are taking points to the right of this line.

Do I take all the points all the way to infinity? The answer is no. x 1 should be less than 2m minus x 2 (Refer Slide Time: 10:33); this. So, I need to plot this line (Refer Slide Time: 10:39) 2m minus x 2. How will this look like? This is the straight line with a vertical intercept of 2m. So, it will start from here. Slope of minus 1. So, this line will go through these 3 points (Refer Slide Time: 11:00). So, this is the line x 1 is equal to 2m minus x 2. In this case, x 1 should be less than 2m minus x 2. So, I cannot take points on this line, but I have to take points to the left of this line. So, this is the best response function in this case that x 2 is less than m.

If x 2 is equal to m, I have got x 1 is equal to m. So, I pick up this point (Refer Slide Time: 11:37). This point is on the best response function. If x 2 is greater than m, this is the case (Refer Slide Time: 11:47). Then, x 1 should be less than x 2. So, all these points will be included (Refer Slide Time: 11:59). x 1 should be greater than 2m minus x 2. So, all these points to the right of this line should be included. So, we have this horizontally shaded region as the best response function of player 1.

Similarly, I can show that if x 1 is less than m, I have got x 2 should be greater than x 1. So, all these points will be included (Refer Slide Time: 12:38). x 2 should be less than 2m minus x 1. So, all these points will be included. Similarly, the last part will be depicted by this shaded region (Refer Slide Time: 12:58). If x 1 is equal to m, x 2 is equal to m. So, again we are picking up this same point in between.

What are the cracks of the story? The cracks of the story are that we are having this shaded region (Refer Slide Time: 13:20); both kind of shades as the best response function of these two players. Horizontal shades represent the best response function of player 1 and vertical shades represent the best response function of player 2. They are not having any intersection point expect for this central point (Refer Slide Time: 13:43). So, this is the only Nash equilibrium (m,m), which means that both the candidates are announcing the same position as their policy position. If they announce the same policy position, then obviously the voters will be equally divided between these 2 candidates.

This means no one is winning out rightly; it is a tie. At the same time, no one is losing either. So, this is the only Nash equilibrium situation here.

When the economist Hotelling proposed this model, he was basically analyzing the political map. His conclusion was that in the US, as you know there are two major parties: the republicans and the democrats. There are other small parties, but they are not very significant. So, the fact that we have two important parties is reflected in this model. The conclusion of this model is that they are going to announce the same policy.

Now, in real life, the democrats and the republicans do not announce identical same policy positions. However, as many political observers feel that their political positions or their announcements of what they will do after elected to the office are not radically different from each other. Maybe there will be some minor differences, but most of these two parties announce positions, which are largely similar, which is reflected by this (m,m) (Refer Slide Time: 15:52). That is how the economic model or game theoretic model is having a real life counterpart.

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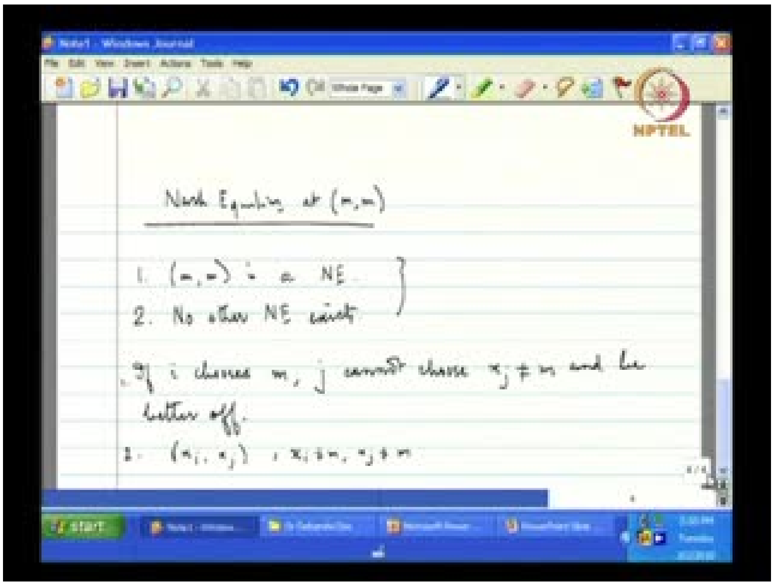

This fact that we are having a Nash equilibrium at (m,m). We have proved this by constructing best response functions (( )) However, the same demonstration could have been done in a more direct fashion. If you remember in the case of Bertrand model, we have tried to show that… There also, cc equilibrium could have been shown in a more direct argument.

Here what one can do is that first show (m,m) is a Nash equilibrium. Second step will be to show that no other Nash equilibrium exists. So, both of them together will mean that (m,m) is the only unique Nash equilibrium. The first part is not difficult to prove; we have actually given the argument already.

If i chooses m, j cannot choose x j not equal to m and be better off. So, if the other player is choosing m, I have to choose m. If I do not choose m, I will lose. If I choose m, of course, there is going to be tie. Tying is better than losing. So, by deviating, one is not better off; one is in fact strictly worse off. So, this is a strict Nash equilibrium. So, first part is proven, what about the second part?

No other Nash equilibrium exists, which means that if (x i, x j), where x i is not equal to m, x j is not equal to m, then this is not a Nash equilibrium. How do we prove this?

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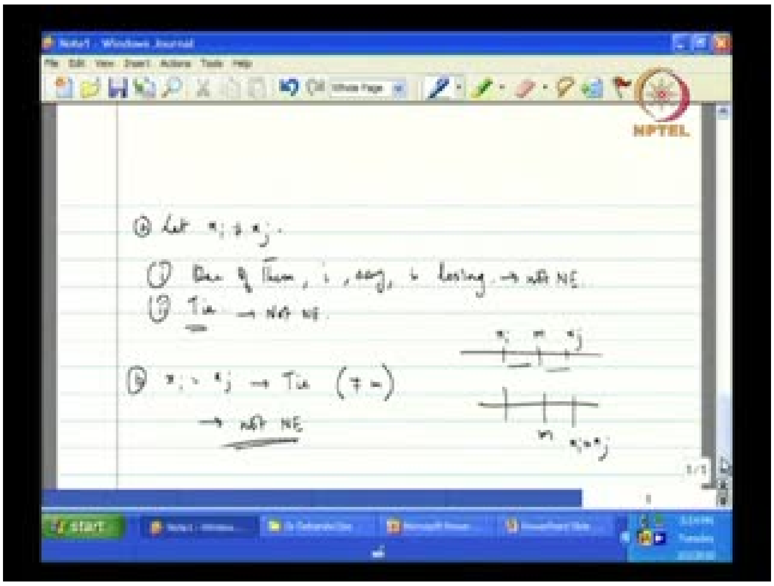

Let x i is not equal to x j. There can be 2 possibilities. One is x i is not equal to x j and the second is that x i is equal to x j. These two exhaust the all possibilities.

Now if x i is not equal to x j, it so happens that suppose there is a possibility here that one of them, let us say i is losing. The second is that nobody is losing, which means this is tie. We have to consider these cases. We have to show that none of these cases can be a Nash equilibrium. If none of these cases can be a Nash equilibrium, obviously we have exhausted all possible cases. So, the only Nash equilibrium that remains is that x i is equal to x j is equal to m. That is the only Nash equilibrium.

Now, if the x i not equal to x j and one of them loses. Can that be a Nash equilibrium? The answer is no. The reason is that i, who is losing will then be deviating and be better off. For example, he can at least announce m. If he announces m, then either he is going to have a tie because if x j is also equal to m, then there is going to be tie. Alternatively, he can out rightly win because if x j is not equal to m, then i by announcing m is going to win the election. So, both of these possibilities are better than losing. So, there is profitable deviation and there is a tie. How can there be a tie if x i is not equal to x j? (Refer Slide Time: 21:07) This is the case where there can be tie; where this distance is equal to this distance. The distance between x i and m is same as distance between m and x j. In this case, there is going to be tie.

Again, this is not a Nash equilibrium because (Refer Slide Time: 21:32) either of the candidates can choose to announce m and win out rightly. Each of them can have a better option of announcing m. If any candidate announces m, he is winning out rightly, which is better than tie. So, again deviation is profitable. So, this is not a Nash equilibrium.

What can be the other possibility? Possibility b is that x i is equal to x j. Now, if x i is equal to x j, it means that it is a tie. Both the players are announcing the same position. So, it is a tie. Now, if there is a tie, is this a Nash equilibrium? The answer is no because remember that we have ruled out the possibility that none of them is equal to m. If none of them is equal to m, then what it means is that they are having a tie somewhere here (Refer Slide Time: 22:44). So, here x i is equal to x j; somewhere here. That does not matter. So, from here also we can find at least one profitable deviation that any of them announces m.

Suppose instead of x i, i announces m. Then, he will win out rightly. So, we have exhausted all these possibilities and we have seen that none of them is a Nash equilibrium. So, the only Nash equilibrium we are left with is the (m,m) Nash equilibrium. So, it is a unique Nash equilibrium.

Let us now look at some other properties of this Hotelling's model. One assumption of this model is that the preference of the voters was symmetric. Now, this is a particular assumption, but it may not be very realistic. It may happen that a voter may dislike the candidate to his left more than the candidate to his right. So, he can have a stronger dislike for leftist than for rightist and vice versa. We are going to show that. That is not going to change the conclusion of the model.

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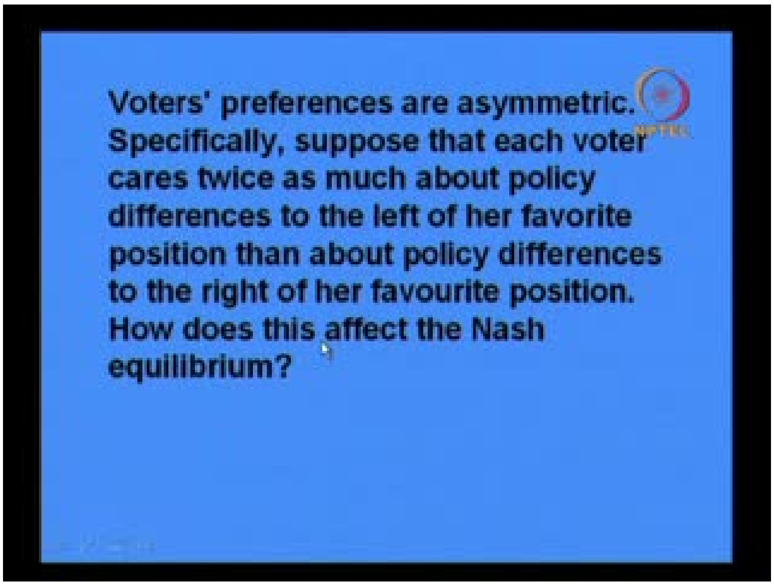

This is the exercise. Voters' preferences are asymmetric. Specifically, suppose each voter cares twice as much about the policy differences to the left of her favorite position than about policy differences to the right of her favorite position. How does this affect the Nash equilibrium?

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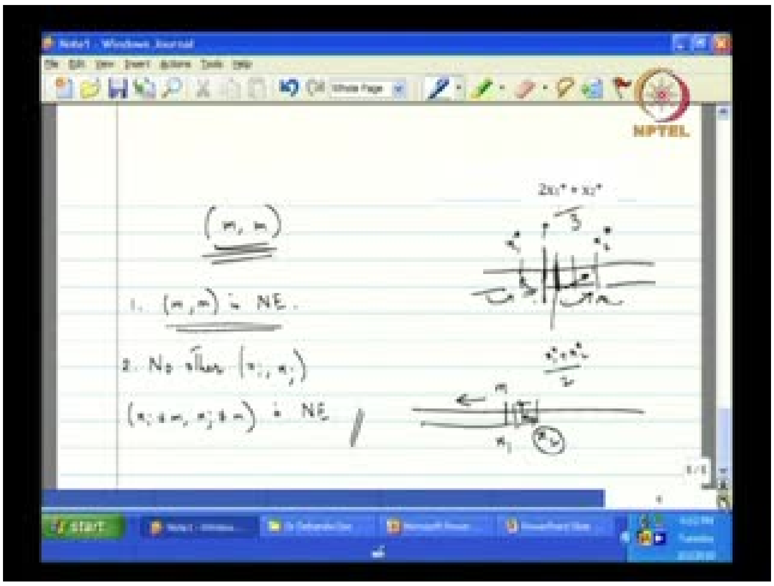

Here the story is the following: Suppose there are these two candidates and they are announcing x 1 star and x 2 star. We know that these voters will vote for him and these voters will vote for him. What happens to the voters in between? Suppose I take the middle voter. This is x 1 star plus x 2 star divided by 2. In the assumption so far with the symmetric preference, this way voter was indifferent. He could either vote for this person or this person. Now, with this new assumption, he will not be indifferent. What is being said is that if this fellow is having the same distance as this fellow; x 1 star is having the same distance as x 2 star, his dislike for x 1 star will be twice as the dislike for x 2 star. This means that this person is going to definitely vote for x 2 star.

If I have to find out which voter will be indifferent between x 1 star and x 2 star, how can I find that person out? This will be given by a person here. Suppose this is x 1 star (Refer Slide Time: 26:23) divided by 3. I am dividing this difference between x 1 star and x 2 star into 3 equal parts. After this first part, I am getting this part - x 1 star plus twice x 2 star divided by 2.

Now, for this person, (Refer Slide Time: 26:44) the distance between this person and x 1 star is half of the distance between his position and x 2 star since for equal distance, the dislike is twice for the leftist candidate than to the rightist candidate. Here since the differences are twice for the rightist candidate, his dislike for x 1 star and x 2 star will be the same. This means that now, all these voters (Refer Slide Time: 27:18) will vote for him and all these voters will vote for him. So, this is the difference that we have in the setup. Question is that does this affect the Nash equilibrium? Remember: The Nash equilibrium in the previous model was (m,m). Question is that does this still remain a Nash equilibrium?

We can proceed as we have done just now in the direct argument case. First we show that this is Nash equilibrium (Refer Slide Time: 27:52) and the procedure is very simple. If someone announces m, what can the other guy do? The other guy can announce more than m; he loses. If he announces x 2 and x 1 at m, then all these voters (Refer Slide Time: 28:19) are getting more than half of the distance. However, still this other fellow at m is getting more than half of the total votes and he is winning. No matter how close you come to m this fellow (Refer Slide Time: 28:35) is never going to win or tie with the other candidate as long as his position is more than the other person’s position. Similar argument can be made for any position to the left of m. So, the argument that was offered before that if the other person is announcing m, any candidate cannot do any better, but to announce m still holds. This means that this remains the Nash equilibrium (Refer Slide Time: 29:08).

Second step was to show that no other pair is Nash equilibrium, where x i is not equal to m or x j is not equal to m. Here also, the argument will be roughly similar; they can be different or they can be same. If they are different, then any of the candidates, who is losing… Suppose a candidate is losing can do better by announcing m; either he will tie or he will win. If x 1 and x 2 are different and they are tying, then again any of the candidates can go announce m and he will win out rightly. The third possibility is that these two candidates are announcing policies, which are same. If their announcements are the same, then they are having a tie. If they are having a tie, then any of the candidates can deviate and announce m and win. So, this means that no other Nash equilibrium (Refer Slide Time: 30:40) is possible.

Look here this fact (Refer Slide Time: 30:43) that hatchet for the leftist candidates is twice. It does not make any difference. So, the result that (m,m) is the unique equilibrium is the robust result. We can look at other aspects of this model. For example, here the model had only two candidates. We also had the requirement that any candidate, who is a candidate, cannot have the option of staying out; not competing if your candidate is contesting. However, this is not very natural. For example, a candidate instead of contesting may like to opt out. In particular, it may happen that if he contests, he will lose for sure. In that case, why should he contest? He will better stay out of the contest. So, if we give that option to a candidate, does the result remain the same?

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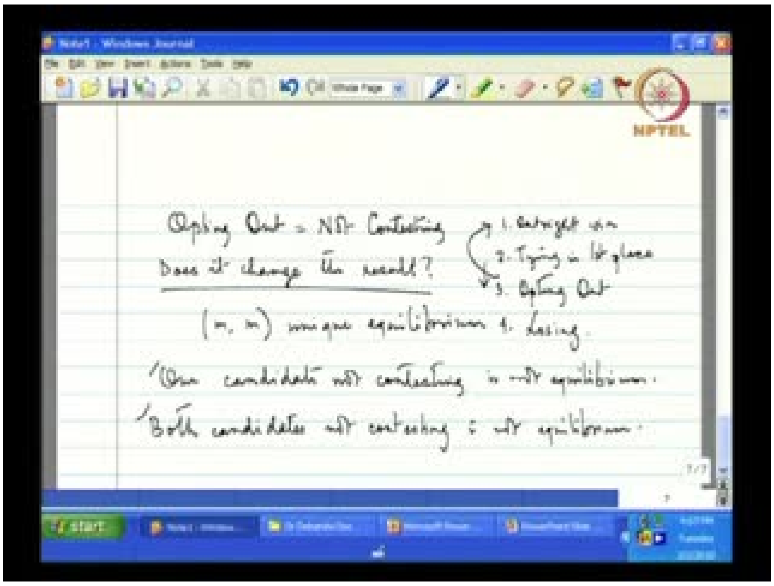

Opting out: That is, not contesting; Question is - does it change the result? The answer to that question is again no. The result that we had (m,m) as the unique equilibrium stays even if we have this additional option for the candidate to stay out. The proof is the following: The fact that someone is not contesting can change the result by two ways. It can happen that in equilibrium, one candidate is contesting the other candidate is not contesting. So, he is opting out. Alternatively, it can happen that both the candidates are not contesting the election. That can also make a difference because that is something, which we did not consider before.

Question is - one candidate opting out or not contesting is not equilibrium. Why we are saying that one candidate not contesting is not equilibrium? The reason is that if I am not contesting, the other candidate is contesting and winning. What are my preferences? My preferences are the following: First - outright win. This is very obvious like before. Second is tying in first place maybe with other candidates; not outright win. Third is opting out; I am not contesting. Fourth is losing.

Now, third is worse than second. If third is worse than second, then the candidate, who is not contesting can do better by contesting and at least for example, announce m. If he announces m, then he either wins or ties, which both of them are better than opting out. So, one candidate contesting and the other candidate not contesting cannot be equilibrium. Can it happen that both candidates not contesting be equilibrium? The answer is again no. If both the candidates are not contesting that cannot be equilibrium because from the point of view of each candidate, opting out is worse than winning. So, each of the candidate will think that better than opting out, I should enter the race and win out rightly. Again that is not equilibrium. So, the only equilibrium that can happen is that both candidates are contesting. If both the candidates contest, we are back to the old situation, where both the candidates are contesting and we know that the only equilibrium is this equilibrium. So, the fact that a candidate has the option of staying out and not contesting is not making any difference to the result.

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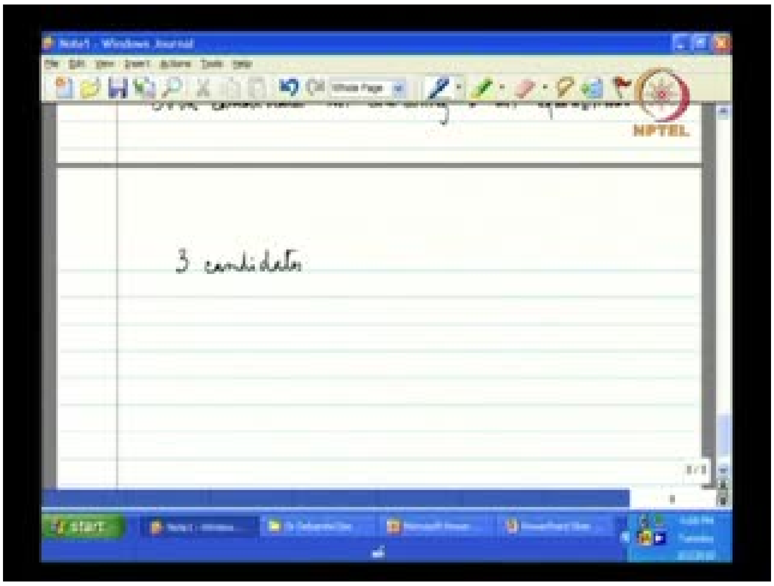

Let us not try to extend this model. Suppose we have three candidates instead of two, then does the result still hold?

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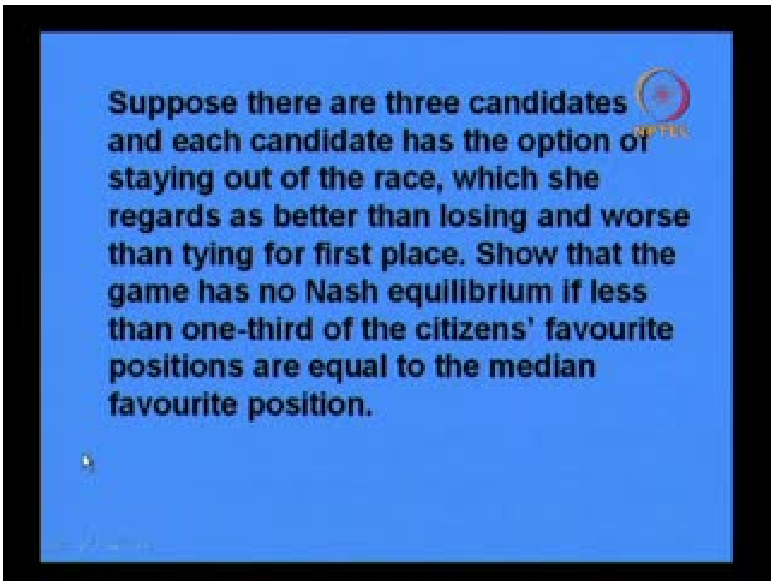

This is the exercise that we are interested in. Suppose there are three candidates and each candidate has the option of staying out of the race, which he regards as better than losing and worse than tying for first place. This is something that we have already seen. Show that the game has no Nash equilibrium if less than one-third of the candidates' favorite positions are equal to the median favorite position.

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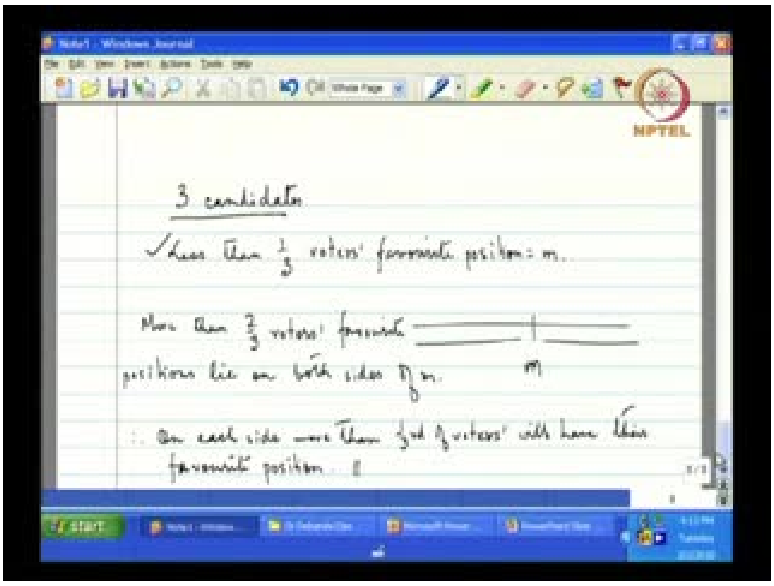

This is one additional assumption. The second is that less than one-third voters' favorite position is equal to m. How to prove that in this particular situation that there is no Nash equilibrium? However, before we start to prove this, let us look at the significance of this assumption.

Here is m. Less than one-third of the voters' have their favorite at m, which means that more than two-third voters' favorite positions lie on both sides of m. So, if I take the total number of voters, whose favorite positions are to the left or to the right of m, then their number will be more than two-third. This is because less than one-thirds are having their favorite position at m. This means that more than one-third of the voters' favorite positions will lie to the left and to the right. This is because this m (Refer Slide Time: 38:55) is dividing the total number of voters into two; equal halves. The total is more than two-third. So, in each half, there will be more than one-third (Refer Slide Time: 39:09). So, this is something that we have to keep in mind. This is something that we are going to apply later.

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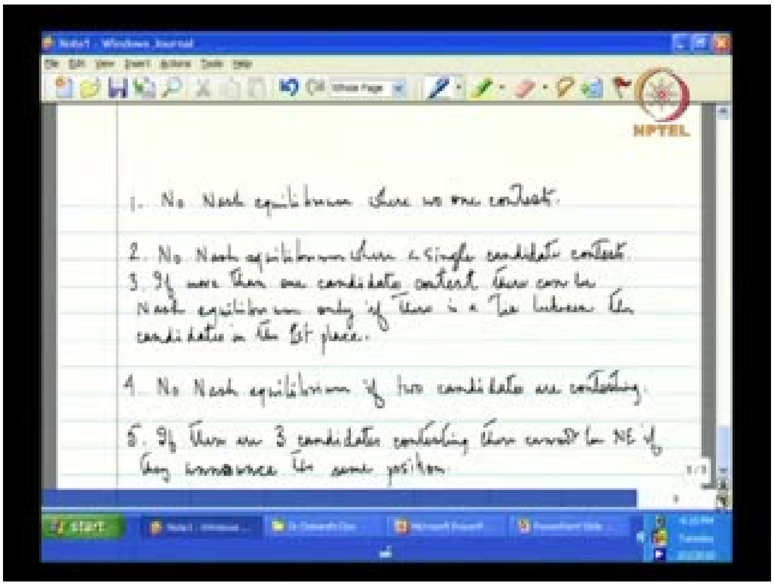

How to prove this that there is no Nash equilibrium in this case? What we are going to do is that we are going to do it step by step. Firstly, we prove that there is no Nash equilibrium in which none contests. This is the first proposition. Second is that no Nash equilibrium where a single candidate contests; there cannot be Nash equilibrium where only one candidate is contesting and the other two are staying out. Thirdly, if more than one candidate contest, then there can be Nash equilibrium only if there is a tie between the candidates. So, what we are saying is that suppose there is Nash equilibrium, in that

Nash equilibrium, more than one candidate has entered the race. Then, there can be such Nash equilibrium only if there is a tie between the candidates, who have entered the race in the first place.

Fourthly, we are going to propose that no Nash equilibrium if two candidates are contesting. So, if there are only two candidates, who are contesting the elections, then that cannot be Nash equilibrium. Fifthly, we show that if there are 3 candidates and all the 3 have entered the race, then there cannot be Nash equilibrium in which they have chosen the same position.

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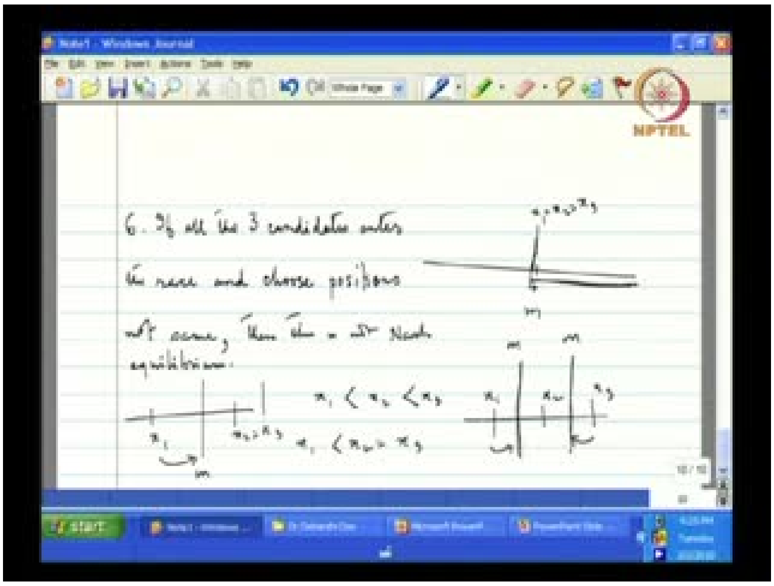

So, what are we left with, what is the only possibility of Nash equilibrium? That if all the three candidates enter the race and choose positions not same as m. It means that is it possible that all the candidates are entering the race? They are choosing different positions; maybe two of them are same and one is different, can that be Nash equilibrium? We are going to show that if all the three candidates enter the race and choose positions not same, then this is not Nash equilibrium. So, what we have done by all these six steps is the following:

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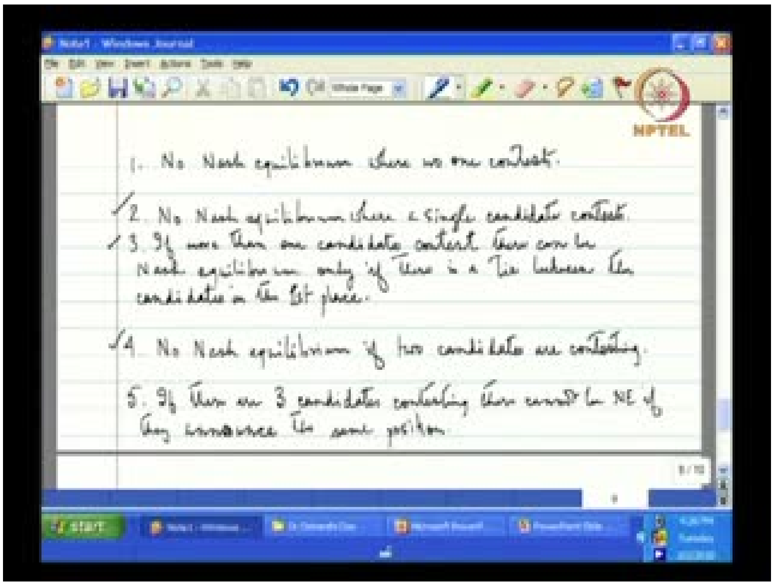

Let me go through the logic once again. Basically, we have ruled out all possibilities. First, we have said that there is no Nash equilibrium if nobody is contesting, which means that it leaves out the possibility that there can be Nash equilibrium if one candidate is contesting, two candidates are contesting, or three candidates are contesting. By 2, (Refer Slide Time: 46:11) we are ruling out the first possibility that there is only one candidate contesting. Then, we are saying that if suppose there are more than one candidate, who are contesting, then there can only be a tie between them.

By the fourth proposition, we are saying that there cannot be Nash equilibrium where only two candidates are contesting. So, that leaves the possibility that all the three candidates are contesting and there is Nash equilibrium. By the fifth proposition, we are saying that if three candidates are contesting, then there cannot be Nash equilibrium, where their positions are the same; they are choosing the same position. So, the only possibility that remains is that all the candidates are contesting and they are choosing positions, which are different; at least one of them is different. By sixth proposition (Refer Slide Time: 47:06), we are ruling out that also.

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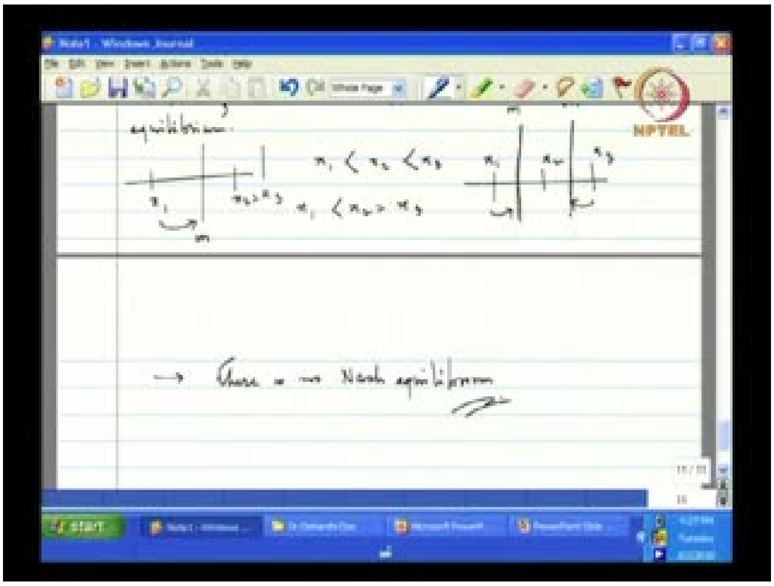

So, in some, we are ruling all possibilities of Nash equilibrium, which means that there is no Nash equilibrium. So, that was the (( )).

We shall stop here in this lecture. To finish what we have done is that we have looked into different aspects of this Hotelling's game.

We have seen that in the Hotelling's game, if we have two candidates and standard assumptions, then (m,m) is the only Nash equilibrium, where each candidate is announcing the median position as the policy decision. We have seen that if we change the assumptions like if we take into account the asymmetric preferences or if we bringing the assumption of opting out by the candidates, even then the result holds. However, if we have three candidates and an additional assumption that less than one-third of the voters have their favorite positions at m, then the result no longer holds. In that case, there will be no Nash equilibrium.

We shall pick up the thread in the next class. Thank you.

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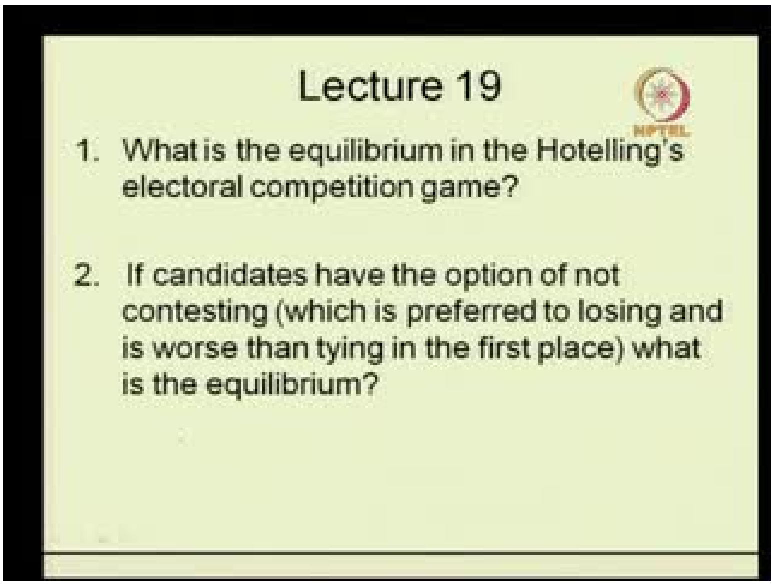

What is the equilibrium in the Hotelling's electoral competition game?

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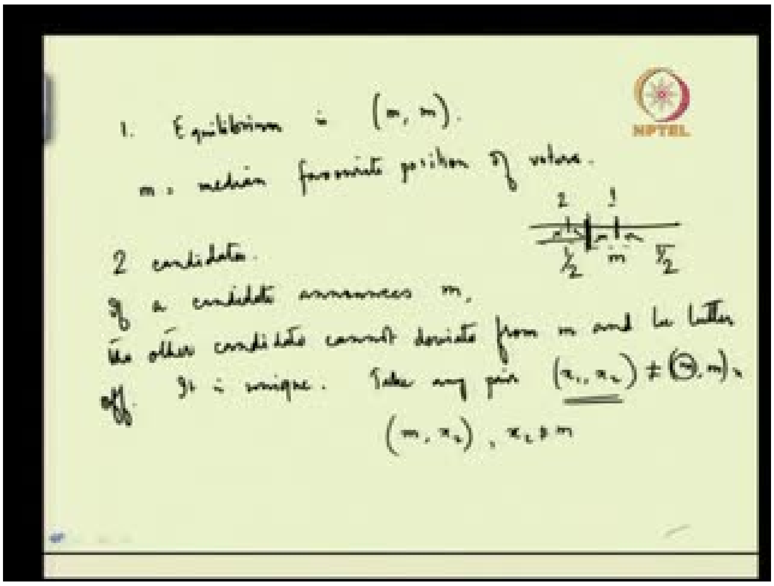

The equilibrium is at (m,m), where m is the median favorite position of voters. This means that half of the voters have their favorite positions to the left of the median and half of the voters have their favorite positions to the right of the median. There could be some overlap at median also. To be more precise, half of the voters have their favorite positions either less than m or equal to m and half of the voters have their favorite positions either equal to m or more than m. Why? This is equilibrium.

We have two candidates here. It is just like the Bertrand game. If a candidate announces m as his policy, the other candidate cannot deviate from m and be better off. The reason is very simple. If someone is announcing m, suppose 1 is announcing m; if 2 is announcing somewhere here (Refer Slide Time: 50:33) as his policy, then all these voters… This is the half amount between this position and this position. All these voters will vote for 2 and all these voters will vote for 1, but these voters are more in number than these voters. So, 1 will win here if 2 announces something to the left of m.

Similarly, if 2 announces something to the right of m, again 2 will lose and the same logic holds for 1 also. If 2 announces m, 1 cannot announce something other than m and win. So, given, any player is announcing m, the other players' best action will be to announce m. Therefore, (m,m) is Nash equilibrium. However, it is unique. What is the reason? Why this is unique?

Suppose we take any pair (x 1,x 2). Suppose x 1 and x 2 are not equal to m; (Refer Slide Time: 52:02) each of x 1 and x 2 is different from m. In this case, either one is losing. If 1 is losing, then what 1 can do is to announce m and that will make 1 the winner. If there is a tie here, even then one of them could go to m and announce m and be the winner. So, this cannot be Nash equilibrium. Can there be Nash equilibrium where (Refer Slide Time: 52:41) the positions are like one is m and the other is x 2; where x 2 is not equal to m? We have seen that this again is not Nash equilibrium because 2 is losing here. So, the best thing for 2 to do is to announce m rather than announcing x 2. Therefore, all other cases of pairs of announcements, where the numbers are different from m are not Nash equilibrium.

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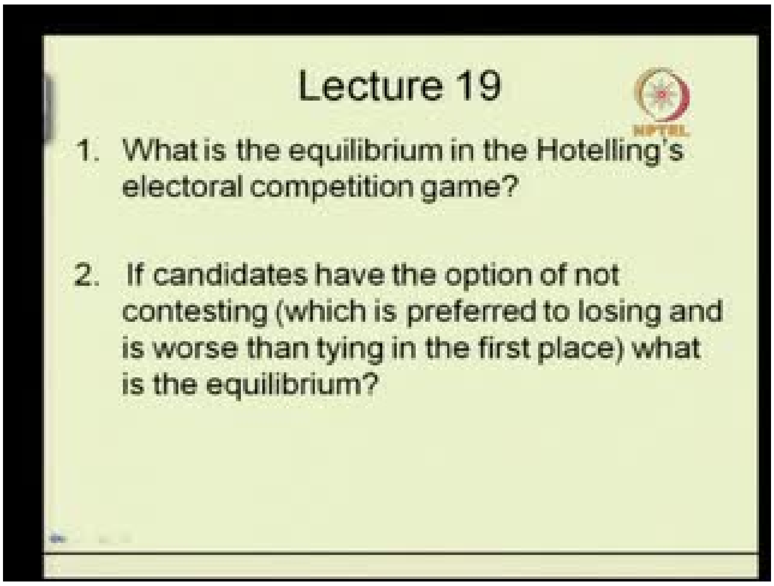

If candidates have the option of not contesting, which is preferable to losing and worse than tying in the first place; what is the equilibrium?

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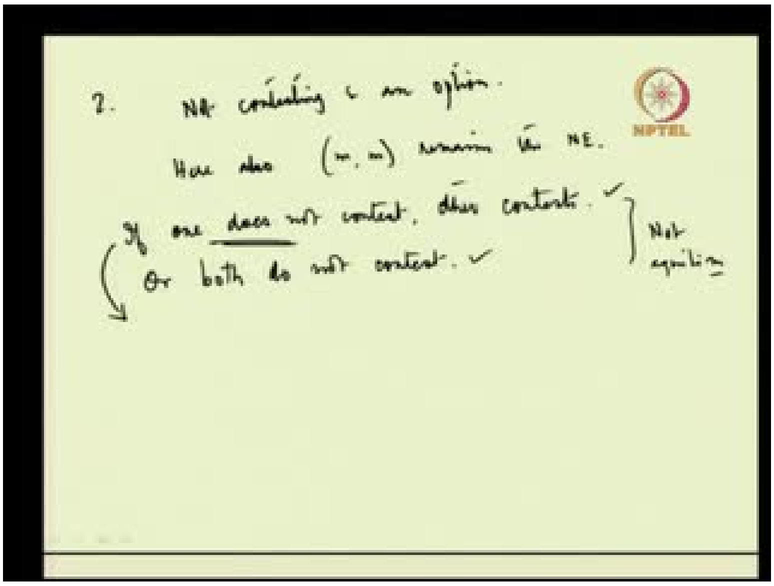

Not contesting is an option. Our claim is that here also, (m,m) remains the Nash equilibrium. The reason is - this when not contesting can make a difference if one does not contest in equilibrium and the other contests or both do not contest. These are the cases where not contesting option makes a difference; otherwise, both the players are contesting. So, we are back to the old game. In that case can these two be the equilibrium positions?

Let us think about this - one is contesting the other is not contesting. If the player who is not contesting sees the game, then he knows that the other player contests. If there is single player who contests, then the other player wins. So, it is better for this player (Refer Slide Time: 54:58) to contest. Maybe announce the same position as the first player. Then, there will be a tie, which is better than not contesting. Both do not contest - again cannot be an equilibrium because any one of them can contest and win the election out rightly, which is worse than not contesting. So, these are (Refer Slide Time: 55:19) not equilibrium. Therefore, not contesting option does not make any difference.

Thank you.