Game Theory and Economics

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Module No. # 03

Illustrations of Nash Equilibrium

Lecture No. # 08

Hotelling Model : Concluding Remarks

Welcome to lecture 8 of module 3 of the course called Game Theory and Economics. What we have done so far is that we have been applying game theory in various situations of economics that are pertaining to the political science. Currently, we have been discussing how the concept of Nash equilibrium in strategic game can be used in the context of elections. We have seen that in the standard model, which is known as the hotelling model, both the candidates will announce the position, which is the median favorite position of all the voters. So that is one fundamental result that we have got. We have been looking into other variations of this model. Suppose, instead of two candidates, we have three candidates, what is the result?

Does the result change compare to the basic hotelling result? For example, if the preferences of the voters are little different and not symmetric, but asymmetric. Does the result change like that? We have been discussing the various aspects and variations of the original hotelling model.

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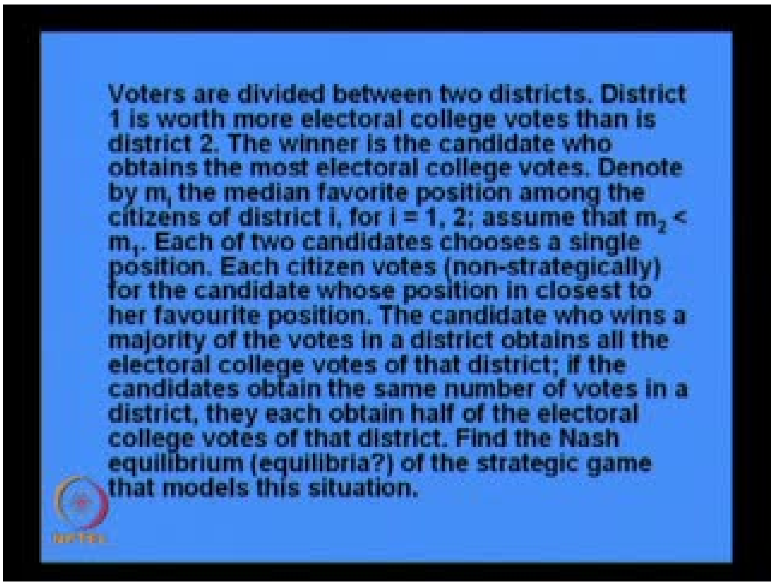

Today, we shall also look into some other variations of this basic model. The first variation is the following. Suppose that the voters are divided between two districts, district 1 is worth more electoral college votes than district 2. The winner is the candidate, who obtains the most electoral college votes. Basically, the winner is decided not directly by the voters, but by how many electoral college votes the candidate gets. There are two electoral colleges 1 and 2 and1 is having more electoral college votes than 2.

The median favorite position among the citizens of district is denoted by m i. The median favorite position among the citizens of district i, for i equal to 1, 2. Assume that m 2 is strictly less than m 1. Each of two candidates chooses a single position and each citizen votes non-strategically for the candidate, whose position is closest to her favorite position. The candidate, who wins in a majority of the votes in a district, obtains all the electoral college votes of that district. If the candidates obtain the same number of votes in a district, each obtain half of the electoral college votes in that district and find the Nash equilibrium or equilibria of the strategic game that models this situation.

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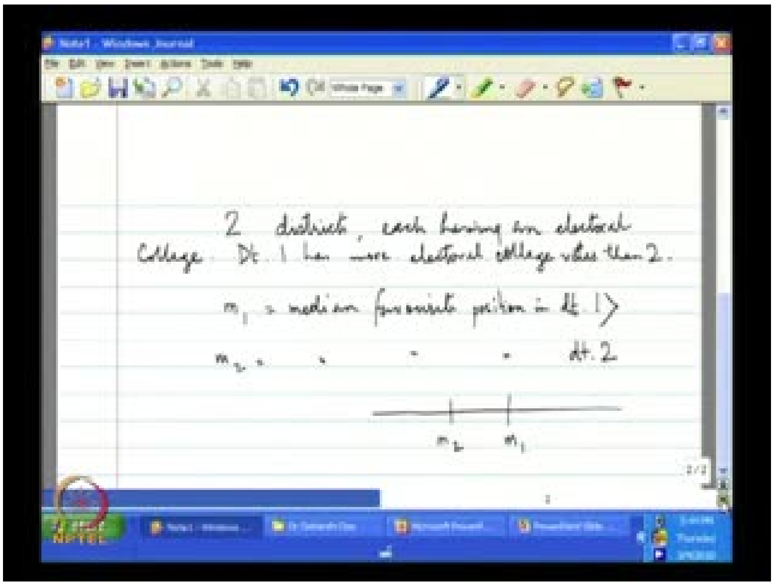

In simple language, what we have is the following. There are two districts and in Indian context, you can think of any two states. However, there are differences of the Indian electoral system. From this system, it is described in this question that each have an electoral college and district 1 has more electoral college votes than 2.

By the Indian semi league, an electoral college is like a Vidhan Sabha, the state legislature. In district 1, the state legislature is bigger and it has more states than district 2, whoever the candidate is and who gets the more electoral college votes is the winner. So, he is not elected directly by the people, but by the people, who are being elected to the electoral college. If a candidate gets the majority of the votes in a particular district, he gets all the electoral college votes of that district. All the people in the electoral college will vote for him in that case and that can be a visualize in that sense. So, this is the setting.

We know that m 1, the median favorite position in district 1 is more than m 2, which is the median favorite position in district 2. Each candidate announces a single position and based on that people in both districts decides whom to vote for. They can either vote for player 1 and player 2. So, I can think of this in the previous manner as this number line, on which the favorite positions of all the voters are represented. Here, m 2 is to the left of m 1 and m 1 is higher question. It is the Nash equilibrium in this game. By the way, if in a district, the two players get the same number of votes. Half of the voters are voting for player 1 and half of the voters are voting for player 2, then the electoral college votes of that district are equally divided between these two candidates. Otherwise, if I get even more than half more than my rival, then I get all the electoral college votes of that district and what is the Nash equilibrium?

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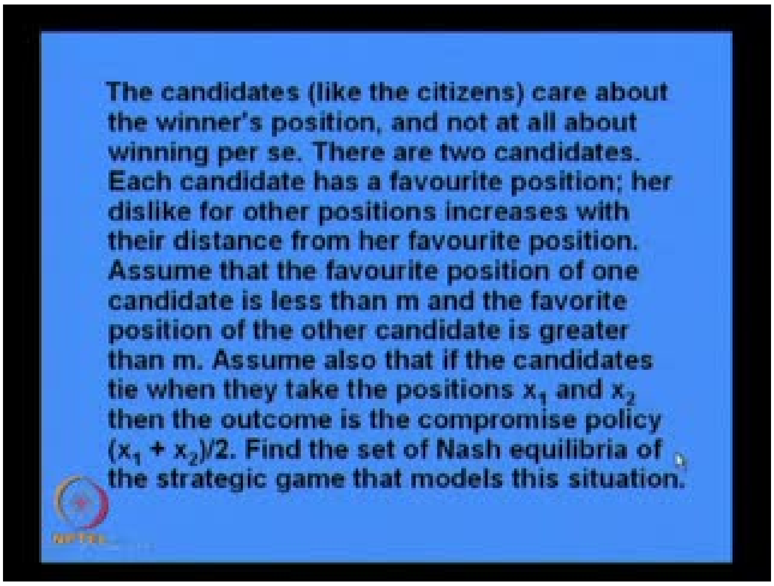

Let us look now at another aspect, which is an important aspect of this electoral games that we are studying. This is the aspect, where we talk about the fact that like the citizens or the candidates pay are also ideologically driven in the sense that they also may care about the positions that they take. So far, we have assumed that the position that the candidates take do not matter to the candidates. What matters to the candidates is whether they are winning or not. So that is what we can relax.

It may happen that like the citizens, candidates also care about positions that they are taking. Not only what position they are taking, can they take a more stronger case. They care about the position of the winner itself. It may happen that they lose; a candidate may lose, but if the winner's position is closer to my favorite position that could be the good thing for me. So that is what we are going to look at in this problem. Here is the problem, the candidates like the citizens care about the winner's position and not at all about winning per se. So, this is just an extremely opposite case. There are two candidates and each candidate has a favorite position. Her dislike for other positions increases with their distance from her favorite positions.

Assume that the favorite position of one candidate is less than m and the favorite position of the other candidate is greater than m. Assume that if candidates tie, they take the positions x 1 and x 2. The outcome is the compromise policy x 1 plus x 2 divided by 2. Find the set of Nash equilibria of the strategic game and that models this situation.

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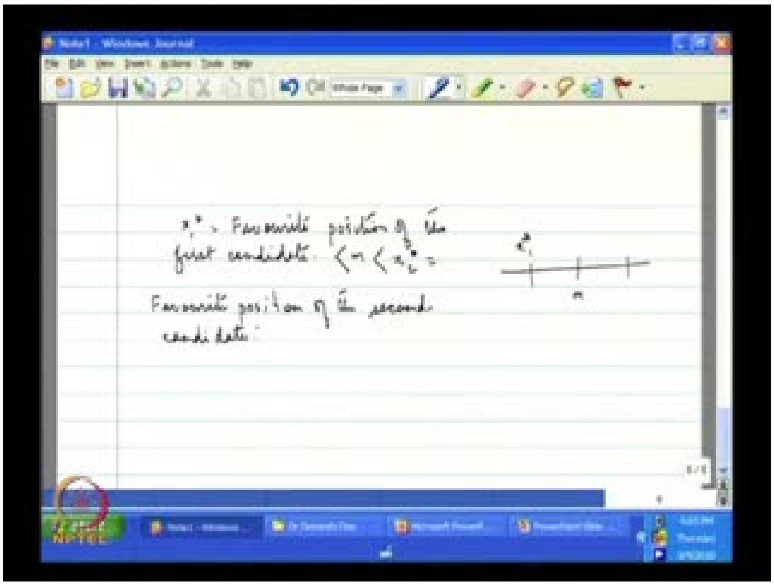

Here, the setting is as the original setting. I have the median favorite position of all the voters. I have to demark it as two favorite positions of two candidates. It is given that x 1 star is the favorite position of the first candidate. Suppose it less than m and less than x 2 star, which is the favorite position of the second candidate. So, x 2 star is here somewhere. Here, m is the favorite median favorite position of the voters.

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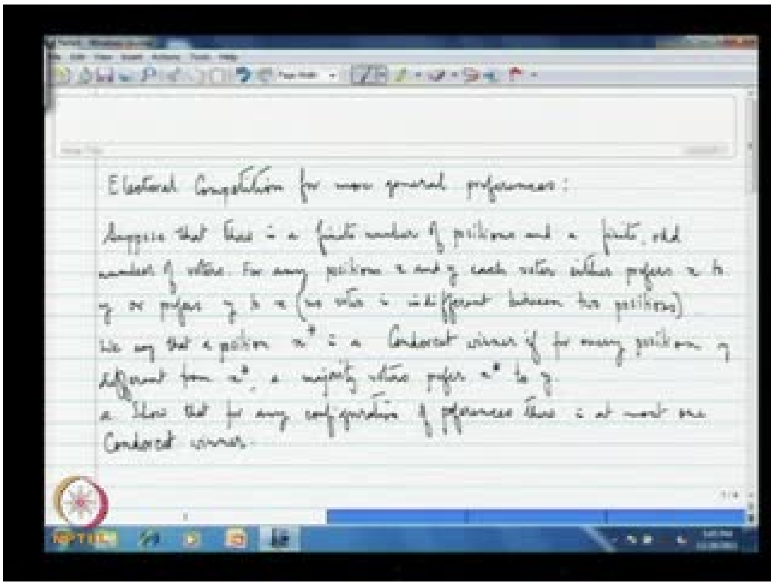

This exercise is called electoral competition. For more general preferences, let me first write down the question and we shall try to solve the exercise. So, this is the question and this is the description of the problem - suppose, there is a finite number of positions and finite odd number of voters. For any positions x and y, each voter either prefers x to y or prefers y to x. No voter is indifferent between two positions. We say that a position x star is a condorcet winner, if for every position different from x star and every position y different from x star. A majority voters prefer x star over y, so this how it defines a condorcet winner. Now, there are three questions. One shows that for any configuration of preferences, there is at most one condorcet winner, which basically means that either there is just one winner or there is no winner.

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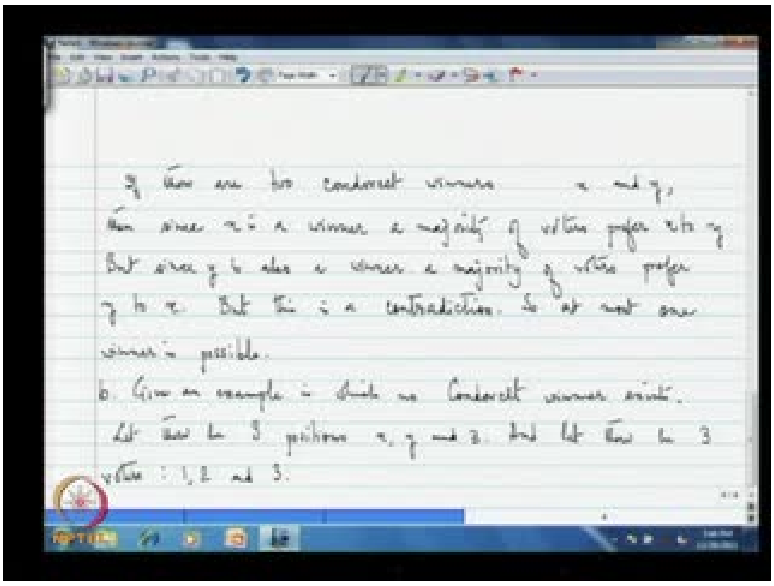

What is the proof? If there are two condorcet winners, x and y. Since x is a winner, majority of voters prefer x to y, but since y is also a winner, a majority of voters prefer y to x, but this is a contradiction. At most, one winner is possible, so that is the proof. Here, b is saying that given example, in which no condorcet winner exists. So, we have to just construct an example, where there are no condorcet winners.

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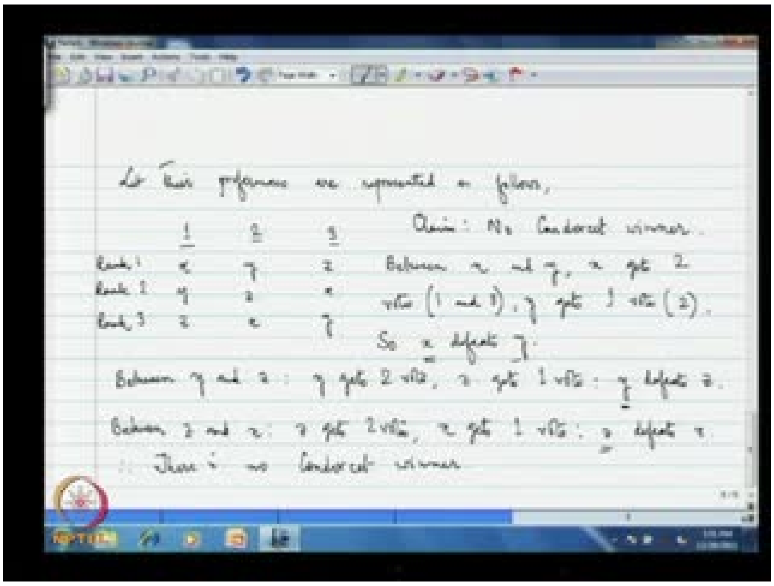

Let there be three positions and let three voters 1, 2 and 3. So, these voters are called voter 1, voter 2 and voter 3. Let the preferences of the voters are as follows: This is the preference of the first voter; this is rank 1, rank 2 and rank 3. So, he prefers x over y and y over z. We shall be assuming that they follow transitivity that is if x is preferred to y and y is preferred to z, then x is preferred to z. We have player 2 and whose preference is the following: here, y is appearing in rank 1, z is appearing in rank 2 and x is appearing in rank 3. So, player 2 prefers y to z and z to x. We have player 3, whose preference is the following and he prefers z to x and x to y.

Now, in this case, I am claiming that there is no condorcet winner, why? Let us pick up one pair between x and y, who gets how much vote? Player 1 prefers x to y, player 2 prefers y to x and player 3 prefers x to y. So, between x and y, x gets two votes - 1 and 3 and y gets just one vote – 2. So, in this case, x is preferred as x defeats y. Alright, between x and y, x is better. X is the majority of the votes between y and z. Let us look at y and z. How many votes y gets? 1 prefers y over z, 2 prefers y over z, 3 prefers z over y. So, y gets two votes, z gets one vote. So, y defeats z, x defeats y and y defeats z.

Now, let us look at between z and x. X is preferred to z by x, but for 2 and 3, z is preferred to x. So, z gets 2 votes, x gets 1 vote. So, z defeats x and it basically means that there is no condorcet winner. Between x and y, x wins, but between x and z, z wins again. Z is the condorcet winner, no z is again defeated by y. y is the condorcet winner, no y is defeated by x. So, we have a kind of circular pattern that x is defeating y, y is defeating z, but z is defeating x. So, there is no condorcet winner in this preference.

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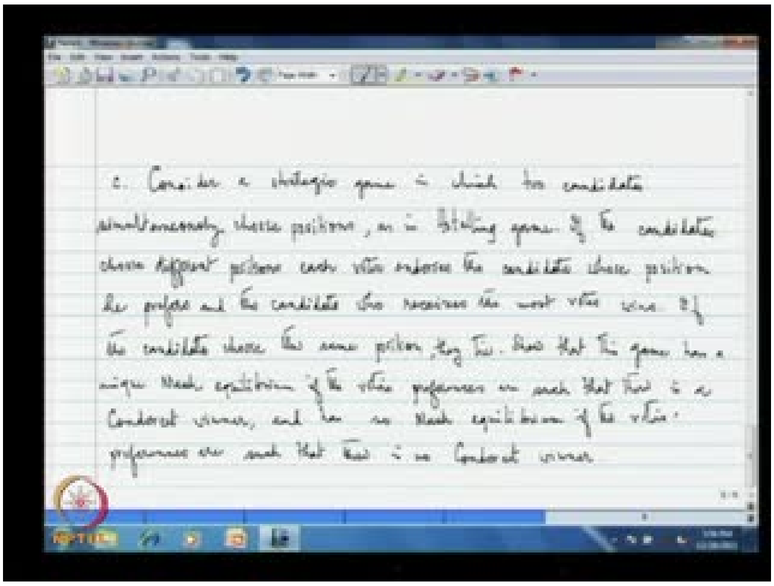

Let us look at last part of the question. So, this is the question - consider a strategic game, in which two candidates simultaneously choose positions. As in hotelling's game, if the candidates choose different positions, each voter endorse the candidate, whose position he prefers and the candidate, who receives the most votes wins.

If the candidates choose the same position, they tie. It shows that the game has a unique Nash equilibrium, if the voter's preferences as such that there is a condorcet winner. It has no Nash equilibrium, if the voter's preferences are such that there is no condorcet winner.

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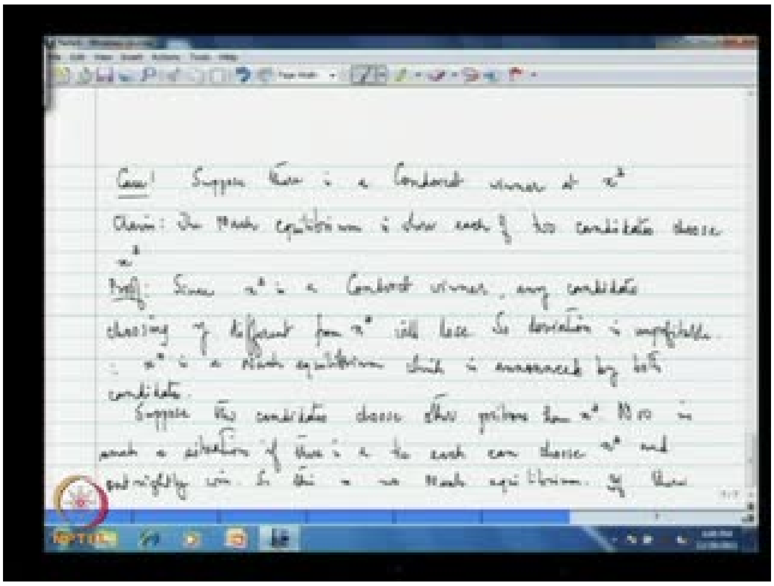

We shall start with first case. Case 1: suppose, there is a condorcet winner at x star to show that there is a Nash equilibrium. In this case, my claim is that the Nash equilibrium is where, each of two candidates choose x star and this is Nash equilibrium. Now, proof is since x is a condorcet winner, any candidate choosing y different from x star will lose. So, deviation is unprofitable. In fact, this is strictly bad. Now, this proves that x star is a Nash equilibrium, but what is the proof that this is the only Nash equilibrium? Is there any other Nash equilibrium? Suppose, the candidates choose other positions, other than x star, in such a situation, if there is a tie, each can choose x star and out rightly win. So, this is not Nash equilibrium.

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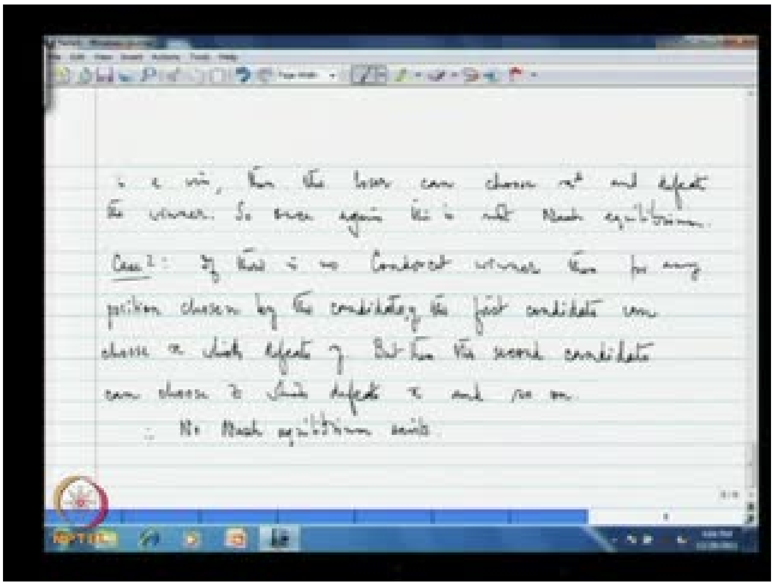

If there is a win, then the loser can choose x star and defeat the winner. So, once again this is not Nash equilibrium. We have ruled out all possibilities and we have seen that x star is the only Nash equilibrium. The second part is case 2 that there is no condorcet winner. If there is no condorcet winner, then for any position chosen by the candidate y, first candidate can choose x, which defeats y, but then, the second candidate can choose z, which defeats x and so on.

In this case, since there is no condorcet winner, there is no Nash equilibrium for any configuration opposition chosen by the two candidates. A person, who is losing can change his position and pick a Condorcet. He can pick another position to defeat the winner and so on, if they are tied at a particular position. Again that is not an equilibrium because the position, which defeats this position can be chosen by any of the candidates. Again, he can do better for himself and therefore, in this case, no Nash equilibrium exists.

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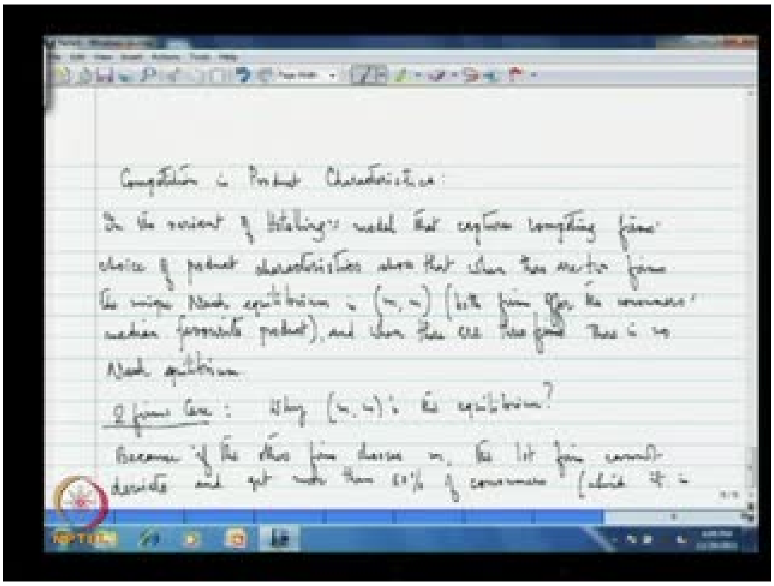

This is an exercise about product characteristics. This again is a variant of the hotelling's model. This is the question in the variant of the hotelling's model that captures competing firms. Choice of product characteristics show that when there are two firms, the unique Nash equilibrium is m, m. Both firms offer the consumer's median favorite product. When there are three firms, there is no Nash equilibrium. So, there are two parts to this question. First - two firm's case- In this case, we have to show that the unique Nash equilibrium is at m, m.

Here, m is the median of the favorite positions of the consumers. How to show this? Suppose, first we show that m, m is the equilibrium and then we show that there is no other equilibrium. Why m, m is the equilibrium? Because, if the other firm chooses m, the first firm cannot deviate and get more than 50 percent of consumers that it is getting right now.

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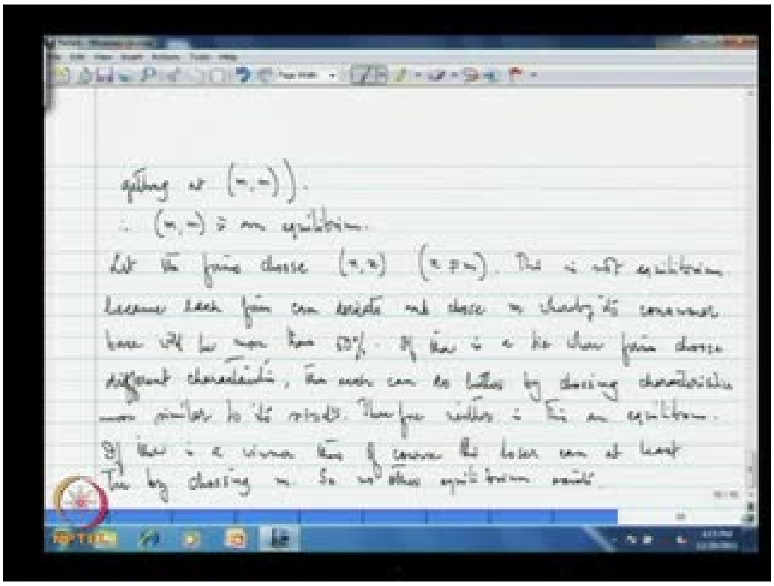

At m, m, every firm is getting 50 percent of the consumers. Now, if any firm deviates and chooses some other characteristic other than m, then the total percentage of consumers that it gets will surely drop from 50 percent to and get less than 50 percent. Therefore, m, m is an equilibrium. Why there is no other equilibrium? Well, let the firms choose x, x, where x is not equal to m. Is this equilibrium? This is not equilibrium because each firm can deviate and choose m, where by its consumer base will be more than 50 percent. We are getting 50 percent x, x. So, this is not a equilibrium.

If there is a tie, where firms choose different characteristics, then each can do better by choosing characteristics more similar to its rival. Therefore, is this equilibrium? If there is a win or loss situation, if there is a winner, then of course the loser can at least tie by choosing m. So, no other configuration of actions is in equilibrium and no other equilibrium exists. So, first part is done and what about the second part? In Second part, we have three firms.

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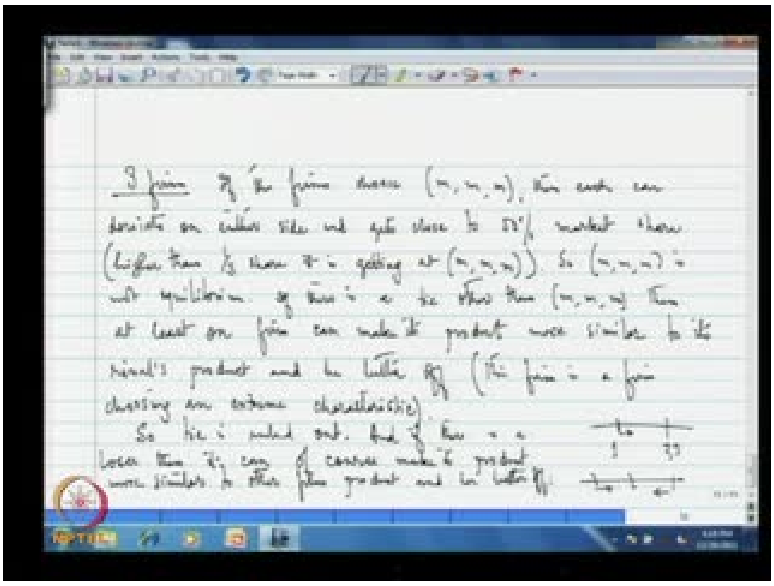

Now, if we have three firms, if the firms choose m and all the firms are choosing the median position, then each can deviate on either side and get close to 50 percent market share. This is higher than one-third share; it is getting at m, m, m. So, profitable deviation exists and m, m, m is not equilibrium. If there is a tie other than m, m, m, then at least one firm can make its product more similar to its rival’s product and be better off. This firm is a firm, choosing an extreme characteristic. So, if there is a tie and the firms are choosing different characteristics, then at least one firm will be there, who is choosing a characteristics to one side. This is why we are calling it as an extreme characteristic to change its product and make its product more similar to the rival to its rival’s product. If he does so, then it will be better off like this.

Suppose, this firm 1 is choosing here and firm 2 or 3 are choosing someone here (Refer Slide Time: 50:23) having same position 2 and 3. There is a tie and this firm can go to this side and capture a high share of the market. If they are choosing all different positions and sharing the market equally, both these firms can come inside and increase the market share. So, tie is not possible and tie is ruled out. If there is a looser and if there is no tie, then it can of course or make its product more similar to other firm’s product and be better off.

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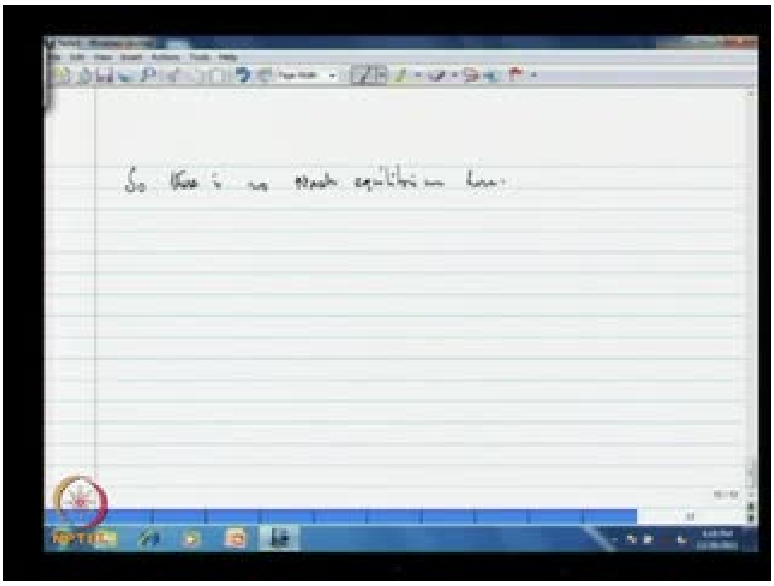

Therefore, we see that there is no Nash equilibrium in this game. We shall talk about some other application and some Nash equilibrium in the next lecture. Thank you.

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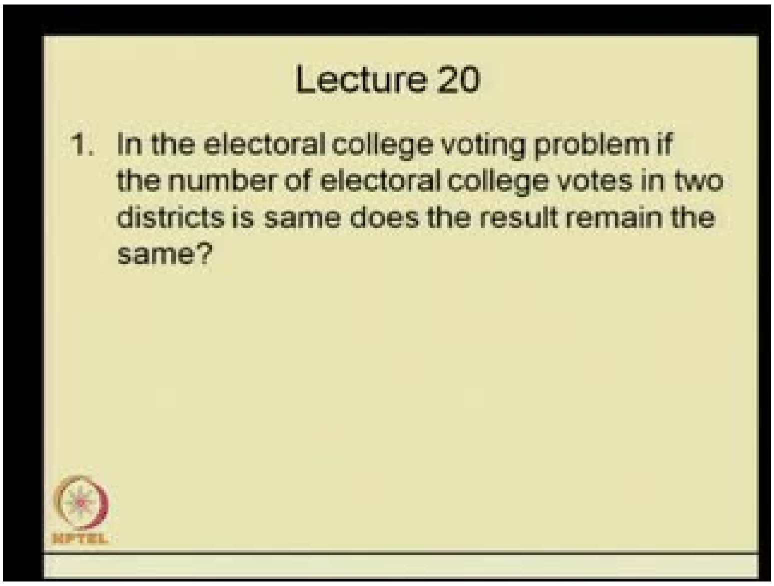

In the electoral college voting problem, if the number of electoral college votes in two districts is same, Does the result remain the same?

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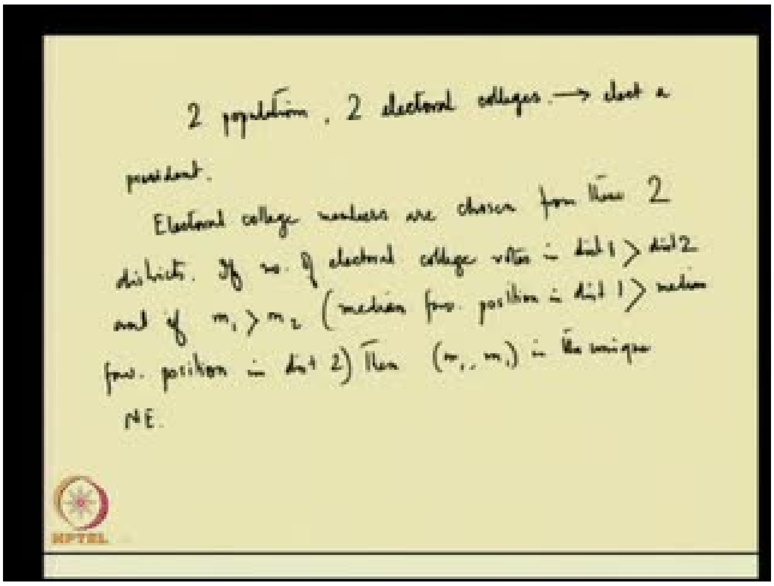

Just to refresh our memory, what was the electoral college voting problem? Two populations are there and therefore, two electoral colleges are there. These electoral college members elect a president, how are these electoral college members chosen? These electoral college members are chosen from these two populations or in districts. We have seen that if the number of electoral college votes in district 1 is more in than in district 2. If m 1 is greater than m 2, then median favorite position in district 1 is more than median favorite position in district 2 and then m 1, m 1 is the unique Nash equilibrium. So, this is what we have seen.

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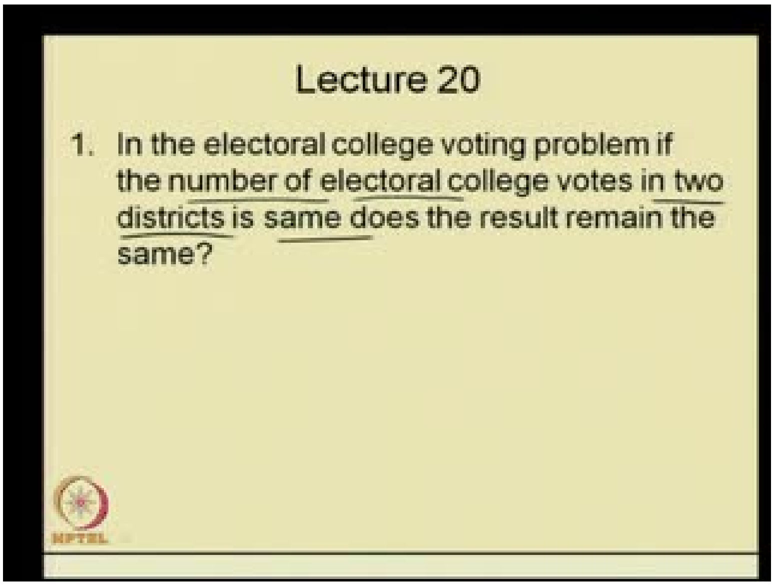

Here, the question is the following that number of electoral college votes in two districts is the same.

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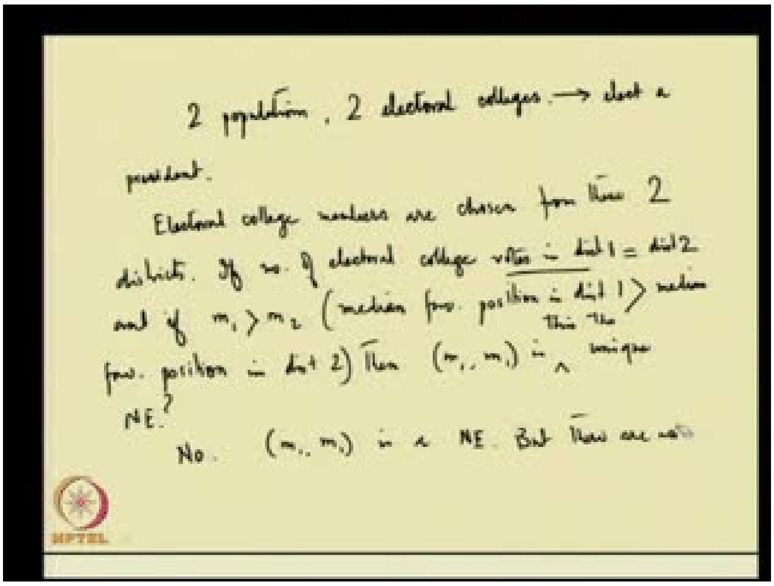

We have a change in assumption here. Here, this is equal to this and other things remain the same. Then, is this the unique Nash equilibrium that is the question and the answer is no. The complete answer is the following: m 1, m 1 is Nash equilibrium, but there are also other Nash equilibrium.

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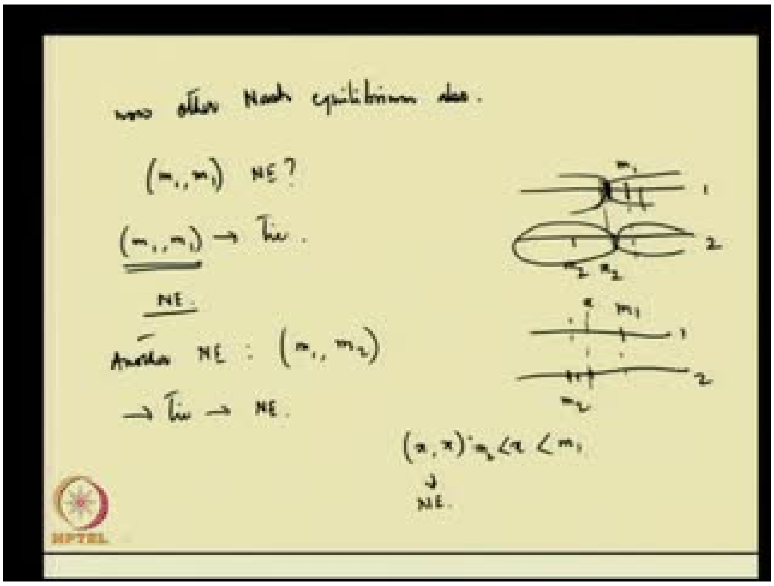

How do I know that? Well, let us look at m 1, m 1. So, this is m 2 and this is m 1; this is district 1 and this is district 2. Now, why m 1, m 1 is still a Nash equilibrium? The reason is that if both the candidates announce m 1, then obviously they are announcing the same position and there is a tie. Suppose, player 2 is considering a deviation and player 2 announces something less than m 1. What happens then? Well, player 2 gets all these votes of district 2 and a player 1 gets all these votes of district 2, which means player 2 wins district 2. However, player 1 wins district 1 and player 1 wins district 1. So, player 2 wins district 2, player 1 wins district 1.

We know that the number of electoral college votes in both the districts is the same, so the tie remains. There was a tie m 1, m 1 and there is a tie now, if player 2 deviates to something less than m 1. So, m 1, m 1 remains an equilibrium because there could be another deviation. Player 2 could deviate to the right; however, if player 2 deviates to the right, we can show that player 2 loses both the districts. Player 1 becomes the outright winner, so deviation by player 2 is not profitable. Therefore, m 1, m 1 is a Nash equilibrium. This demonstration itself shows us that there could be other Nash equilibrium also. For example, one Nash equillibrium could be m 1, m 2.

Here, player 1 is announcing m 1 and player 2 is announcing m 2. This means that player 2 will win district 2 and player 1 will win district 1. So, there will be equal division of the electoral college votes and that will be a tie. This is a Nash equilibrium because we cannot deviate and win outright and none of the players can do that. In fact, any player, which lies in this range from m 2 to m 1 can be thought to be in equilibrium. This can be done as an exercise that if I take any player, suppose x, x, which is such that x lies between m 1 and m 2. This is Nash equilibrium.

In this case also, there is going to be tie, but what is happening here is that suppose, this is x. If player 2 deviates to the left of x, he wins district 2, but loses district 1 at x, x. They are in a tying position in both the districts. There is an overall tie, so there is infinite number of equilibrium in this case. Thank you.